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SOLVING INVERSE NODAL PROBLEM WITH FROZEN ARGUMENT BY USING SECOND CHEBYSHEV WAVELET METHOD

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Abstract. We consider the inverse nodal problem for Sturm-Liouville (S-L) equation with frozen argument. Asymptotic behaviours of eigenfunctions, nodal parameters are represented in two cases and numerical algorithms are produced to solve the given problems. Subsequently, solution of inverse nodal problem is calculated by the second Chebyshev wavelet method (SCW), accuracy and effectiveness of the method are shown in some numerical examples.

Keywords: Sturm-Liouville equation; inverse nodal problem; Frozen argument; nodal parameters; SCW method

MSC 2020: 34A55, 34B99, 34L40, 35R30, 35Q60

1. Introduction

The Sturm-Liouville problem with frozen argument under following boundary conditions has considerable significance in mathematical physics. Such an important problem has been solved by an appropriate method and the effectiveness of the method has been demonstrated.

Consider the differential equation

$$-y''(x) + q(x)y(a) = \lambda y(x), \quad 0 < x < \pi, \tag{1.1}$$

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under separated boundary conditions

$$y^{(\alpha)}(0) = y^{(\beta)}(\pi) = 0, \quad \alpha, \beta \in \{0, 1\}, \tag{1.2}$$

where $q(x) \in W_2^1(0, \pi)$ is a real valued potential function, $\lambda = \varrho^2$ is spectral parameter in (1.1), and $a \in (0, \pi)$ is called the frozen argument. The inverse spectral problem of S-L equation with frozen argument has attracted numerous physicists and mathematicians, [3], [8], [11], [19], [34]. For the case $a/\pi = 1/k$, $k \in \mathbb{N}$, Bondarenko et al. [3] recovered $q(x)$ on $[0, \pi]$ from the spectrum for non-degenerate or degenerate cases together with an additional condition on $q(x)$. The generalized results for $a/\pi = j/k$, $j+1, k \in \mathbb{N}$ were found in [8], [11]. For irrational case $a/\pi \in \mathbb{Q}^c$, Wang et al. [34] reconstructed $q(x)$ from the spectrum without any restriction. In this way, the authors responded to some open problems posed by previous studies on this topic.

Inverse spectral problem for differential operators is to construct this operator from its spectral data. Eigenvalues, spectral functions or nodal parameters can be used to determine the operator being studied. Setting $y(x)$ instead of $y(a)$ in (1.1), (1.1)–(1.2) becomes the well known S-L problem. Inverse spectral problems for usual S-L operator have been studied almost completely (see [6], [27], [28], [40]), and numerical solutions of inverse problems for S-L operator are investigated in [1], [15], [17], [24], [35]. Lately, inverse spectral problems of diverse differential operators have been examined widely, like differential operators with a constant delay (see [5], [4], [9], [12], [26], [30], [32]), with integral delay (see [20], [40]), non-local differential operators (see [2], [25]), and others.

While two pairs of information sequences are needed in other types of inverse problems, a single nodal point sequence is sufficient to determine the operator in inverse nodal problems. This type of problem is a very new method compared to the others. So far, many researchers have studied inverse nodal problems for different types equations (see [7], [10], [13], [14], [16], [21], [22], [23], [29], [31], [33], [36], [38], [37], [39]). In [18], Hu et al. considered BVP (1.1)–(1.2) and computed the asymptotic expressions of eigenfunctions and nodal points for $\alpha = \beta = 0$ and gave a constructive process to solve the inverse nodal problem. In [1], Akbarpoor et al. perused the S-L equation under spectral parameter dependent boundary conditions and applied FCW and Chebyshev interpolation methods to compute the approximate solution of the inverse nodal problem. Numerical solutions for fractional integro-differential equations with a weakly singular kernel was studied by Wang and Zhu [35] from the SCW method.

Here, we focus on problem (1.1)–(1.2) in two cases $\alpha = \beta = 0$ and $\alpha = 1$, $\beta = 0$, present the asymptotic expressions of eigenfunctions and nodal points and solve the inverse nodal problem by the SCW method. The cases $\alpha = \beta = 1$ and $\alpha = 0$, $\beta = 1$ require the condition $q \in W_2^2(0, \pi)$ on q, however the method is similar to that in this paper. The potential $q(x) \in L^2(0, \pi)$ requires a separated investigation.

The rest of this work is organized as follows. In Section 2, asymptotic forms of eigenfunctions, nodal points and uniqueness theorems are revealed. In Section 3, we present numerical algorithms to get the solution of inverse nodal problem, approximation of solution for the given problem by using SCW method. Consequently, some numerical examples are considered in Section 4.

