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# A NEW APPROACH TO CONSTRUCT UNINORMS VIA UNINORMS ON BOUNDED LATTICES

Zhen-Yu Xiu and Xu Zheng

In this paper, on a bounded lattice $L$, we give a new approach to construct uninorms via a given uninorm $U^*$ on the subinterval $[0,a]$ (or $[b,1]$) of $L$ under additional constraint conditions on $L$ and $U^*$. This approach makes our methods generalize some known construction methods for uninorms in the literature. Meanwhile, some illustrative examples for the construction of uninorms on bounded lattices are provided.

*Keywords:* bounded lattices, $t$-norms, $t$-conorms, uninorms

*Classification:* 03B52, 06B20, 03E72

## 1. INTRODUCTION

Triangular norms ($t$-norms for short) and triangular conorms ($t$-conorms for short) on the unit interval, introduced by Menger [37], Schweizer and Sklar [41, 42], play an important role in many fields, such as fuzzy set theory, fuzzy logic and so on (see, e. g., [3, 26, 36, 43, 46, 53]). As an important generalization of $t$-norms and $t$-conorms, uninorms on the unit interval $[0,1]$ were introduced by Yager and Rybalov [50]. Since then, uninorms have been proved to be useful in several fields, such as expert systems, neural networks, fuzzy logics and so on (see, e. g., [17, 24, 25, 40]).

In recent years, the study of these aggregation operators on the unit interval has already been extended to bounded lattices. In fact, $t$-norms ($t$-conorms) have been widely studied on bounded lattices by many authors (see, e. g., [8, 12, 18, 21, 22, 23, 30, 34, 35, 38, 44, 45]), including the constructions of $t$-norms ($t$-conorms), especially in the construction of ordinal sums of $t$-norms ($t$-conorms). Then, uninorms have been extensively investigated on bounded lattices [7] and the constructions of uninorms are usually based on these tools, such as $t$-norms ($t$-conorms) (see, e. g., [1, 4, 5, 6, 9, 10, 11, 13, 15, 19, 20, 32, 33, 47, 49]), $t$-subnorms ( $t$-superconorms) (see, e. g., [29, 31, 49, 52]), closure operators (interior operators) (see, e. g., [14, 27, 39, 51]), additive generators [28] and uninorms (see, e. g., [16, 48]).

More specifically, in [16] and [48], the researchers introduced new approaches to construct uninorms on $L$ via given uninorms defined on a subinterval of $L$, respectively.

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In [16], G.D. Çaylı et al. obtained new uninorms based on the presence of $t$-norms ($t$-conorms) and uninorms defined on a subinterval of $L$ under some additional constraints on $L$. In [48], Xiu and Zheng gave new methods to yield uninorms on bounded lattices using the presence of $t$-superconorms ($t$-subnorms) and uninorms under some additional constraints. As we see, in fact, the above methods to construct uninorms both started from a given uninorm on a subinterval of a bounded lattice and then provided a novel perspective to study the constructions of uninorms.

In this paper, we give a new approach to construct uninorms via uninorms on bounded lattices under additional constraint conditions. More specifically, it can be understood that we first fix the given uninorm on $[0, a]$ (or $[b, 1]$) and then extend it to the bounded lattice $L$. These construction methods generalize some existing construction methods for uninorms and also extend the construction methods for $t$-norms and $t$-conorms.

The structure of this paper is as follows. In Section 2, we recall some basic concepts and properties related to lattices and aggregation operators on bounded lattices. In Section 3, we propose new construction methods of uninorms on bounded lattices via given uninorms and discuss the relationship between the new uninorms and some known uninorms. Moreover, we provide some examples to illustrate the construction methods of uninorms. In Section 4, some conclusions are added.

## 2. PRELIMINARIES

In this section, we recall some basic notions of lattices and aggregation operators on bounded lattices.

**Definition 2.1.** (Birkhoff [2]) A lattice $(L, \leq)$ is bounded if it has top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$ for all $x \in L$.

Throughout this article, unless stated otherwise, we denote $L$ as a bounded lattice with the top and bottom elements 1 and 0, respectively.