2. Some uniqueness theorems

Here, we suppose that $0 < a \leqslant \pi/2$ without loss of generality. Let $C(x,\lambda)$ and $S(x,\lambda)$ be the solutions of (1.1) with the initial conditions (see [3])

$$C(a,\lambda) = S'(a,\lambda) = 1, \quad S(a,\lambda) = C'(a,\lambda) = 0.$$

Thus, we can write

$$S(x,\lambda) = \frac{\sin\varrho(x-a)}{\varrho}, \quad S'(x,\lambda) = \cos\varrho(x-a), \tag{2.1}$$

$$C(x,\lambda) = \cos\varrho(x-a) + \int_a^x q(t)\frac{\sin\varrho(x-t)}{\varrho}\,\mathrm{d}t, \tag{2.2}$$
$$C'(x,\lambda) = -\varrho\sin\varrho(x-a) + \int_a^x q(t)\cos\varrho(x-t)\,\mathrm{d}t.$$

Denote the characteristic functions $\Delta_{0,0}(\lambda)$ and $\Delta_{1,0}(\lambda)$, respectively, by

$$\begin{aligned} \Delta_{0,0}(\lambda) &= C(0,\lambda)S(\pi,\lambda) - S(0,\lambda)C(\pi,\lambda), \\ \Delta_{1,0}(\lambda) &= C'(0,\lambda)S(\pi,\lambda) - S'(0,\lambda)C(\pi,\lambda). \end{aligned} \tag{2.3}$$

It follows from (2.1)–(2.3) that

$$\begin{aligned} \Delta_{0,0}(\lambda) = {} & \frac{\sin(\varrho\pi)}{\varrho} + \frac{\sin(\varrho(\pi-a))}{\varrho^2}\int_0^a q(t)\sin(\varrho t)\,\mathrm{d}t \\ & + \frac{\sin(\varrho a)}{\varrho^2}\int_a^\pi q(t)\sin(\varrho(\pi-t))\,\mathrm{d}t, \\ \Delta_{1,0}(\lambda) = {} & -\cos(\varrho\pi) - \frac{\sin(\varrho(\pi-a))}{\varrho}\int_0^a q(t)\cos(\varrho t)\,\mathrm{d}t \\ & - \frac{\cos(\varrho a)}{\varrho}\int_a^\pi q(t)\sin(\varrho(\pi-t))\,\mathrm{d}t. \end{aligned}$$

The asymptotic form of eigenvalues, for sufficiently large n, can be denoted by (see [18])

(2.4)

$$\varrho_{\alpha,\beta,n} := \sqrt{\lambda_{\alpha,\beta,n}} = \begin{cases} n + \dfrac{\xi_{0,0,n}}{n^2\pi} + \dfrac{\eta_{0,0,n}}{n^2}, & \alpha = 0,\ \beta = 0,\ \{\eta_{0,0,n}\} \in l^2, \\ n - \dfrac{1}{2} + \dfrac{\xi_{1,0,n}}{(n-1/2)^2\pi} + \dfrac{\eta_{1,0,n}}{n^2}, & \alpha = 1,\ \beta = 0,\ \{\eta_{1,0,n}\} \in l^2, \end{cases}$$

where

$$\xi_{\alpha,\beta,n} = \begin{cases} \sin(na)((-1)^{n+1}q(\pi) + q(0)), & \alpha = 0,\ \beta = 0, \\ (-1)^{n+1}\cos((n-1/2)a)q(\pi), & \alpha = 1,\ \beta = 0. \end{cases}$$

Now, we present the asymptotic forms of the eigenfunctions and nodal points for the cases $\alpha = \beta = 0$ and $\alpha = 1$, $\beta = 0$.

Case A. Let $(\alpha, \beta) = (0, 0)$: In this case, it is easy to show that

$$y(x, \lambda_{0,0,n}) = C(0, \lambda_{0,0,n})S(x, \lambda_{0,0,n}) - S(0, \lambda_{0,0,n})C(x, \lambda_{0,0,n})$$

is the eigenfunction corresponding to eigenvalue $\lambda_{0,0,n}$. Taking (2.1)–(2.2) into account, we obtain (see [18])

$$\begin{aligned} (2.5)\quad y(x, \lambda_{0,0,n}) = {}& \frac{\sin(\varrho_{0,0,n}x)}{\varrho_{0,0,n}} + \frac{1}{\varrho^3_{0,0,n}}[q(0)\sin(\varrho_{0,0,n}(x-a)) - q(a)\sin(\varrho_{0,0,n}x)] \\ & + \frac{1}{\varrho^3_{0,0,n}}q(x)\sin(\varrho_{0,0,n}a) + \mathrm{o}\Big(\frac{1}{\varrho^3_{0,0,n}}\Big) \end{aligned}$$

for sufficiently large n. Let $0 < x^1_{0,0,n} < x^2_{0,0,n} < \ldots < x^{n-1}_{0,0,n} < \pi$ be the nodal points of the nth eigenfunction. Then for $j = 1, 2, \ldots, n-1$, the nodal points satisfy the asymptotic formula

$$(2.6)\qquad x^j_{0,0,n} = \frac{j\pi}{\varrho_{0,0,n}} + \frac{\sin(\varrho_{0,0,n}a)}{\varrho^3_{0,0,n}}\Big[(-1)^{j+1}q\Big(\frac{j\pi}{\varrho_{0,0,n}}\Big) + q(0)\Big] + \mathrm{o}\Big(\frac{1}{\varrho^3_{0,0,n}}\Big).$$

Furthermore, the more precise asymptotic can follow from (2.4) and (2.5) for $x^j_{0,0,n}$ in the form

$$x^j_{0,0,n} = \frac{j\pi}{n} + \frac{1}{n^3}\Big[(-1)^{j+1}q\Big(\frac{j\pi}{n}\Big)\sin(na) + q(0)\sin(na) - \frac{j\xi_{0,0,n}}{n}\Big] + \mathrm{o}\Big(\frac{1}{n^3}\Big).$$

Similarly to Case A, by some calculations of (2.1)–(2.3), we have:

Case B. Let $(\alpha, \beta) = (1, 0)$: Then

$$y(x, \lambda_{1,0,n}) = S'(0, \lambda_{1,0,n})C(x, \lambda_{1,0,n}) - C'(0, \lambda_{1,0,n})S(x, \lambda_{1,0,n})$$

is the eigenfunction related to eigenvalue $\lambda_{1,0,n}$. Hence, from (2.1)–(2.2) it yields that

(2.7)

$$y(x, \lambda_{1,0,n}) = -\cos(\varrho_{1,0,n}x) + \frac{1}{\varrho^2_{1,0,n}}[q(a)\cos(\varrho_{1,0,n}x) - q(x)\cos(\varrho_{1,0,n}a)] + \mathrm{o}\Big(\frac{1}{\varrho^2_{1,0,n}}\Big).$$

Consequently, for $j = 1, 2, \ldots, n-1$, the asymptotic forms of nodal points satisfy the following formula:

$$x^j_{1,0,n} = \frac{(j-1/2)\pi}{\varrho_{1,0,n}} + \frac{(-1)^{j+1} q((j-1/2)\pi\varrho^{-1}_{1,0,n}) \cos(\varrho_{1,0,n} a)}{\varrho^3_{1,0,n}} + \mathrm{o}\Big(\frac{1}{\varrho^3_{1,0,n}}\Big). \tag{2.8}$$

Also, by (2.4) and (2.7), we can obtain the asymptotic formula
(2.9)

$$\begin{aligned} x^j_{1,0,n} &= \frac{(j-1/2)\pi}{n-1/2} + \frac{1}{(n-1/2)^3} \\ &\quad \times \Big[(-1)^{j+1} q\Big(\frac{(j-1/2)\pi}{n-1/2}\Big) \cos\Big(\Big(n-\frac{1}{2}\Big)a\Big) - \frac{(j-1/2)\xi_{1,0,n}}{n-1/2}\Big] + \mathrm{o}\Big(\frac{1}{n^3}\Big). \end{aligned}$$

Let $X_{\alpha,0} := \{x^j_{\alpha,0,n}\}$, $\alpha = 0, 1$. Then $X_{\alpha,0}$, $\alpha = 0, 1$, is a dense-nodal set on $(0, \pi)$. Now, we determine the reconstruction formula of $q(x)$ on $(0, \pi)$.

R e m a r k 2.1 ([18], Theorem 4.2). Hu et al. proved that $q(x)$ on $(0, \pi)$ is specified by the nodal set $X_{0,0}$ with an algorithm where a is known.

Theorem 2.1. *Let $x^{j_k}_{1,0,n_k} \in X_{1,0}$ be chosen such that $\lim\limits_{k\to\infty} x^{j_k}_{1,0,n_k} = x$. Then there exist the functions of $f_{0,0}$ and $f_{1,0}$ which are obtained from the finite limit*

$$\lim_{k\to\infty} \frac{(n_k-1/2)^3}{\cos((n_k-1/2)a)} \Big(x^{j_k}_{1,0,n_k} - \frac{(j_k-1/2)\pi}{n_k-1/2}\Big)$$

for odd and even j_k, respectively. In addition, the potential function $q(x)$ on $(0, \pi)$ is uniquely reconstructed by the nodal set $X_{1,0}$ for problem (1.1)–(1.2) as

$$q(x) = \frac{1}{2}(f_{1,0}(x) - f_{0,0}(x)).$$

P r o o f. For every fixed $x \in (0, \pi)$, we choose $x^{j_k}_{1,0,n_k} \in X_{1,0}$ such that $\lim\limits_{k\to\infty} x^{j_k}_{1,0,n_k} = x$. This implies

$$\lim_{k\to\infty} \frac{(j_k-1/2)\pi}{n_k-1/2} = x.$$

Choose a subsequence of $\{n_k\}_{k=1}^\infty$, which is still denoted as $\{n_k\}_{k=1}^\infty$, such that

$$\cos\Big(\Big(n_k-\frac{1}{2}\Big)a\Big) \neq 0.$$

If j_k is odd, then $j_k \pm 1$ are even. Therefore, we assume all j_k and n_k are even in the case. It follows from (2.9) that

$$
\begin{aligned}
(2.10) \qquad f_{0,0}(x) &:= \lim_{k\to\infty} \frac{(n_k-1/2)^3}{\cos((n_k-1/2)a)}\Big(x^{j_k}_{1,0,n_k} - \frac{(j_k-1/2)\pi}{n_k-1/2}\Big) \\
&= \lim_{k\to\infty}\Big(-q\Big(\frac{(j-1/2)\pi}{n-1/2}\Big) - \frac{(j-1/2)\pi q(\pi)}{n-1/2} + \mathrm{o}(1)\Big) \\
&= -q(x) + q(\pi)x.
\end{aligned}
$$

Similarly, we assume all j_k are odd and n_k are even in the case. So, we have

$$
\begin{aligned}
(2.11) \qquad f_{1,0}(x) &:= \lim_{k\to\infty} \frac{(n_k-1/2)^3}{\cos((n_k-1/2)a)}\Big(x^{j_k}_{1,0,n_k} - \frac{(j_k-1/2)\pi}{n_k-1/2}\Big) \\
&= \lim_{k\to\infty}\Big(q\Big(\frac{(j-1/2)\pi}{n-1/2}\Big) - \frac{(j-1/2)\pi q(\pi)}{n-1/2} + \mathrm{o}(1)\Big) \\
&= q(x) + q(\pi)x.
\end{aligned}
$$

It follows from (2.10) and (2.11) that

$$q(x) = \frac{1}{2}(f_{1,0}(x) - f_{0,0}(x)).$$

□

3. Numerical solution by using SCW method

In this part, we define the inverse nodal problem, present SCW method and then solve this problem by the presented method.