**Definition 2.2.** (Birkhoff [2]) Let $L$ be a bounded lattice, $a, b \in L$ with $a \leq b$. A subinterval $[a, b]$ of $L$ is defined as

$$[a, b] = \{x \in L : a \leq x \leq b\}.$$

Similarly, we can define $[a, b) = \{x \in L : a \leq x < b\}, (a, b] = \{x \in L : a < x \leq b\}$ and $(a, b) = \{x \in L : a < x < b\}$. If $a$ and $b$ are incomparable, then we use the notation $a \parallel b$.

In the following, $I_a$ denotes the set of all incomparable elements with $a$, that is, $I_a = \{x \in L \mid x \parallel a\}$. $I_a^b$ denotes the set of elements that are incomparable with $a$ but comparable with $b$, that is, $I_a^b = \{x \in L \mid x \parallel a \textit{ and } x \nparallel b\}$. $I_{a,b}$ denotes the set of elements that are incomparable with both $a$ and $b$, that is, $I_{a,b} = \{x \in L \mid x \parallel a \textit{ and } x \parallel b\}$. $I^{a,b}$ denotes the set of elements that are comparable with both $a$ and $b$, that is, $I^{a,b} = \{x \in L \mid x \nparallel a \textit{ and } x \nparallel b\}$. Obviously, $I_a^a = \emptyset$ and $I_{a,a} = I_a$.

**Definition 2.3.** (Saminger [44]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $T : L^2 \to L$ is called a $t$-norm on $L$ if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element $1 \in L$, that is, $T(1, x) = x$ for all $x \in L$.

**Definition 2.4.** (Çaylı et al. [5]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $S : L^2 \to L$ is called a $t$-conorm on $L$ if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element $0 \in L$, that is, $S(0, x) = x$ for all $x \in L$.

**Definition 2.5.** (Karaçal and Mesiar [32]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on $L$ (a uninorm if $L$ is fixed) if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element $e \in L$, that is, $U(e, x) = x$ for all $x \in L$.

**Proposition 2.6.** (Karaçal and Mesiar [32]) Let $U$ be a uninorm with the neutral element $e \in L \setminus \{0, 1\}$ on $L$. Then the following statements hold:

(1) $T_e = U \mid [0, e]^2 \to [0, e]$ is a $t$-norm on $[0, e]$.

(2) $S_e = U \mid [e, 1]^2 \to [e, 1]$ is a $t$-conorm on $[e, 1]$.

**Definition 2.7.** (Zhang et al. [52]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L \setminus \{0, 1\}$. We denote by $\mathcal{U}_{\min}$ the class of all uninorms $U$ on $L$ with neutral element $e$ satisfying the following condition: $U(x, y) = y$, for all $(x, y) \in (e, 1] \times (L \setminus [e, 1])$.

Similarly, we denote by $\mathcal{U}_{\max}$ the class of all uninorms $U$ on $L$ with neutral element $e$ satisfying the following condition: $U(x, y) = y$, for all $(x, y) \in [0, e) \times (L \setminus [0, e])$.

## 3. NEW CONSTRUCTION METHODS FOR UNINORMS ON BOUNDED LATTICES

In this section, we propose new construction methods for uninorms on a bounded lattice $L$ via a given uninorm $U^*$ on the subinterval $[0, a]$ (or $[b, 1]$) under additional constraint conditions on $L$ and $U^*$. For convenience, $\mathcal{U}^*_\perp$ denotes the class of all uninorms $U$ on $L$ with neutral element $e$ satisfying the following condition: $U(x, y) \in [0, e]$ implies $(x, y) \in [0, e]^2$. Similarly, $\mathcal{U}^*_\top$ denotes the class of all uninorms $U$ on $L$ with neutral element $e$ satisfying the following condition: $U(x, y) \in [e, 1]$ implies $(x, y) \in [e, 1]^2$.