Inverse problem. Let $\{x^j_{\alpha,0,n}\}$, $\alpha = 0,1$, be given. Then we shall construct $q(x)$.

Since $\{x^j_{\alpha,0,n}\}$, $j = \overline{1, n-1}$, $n > 1$, $\alpha = 0,1$, are zeroes of the nth eigenfunction $y(x, \lambda_{\alpha,0,n})$, then it follows that

$$y(x^j_{\alpha,0,n}, \lambda_{\alpha,0,n}) = 0, \quad j = \overline{1, n-1}, \quad n > 1, \quad \alpha = 0,1.$$

Therefore, in case A, formula (2.5) yields

$$
\begin{aligned}
(3.1) \quad q(a)\sin(\varrho_{0,0,n}x^j_{0,0,n}) - q(x^j_{0,0,n})\sin(\varrho_{0,0,n}a)& \\
- q(0)\sin(\varrho_{0,0,n}(x^j_{0,0,n} - a)) &\cong \varrho^2_{0,0,n}\sin(\varrho_{0,0,n}x^j_{0,0,n}),
\end{aligned}
$$

and in case B, formula (2.7) yields

$$(3.2) \qquad q(a)\cos(\varrho_{1,0,n}x^j_{1,0,n}) - q(x^j_{1,0,n})\cos(\varrho_{1,0,n}a) \cong \varrho^2_{1,0,n}\cos(\varrho_{1,0,n}x^j_{1,0,n}).$$

In two cases A and B, we can get the solution of the inverse nodal problem by using formulas (3.1) and (3.2). For this purpose, we apply the SCW method, approximate q by the second kind Chebyshev polynomials as fundamental functions and convert formulas (3.1) and (3.2) to linear equations system.

In the interval $(0, \pi)$, arbitrary function $f(t)$ can be expressed by

$$f(t) \cong \sum_{l=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{lm}\psi_{lm}(t) = C^\top \psi(t), \tag{3.3}$$

where

$$\begin{aligned} C &= [c_{10}, c_{11}, \dots, c_{1(M-1)}, c_{20}, \dots, c_{2(M-1)}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}(M-1)}]^\top, \\ \psi(t) &= [\psi_{10}(t), \dots, \psi_{1(M-1)}(t), \psi_{20}(t), \dots, \psi_{2(M-1)}(t), \dots, \\ &\qquad\qquad \psi_{2^{k-1}0}(t), \dots, \psi_{2^{k-1}(M-1)}(t)]^\top, \end{aligned}$$

and functions $\psi_{lm}(t)$, $l = 1, 2, \dots, 2^{k-1}$, $m = 0, 1, \dots, M-1$, are the SCW on the interval $(0, \pi)$ as [22]

$$\psi_{lm}(t) = \begin{cases} 2^{k/2}\widetilde{U}_m\Big(\dfrac{2^k t}{\pi} - 2l + 1\Big), & \dfrac{(l-1)\pi}{2^{k-1}} < t < \dfrac{l\pi}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$\widetilde{U}_m(t) = \sqrt{\frac{2}{\pi}}\, U_m(t),$$

and k can be any positive integer. Second kind Chebyshev polynomials $U_m(t)$ on $[-1, 1]$ are defined by below recursive formula:

$$\begin{cases} U_0(t) = 1, \quad U_1(t) = 2t, \\ U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, \dots \end{cases}$$

Now, we obtain approximation of $q(t)$ by SCW method. For this reason, using (3.3), we write

$$q(t) \cong \sum_{l=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{lm}\psi_{lm}(t). \tag{3.4}$$

Substituting (3.4) into (3.1) and (3.2), we get

$$\sum_{l=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{lm}R_{lm}(x^j_{\alpha,0,n}) \cong g(x^j_{\alpha,0,n}), \quad n > 1, \quad j = \overline{1, n-1}, \quad \alpha = 0, 1,$$

where

$$R_{lm}(x^j_{\alpha,0,n}) = \begin{cases} \psi_{lm}(a)\sin(\varrho_{0,0,n}x^j_{0,0,n}) - \psi_{lm}(x^j_{0,0,n})\sin(\varrho_{0,0,n}a) \\ \qquad\qquad\qquad -\psi_{lm}(0)\sin(\varrho_{0,0,n}(x^j_{0,0,n}-a)), & \alpha = 0, \\ \psi_{lm}(a)\cos(\varrho_{1,0,n}x^j_{1,0,n}) - \psi_{lm}(x^j_{1,0,n})\cos(\varrho_{1,0,n}a), & \alpha = 1, \end{cases}$$

and

$$g(x^j_{\alpha,0,n}) = \begin{cases} \varrho^2_{0,0,n}\sin(\varrho_{0,0,n}x^j_{0,0,n}), & \alpha = 0, \\ \varrho^2_{1,0,n}\cos(\varrho_{1,0,n}x^j_{1,0,n}), & \alpha = 1. \end{cases}$$

Suppose that the nodal points $\{x^j_{\alpha,0,n}\}$, $j = \overline{1, n-1}$, $n > 1$, $\alpha = 0, 1$, are given. To calculate the approximation of $q(t)$, we use the following steps:

1. Choose k, M, a. Then take $n = 2^{k-1}M + 1$.
2. Calculate the vector C by below linear system:

$$AC = B,$$

where

$$A = [A_1, A_2, \ldots, A_{2^{k-1}}], \quad A_l = \begin{pmatrix} a^1_{l0} & a^1_{l1} & \ldots & a^1_{l(M-1)} \\ a^2_{l0} & a^2_{l1} & \ldots & a^2_{l(M-1)} \\ \vdots & & & \vdots \\ a^{m'}_{l0} & a^{m'}_{l1} & \ldots & a^{m'}_{l(M-1)} \end{pmatrix}, \quad l = \overline{1, 2^{k-1}},$$

and

$$a^j_{lm} = \begin{cases} \psi_{lm}(a)\sin(\varrho_{0,0,n}x^j_{0,0,n}) - \psi_{lm}(x^j_{0,0,n})\sin(\varrho_{0,0,n}a) \\ \qquad\qquad\qquad -\psi_{lm}(0)\sin(\varrho_{0,0,n}(x^j_{0,0,n}-a)), & \alpha = 0, \\ \psi_{lm}(a)\cos(\varrho_{1,0,n}x^j_{1,0,n}) - \psi_{lm}(x^j_{1,0,n})\cos(\varrho_{1,0,n}a), & \alpha = 1, \end{cases}$$

$$l = \overline{1, 2^{k-1}}, \quad m = \overline{0, M-1}, \quad m' = 2^{k-1}M, \quad n = m' + 1, \quad j = \overline{1, n-1},$$

$$B = [b_1, b_2, \ldots, b_{n-1}]^\top, \quad j = \overline{1, n-1}, \quad b_j = \begin{cases} \varrho^2_{0,0,n}\sin(\varrho_{0,0,n}x^j_{0,0,n}), & \alpha = 0, \\ \varrho^2_{1,0,n}\cos(\varrho_{1,0,n}x^j_{1,0,n}), & \alpha = 1, \end{cases}$$

and

$$C = [c_{10}, c_{11}, \ldots, c_{1(M-1)}, c_{20}, \ldots, c_{2(M-1)}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}(M-1)}]^\top.$$

3. Get the values of $q(t_i)$, $i = \overline{1, m'}$ by

$$[q(t_i)] = C^\top \Phi,$$

where

$$t_i = \frac{(2i-1)\pi}{2^k M}, \quad i = 1, 2, \ldots, 2^{k-1}M,$$

and

$$\Phi = \Big[\psi\Big(\frac{\pi}{2m'}\Big), \psi\Big(\frac{3\pi}{2m'}\Big), \ldots, \psi\Big(\frac{(2m'-1)\pi}{2m'}\Big)\Big], \quad m' = 2^{k-1}M.$$

4. Numerical examples

Here, SCW method is used to solve inverse nodal problem (1.1)–(1.2) and the accuracy of the presented method is shown by providing a few numerical examples. For solving the numerical examples, we use Matlab software program.

Example 4.1. Let $q(x) = \sqrt{x^2+1} - x/(x^2+1)$, $a = \pi/2$ and $\alpha = \beta = 0$. Let $M = 3$ and $k = 4$, then the numerical values of nodal points x_n^j, $n = 2^{k-1}M+1 = 25$, $j = \overline{1, n-1} = \overline{1, 24}$ are shown in Table 1.

j	1	2	3	4	5	6	7	8
x_n^j	0.1258	0.2513	0.3771	0.5027	0.6284	0.7540	0.8798	1.0053
j	9	10	11	12	13	14	15	16
x_n^j	1.1311	1.2566	1.3824	1.5079	1.6338	1.7593	1.8851	2.0106
j	17	18	19	20	21	22	23	24
x_n^j	2.1365	2.2619	2.3878	2.5132	2.6392	2.7645	2.8905	3.0158

Table 1. Numerical values of nodal points x_n^j in Example 4.1 for $n = 25$.

Assume that the nodal points given in Table 1 and q, with $M = 3$ and $k = 4$, be the input data and unknown function, respectively. We compute the approximate value of q as the solution of inverse nodal problem by SCW method. We draw the exact solution and numerical approximations with $M = 3$ and different k by using SCW shown in Figure 1. It is seen that SCW is an effective way to solve inverse nodal problems.

In addition, the relative L^2-norm errors between approximate and exact solutions with $M = 3$ and different k can be seen in Figure 2.

Example 4.2. Let $q(x) = \sec(x/3) - \csc((x+1)/2)$, $a = 1$, and $\alpha = 1$, $\beta = 0$ be given. Then we obtain the numerical values of nodal points with $M = 3$ and $k = 4$ given in Table 2.

j	1	2	3	4	5	6	7	8
x_n^j	0.0641	0.1924	0.3205	0.4488	0.5770	0.7053	0.8335	0.9617
j	9	10	11	12	13	14	15	16
x_n^j	1.0899	1.2182	1.3464	1.4746	1.6029	1.7311	1.8593	1.9875
j	17	18	19	20	21	22	23	24
x_n^j	2.1158	2.2440	2.3722	2.5004	2.6287	2.7569	2.8852	3.0133

Table 2: Numerical values of nodal points x_n^j in Example 4.2 for $n = 25$.

The exact solution and numerical approximations obtained with $M = 3$ and $k = 3, 4, 5$ and the relative L^2-norm errors between the approximate and exact solutions are shown in Figure 3 and Figure 4, respectively.