Let $a \in L \setminus \{0, 1\}$, $q \in L$ and $U^*$ be a uninorm on $[0, a]$ with a neutral element $e$. We can define a function $U : L^2 \to L$ by

$$U(x, y) = \begin{cases} U^*(x, y) & \text{if } (x, y) \in [0, a]^2, \\ x & \text{if } (x, y) \in (L \setminus [0, a]) \times [0, e], \\ y & \text{if } (x, y) \in [0, e] \times (L \setminus [0, a]), \\ x \vee y \vee q & \text{if } (x, y) \in I_{e,a} \times I_{e,a}, \\ 1 & \text{otherwise.} \end{cases} \tag{1}$$

In the following, we discuss how the function $U$ given by (1) can be a uninorm with $q = 0$ and $q \in I_{e,a}$, respectively.

First, we illustrate that the function $U$ given by (1) with $q = 0$ can be a uninorm on bounded lattices under some conditions.

**Theorem 3.1.** Let $a \in L \setminus \{0, 1\}$, $q = 0$, $U^*$ be a uninorm on $[0, a]$ with a neutral element $e$ and $U_1$ be a function given by (1). Suppose that either $x \vee y = 1$ for all $x, y \in I_{e,a}$ with $x \neq y$ or $x \vee y \in I_{e,a}$ for all $x, y \in I_{e,a}$.

(1) Let us assume that $U^* \in \mathcal{U}^*_\perp$. Then $U_1$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I^a_e$.

(2) Moreover, let us assume that $I^e_a \cup I_{e,a} \cup (a, 1) \neq \emptyset$. Then $U_1$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $U^* \in \mathcal{U}^*_\perp$ and $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I^a_e$.

Proof. (1) Necessity. Let $U_1(x, y)$ be a uninorm on $L$ with a neutral element $e$. We prove that $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I^a_e$.

Assume that there exist $x \in I_{e,a}$ and $y \in I^a_e$ such that $x \nparallel y$, i. e., $y < x$. Then $U_1(x, y) = 1$ and $U_1(x, x) = x$. Since $x < 1$, the increasingness property of $U_1$ is violated. Thus $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I^a_e$.

Sufficiency. By the definition of $U_1$, it is easy to obtain that $U_1$ is commutative and $e$ is the neutral element of $U_1$. Thus, we only need to prove the increasingness and the associativity of $U_1$.

I. Increasingness: We prove that if $x \leq y$, then $U_1(x, z) \leq U_1(y, z)$ for all $z \in L$. It is obvious that $U_1(x, z) \leq U_1(y, z)$ if both $x$ and $y$ belong to one of the intervals $[0, e], I^a_e, (e, a], I^e_a, I_{e,a}$ or $(a, 1]$ for all $z \in L$. The residual proof can be split into all possible cases:

1. $x \in [0, e]$
   1.1. $y \in I^a_e \cup (e, a]$
      1.1.1. $z \in [0, e] \cup I^a_e \cup (e, a]$
      $U_1(x, z) = U^*(x, z) \leq U^*(y, z) = U_1(y, z)$
      1.1.2. $z \in I^e_a \cup I_{e,a} \cup (a, 1]$
      $U_1(x, z) = z < 1 = U_1(y, z)$

   1.2. $y \in I^e_a \cup (a, 1]$
      1.2.1. $z \in [0, e]$
      $U_1(x, z) = U^*(x, z) \leq x < y = U_1(y, z)$
      1.2.2. $z \in I^a_e \cup (e, a]$
      $U_1(x, z) = U^*(x, z) \leq a < 1 = U_1(y, z)$
      1.2.3. $z \in I^e_a \cup I_{e,a} \cup (a, 1]$
      $U_1(x, z) = z \leq 1 = U_1(y, z)$

   1.3. $y \in I_{e,a}$

      1.3.1. $z \in [0, e]$

$U_1(x,z) = U^*(x,z) \leq x < y = U_1(y,z)$

1.3.2. $z \in I_e^a \cup (e,a]$
$U_1(x,z) = U^*(x,z) \leq a < 1 = U_1(y,z)$

1.3.3. $z \in I_a^e \cup (a,1]$
$U_1(x,z) = z \leq 1 = U_1(y,z)$

1.3.4. $z \in I_{e,a}$
$U_1(x,z) = z \leq y \vee z = U_1(y,z)$

2. $x \in I_e^a$

2.1. $y \in (e,a]$

2.1.1. $z \in [0,e] \cup I_e^a \cup (e,a]$
$U_1(x,z) = U^*(x,z) \leq U^*(y,z) = U_1(y,z)$