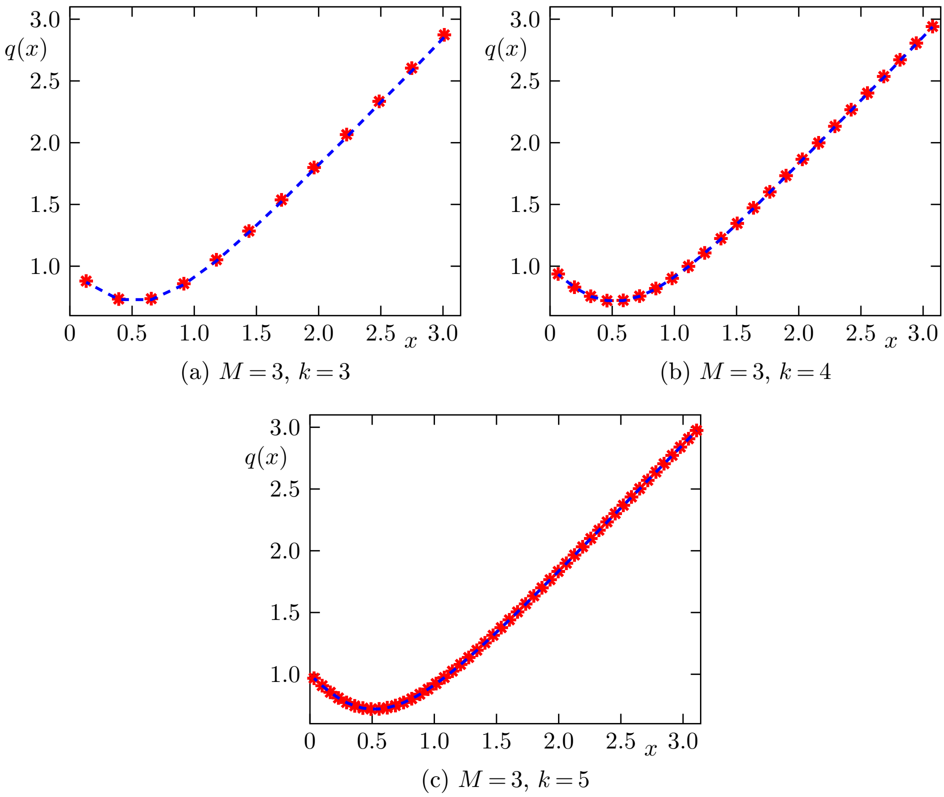


Figure 1. Exact ($***$) and approximate (- - -) solutions of inverse nodal problem in Example 4.1 for different k.

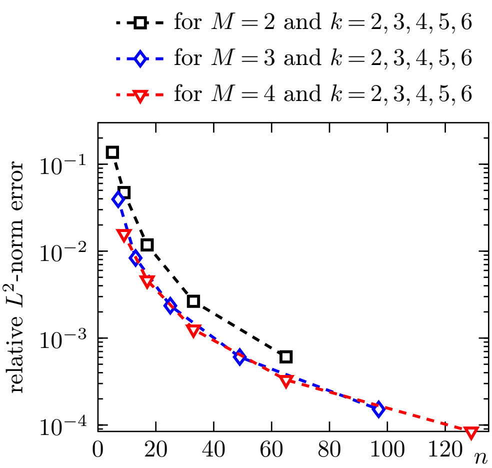
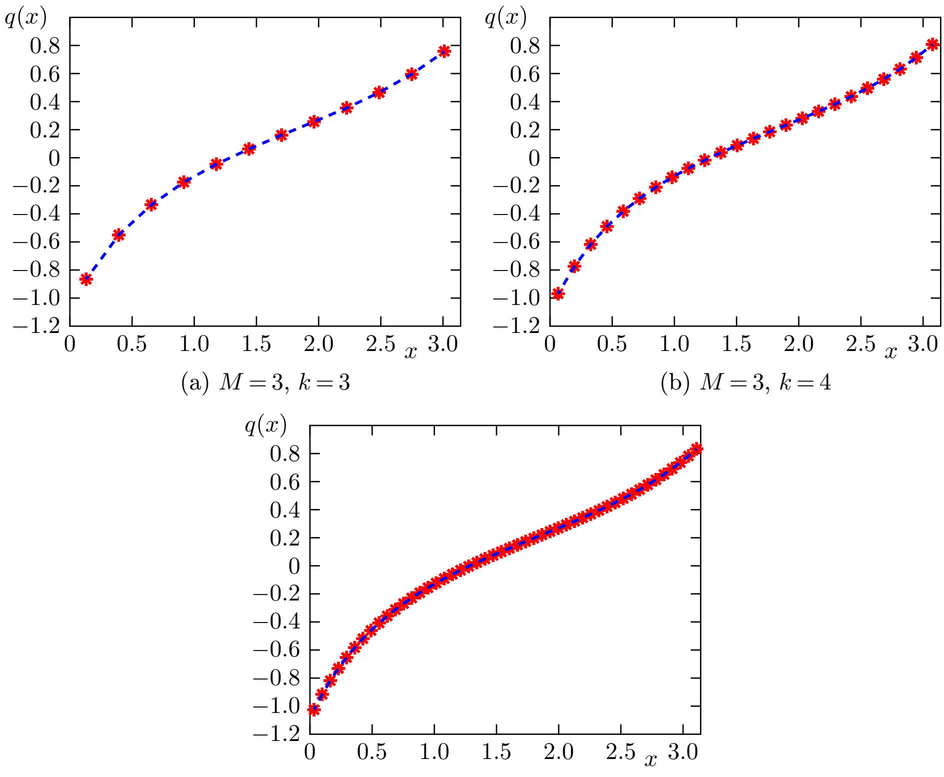


Figure 2. Relative L^2-norm errors between the approximate and exact solutions in Example 4.1.



(a) $M=3,\ k=3$ (b) $M=3,\ k=4$ (c) $M=3,\ k=5$

Figure 3. Exact ($***$) and approximate (- - -) solutions of inverse nodal problem in Example 4.2 for different k.

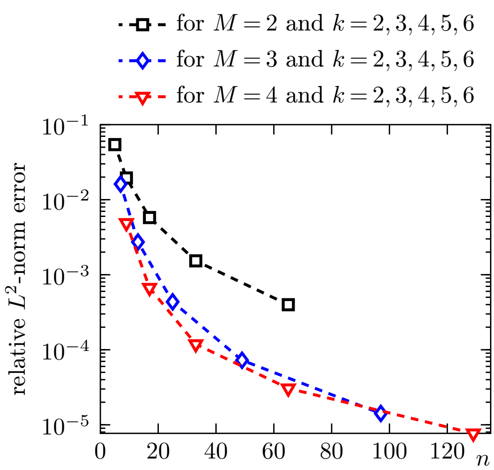


Figure 4. Relative L^2-norm errors between the approximate and exact solutions in Example 4.2.

Example 4.3. Consider $q(x) = \exp x \ln(x+1)$, $a = 0.5$ and $\alpha = 1$, $\beta = 0$. Then the numerical values of nodal points are shown in Table 3.

j	1	2	3	4	5	6	7	8
x_n^j	0.0641	0.1923	0.3206	0.4488	0.5771	0.7052	0.8336	0.9616
j	9	10	11	12	13	14	15	16
x_n^j	1.0901	1.2180	1.3466	1.4744	1.6032	1.7307	1.8597	1.9870
j	17	18	19	20	21	22	23	24
x_n^j	2.1164	2.2433	2.3731	2.4995	2.6298	2.7556	2.8867	3.0115

Table 3. Numerical values of nodal points x_n^j in Example 4.3 for $n = 25$.

The numerical results with $M = 3$ and $k = 3, 4, 5$ are seen in Figure 5 and Figure 6, respectively.

Example 4.4. Assume that $q(x) = (\tanh x)/(x+1)$, $a = 1.2$ and $\alpha = \beta = 0$. Then the numerical values of nodal points are seen in Table 4.

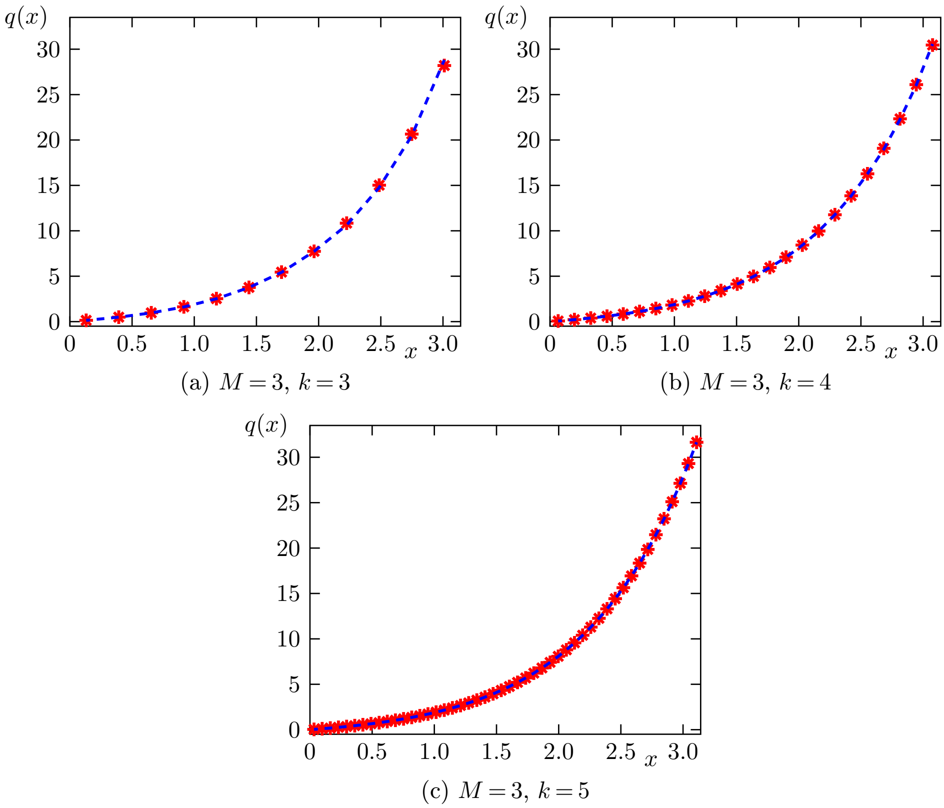
j	1	2	3	4	5	6	7	8
x_n^j	0.1257	0.2513	0.3770	0.5027	0.6283	0.7540	0.8796	1.0053
j	9	10	11	12	13	14	15	16
x_n^j	1.1309	1.2567	1.3823	1.5080	1.6336	1.7593	1.8849	2.0106
j	17	18	19	20	21	22	23	24
x_n^j	2.1363	2.2620	2.3876	2.5133	2.6389	2.7646	2.8902	3.0159

Table 4. Numerical values of nodal points x_n^j in Example 4.4 for $n = 25$.