2.1.2. $z \in I_a^e \cup I_{e,a} \cup (a,1]$
$U_1(x,z) = 1 = U_1(y,z)$

2.2. $y \in I_a^e \cup (a,1]$

2.2.1. $z \in [0,e]$
$U_1(x,z) = U^*(x,z) \leq x < y = U_1(y,z)$

2.2.2. $z \in I_e^a \cup (e,a]$
$U_1(x,z) = U^*(x,z) \leq a < 1 = U_1(y,z)$

2.2.3. $z \in I_a^e \cup I_{e,a} \cup (a,1]$
$U_1(x,z) = 1 = U_1(y,z)$

3. $x \in (e,a], y \in I_a^e \cup (a,1]$

3.1. $z \in [0,e]$
$U_1(x,z) = U^*(x,z) \leq x < y = U_1(y,z)$

3.2. $z \in I_e^a \cup (e,a]$
$U_1(x,z) = U^*(x,z) \leq a < 1 = U_1(y,z)$

3.3. $z \in I_a^e \cup I_{e,a} \cup (a,1]$
$U_1(x,z) = 1 = U_1(y,z)$

4. $x \in I_a^e, y \in (a,1]$

4.1. $z \in [0,e]$
$U_1(x,z) = x \leq y = U_1(y,z)$

4.2. $z \in I_e^a \cup (e,a] \cup I_a^e \cup I_{e,a} \cup (a,1]$
$U_1(x,z) = 1 = U_1(y,z)$

5. $x \in I_{e,a}, y \in I_a^e \cup (a,1]$

5.1. $z \in [0,e]$
$U_1(x,z) = x < y = U_1(y,z)$

5.2. $z \in I_e^a \cup (e,a] \cup I_a^e \cup (a,1]$

$U_1(x,z) = 1 = U_1(y,z)$

5.3. $z \in I_{e,a}$
$U_1(x,z) = x \vee z \leq 1 = U_1(y,z)$

II. Associativity: We demonstrate that $U_1(x, U_1(y,z)) = U_1(U_1(x,y),z)$ for all $x,y,z \in L$. By Theorem 3.12 in [31], we need to consider the following cases:

1. If $x,y,z \in [0,e] \cup I_e^a \cup (e,a]$, then since $U^*$ is associative, we have $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

2. If $x,y,z \in I_a^e \cup (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(1,z) = U_1(U_1(x,y),z)$.

3. Suppose that $x,y,z \in I_{e,a}$.

3.1. If $x_1 \vee y_1 \in I_{e,a}$ for all $x_1,y_1 \in I_{e,a}$, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = x \vee y \vee z = U_1(x \vee y,z) = U_1(U_1(x,y),z)$.

3.2. Assme that $x_1 \vee y_1 = 1$ for all $x_1,y_1 \in I_{e,a}$ with $x_1 \neq y_1$.

3.2.1. If $x \neq y$, $y \neq z$ and $x \neq z$, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = U_1(x,1) = 1 = U_1(x \vee y,z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,x \vee z) = U_1(y,1) = 1$.

3.2.2. If $x = y$ and $x,y \neq z$, then $U_1(x,U_1(y,z)) = U_1(x,U_1(x,z)) = U_1(x,x \vee z) = U_1(x,1) = 1 = x \vee y = U_1(x,y) = U_1(x \vee x,y) = U_1(U_1(x,x),z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(x,U_1(x,z)) = U_1(x,U_1(x \vee z)) = U_1(y,1) = 1$.

3.2.3. If $y = z$ and $y,z \neq x$, then we also have $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$ by the commutativity property of $U_1$.

3.2.4. If $x = z$ and $x,z \neq y$, then we also have $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$ by the commutativity property of $U_1$.

3.2.5. If $x = y = z$, then $U_1(x,U_1(y,z)) = U_1(x,U_1(x,x)) = U_1(x,x) = U_1(U_1(x,x),x) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(x,U_1(x,x)) = U_1(x,x)$.