Also, in Figure 7 and Figure 8, numerical results are shown with $M = 3$ and different k.

5. Conclusion

Frozen-type Sturm-Liouville equation, which has a very important physical value, is discussed. Asymptotic behaviours of the eigenfunctions and nodal points are analyzed for two cases. Then the numerical algorithm is produced to solve the constructed inverse nodal problem. Using some numerical examples, SCW method was shown to be reliable and very effective. Classical inverse problems can be carried forward using such methods. Such studies increase the value of the results obtained before.



(a) $M=3,\ k=3$ (b) $M=3,\ k=4$

(c) $M=3,\ k=5$

Figure 5. Exact ($* * *$) and approximate (- - -) solutions of the inverse nodal problem in Example 4.3 for different k.

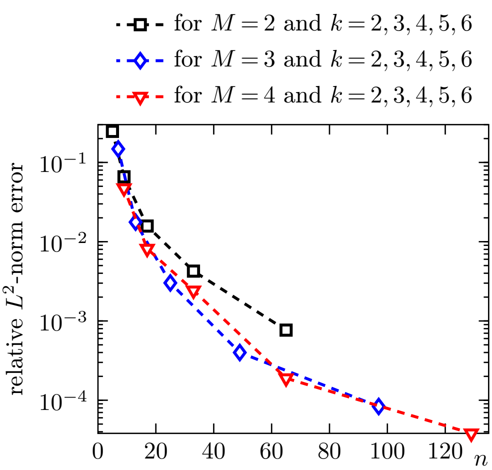
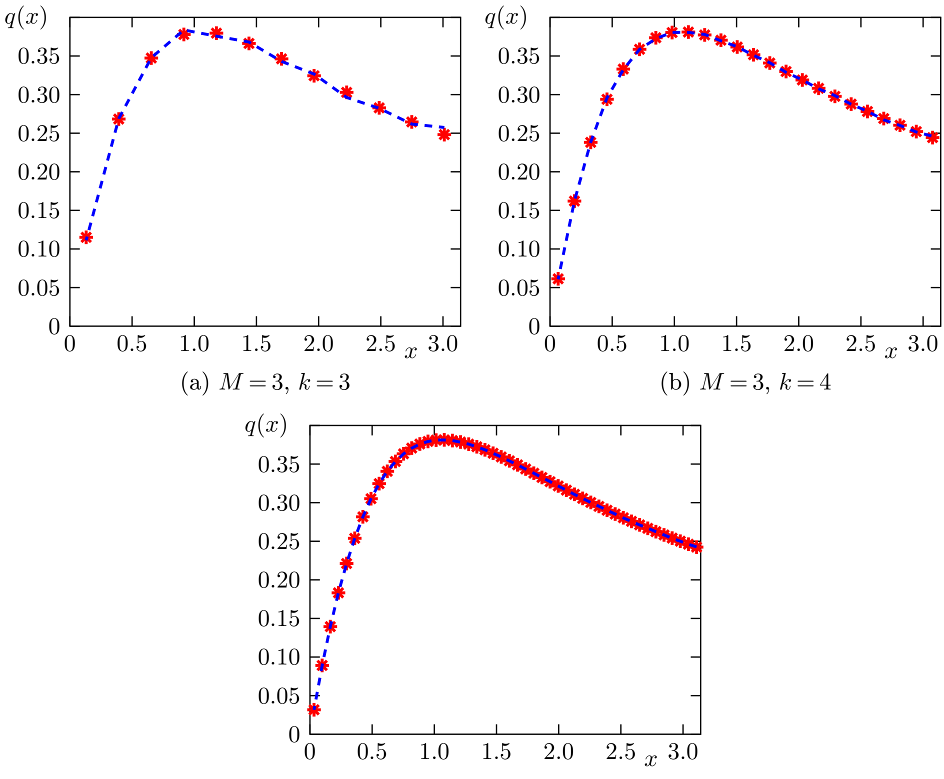


Figure 6. Relative L^2-norm errors between the approximate and exact solutions in Example 4.3.



(a) $M = 3,\ k = 3$ (b) $M = 3,\ k = 4$

(c) $M = 3,\ k = 5$

Figure 7. Exact ($***$) and approximate (- - -) solutions of inverse nodal problem in Example 4.4 for different k.

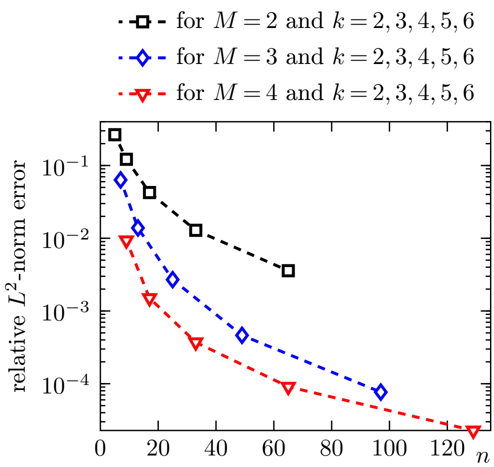


Figure 8. Relative L^2-norm errors between the approximate and exact solutions in Example 4.4.

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