4. If $x,y \in [0,e]$ and $z \in I_a^e \cup I_{e,a} \cup (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,z) = z = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$.

5. If $x,y \in I_e^a \cup (e,a]$ and $z \in I_a^e \cup I_{e,a} \cup (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,1) = 1$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

6. If $x,y \in I_a^e$ and $z \in I_{e,a} \cup (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(1,z) = U_1(U_1(x,y),z)$.

7. If $x,y \in I_{e,a}$ and $z \in (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(x \vee y,z) = U_1(U_1(x,y),z)$.

8. If $x \in [0,e]$ and $y,z \in I_a^e \cup (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(y,z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = 1$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

9. If $x \in [0,e]$ and $y,z \in I_{e,a}$, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = y \vee z = U_1(y,z) = U_1(U_1(x,y),z)$.

10. If $x \in I_e^a \cup (e,a]$ and $y,z \in I_a^e \cup (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(1,z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,1) = 1$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

11. If $x \in I_e^a \cup (e,a] \cup I_a^e$ and $y,z \in I_{e,a}$, then $U_1(x,U_1(y,z)) = U_1(x,y \vee z) = 1 = U_1(1,z) = U_1(U_1(x,y),z)$.

12. If $x \in I_{e,a}$ and $y,z \in (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(1,z) = U_1(U_1(x,y),z)$.

13. If $x \in [0,e], y \in I_e^a \cup (e,a]$ and $z \in I_a^e \cup I_{e,a} \cup (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(U^*(x,y),z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = 1$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

14. If $x \in [0,e], y \in I_a^e$ and $z \in I_{e,a}$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(y,z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = 1$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

15. If $x \in [0,e], y \in I_{e,a}$ and $z \in (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(y,z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,z) = 1$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

16. If $x \in I_e^a \cup (e,a], y \in I_a^e$ and $z \in I_{e,a}$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(1,z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,1) = 1$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

17. If $x \in I_e^a \cup (e,a] \cup I_a^e, y \in I_{e,a}$ and $z \in (a,1]$, then $U_1(x,U_1(y,z)) = U_1(x,1) = 1 = U_1(1,z) = U_1(U_1(x,y),z)$ and $U_1(y,U_1(x,z)) = U_1(y,1) = 1$. Thus $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z) = U_1(y,U_1(x,z))$.

Combining the above cases, we obtain that $U_1(x,U_1(y,z)) = U_1(U_1(x,y),z)$ for all $x,y,z \in L$ by Theorem 3.12 in [31]. Therefore, $U_1$ is a uninorm on $L$ with the neutral element $e$.

(2) Next we just prove that if $I_a^e \cup I_{e,a} \cup (a,1) \neq \emptyset$, then the condition $U^* \in \mathcal{U}_\perp^*$ is necessary for that $U_1$ is a uninorm on $L$.

Suppose that $I_a^e \cup I_{e,a} \cup (a,1) \neq \emptyset$ and $U_1(x,y)$ is a uninorm on $L$ with the neutral element $e$. We prove that if $U^*(x,y) \in [0,e]$, then $(x,y) \in [0,e]^2$. The proof can be split into all possible cases:

(i) $U^*(x,y) \notin [0,e]$ for all $(x,y) \in [0,e] \times (I_e^a \cup (e,a]) \cup (I_e^a \cup (e,a]) \times [0,e]$.

Now we just prove that $U^*(x,y) \notin [0,e]$ for all $(x,y) \in [0,e] \times (I_e^a \cup (e,a])$, and the other case can be proved immediately by the commutativity property of $U^*$. Assume that there exists $(x,y) \in [0,e] \times (I_e^a \cup (e,a])$ such that $U^*(x,y) \in [0,e]$. Take $z \in I_a^e \cup I_{e,a} \cup (a,1)$. Then $U_1(x,U_1(y,z)) = U_1(x,1) = 1$ and $U_1(U_1(x,y),z) = U_1(U^*(x,y),z) = z$. Since $z \neq 1$, the associativity property of $U_1(x,y)$ is violated. Thus $U^*(x,y) \notin [0,e]$ for all $(x,y) \in [0,e] \times (I_e^a \cup (e,a]) \cup (I_e^a \cup (e,a]) \times [0,e]$.

(ii) $U^*(x,y) \notin [0,e]$ for all $(x,y) \in I_e^a \times I_e^a$.

Assume that there exists $(x,y) \in I_e^a \times I_e^a$ such that $U^*(x,y) \in [0,e]$. Take $z \in I_a^e \cup I_{e,a} \cup (a,1)$. Then $U_1(x,U_1(y,z)) = U_1(x,1) = 1$ and $U_1(U_1(x,y),z) = U_1(U^*(x,y),z) = z$.

Since $z \neq 1$, the associativity property of $U_1(x,y)$ is violated. Thus $U^*(x,y) \notin [0,e]$ for all $(x,y) \in I_e^a \times I_e^a$.

(iii) $U^*(x,y) \notin [0,e]$ for all $(x,y) \in (e,a]^2 \cup (e,a] \times I_e^a \cup I_e^a \times (e,a]$.

Now we just prove that $U^*(x,y) \notin [0,e]$ for all $(x,y) \in (e,a]^2 \cup (e,a] \times I_e^a$, and the other case can be proved immediately by the commutativity property of $U^*$. By the increasingness property of $U^*$, we can obtain that $y = U^*(e,y) \leq U^*(x,y)$. Since $y \in I_e^a \cup (e,a]$, we can obtain that $U^*(x,y) \notin [0,e]$. Thus $U^*(x,y) \notin [0,e]$ for all $(x,y) \in (e,a]^2 \cup (e,a] \times I_e^a \cup I_e^a \times (e,a]$.

Hence, $U^*(x,y) \in [0,e]$ implies $(x,y) \in [0,e]^2$. □

If we take $e = 0$ or $a$ in Theorem 3.1, then we can obtain some existing results in the literature.

**Remark 3.2.** (1) If we take $e = 0$ in Theorem 3.1, then we obtain the $t$-conorm $S$ in Theorem 1 of [8].

(2) If we take $e = a$ in Theorem 3.1, then we obtain the uninorm $U_{cl}^e$ in Corollary 4.3 of[27].

The next example illustrates the construction method of uninorms on bounded lattices in Theorem 3.1.

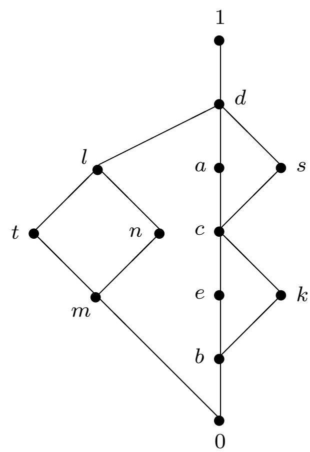


**Fig. 1:** The lattice $L_1$.

**Example 3.3.** Given a bounded lattice $L_1 = \{0, b, e, k, c, a, m, t, n, l, s, d, 1\}$ depicted in Figure 1 and a uninorm $U^* : [0,a]^2 \to [0,a]$ shown in Table 1. It is easy to see that $L_1$ and $U^*$ satisfy the conditions in Theorem 3.1, i. e., $x \vee y \in I_{e,a}$ for all $x, y \in I_{e,a}$ and $U^* \in \mathcal{U}_\perp^*$. By using the construction method in Theorem 3.1, the uninorm $U_{11} : L_1^2 \to L_1$ with the neutral element $e$ is defined as in Table 2.

**Remark 3.4.** In Theorem 3.1, we observe that the condition either $x \vee y = 1$ for all $x, y \in I_{e,a}$ with $x \neq y$ or $x \vee y \in I_{e,a}$ for all $x, y \in I_{e,a}$ can not be omitted, in general. Moreover, we give the following example to show the above fact.

| $U^*$ | $0$ | $b$ | $e$ | $k$ | $c$ | $a$ |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $k$ | $c$ | $a$ |
| $b$ | $0$ | $b$ | $b$ | $k$ | $c$ | $a$ |
| $e$ | $0$ | $b$ | $e$ | $k$ | $c$ | $a$ |
| $k$ | $k$ | $k$ | $k$ | $k$ | $c$ | $a$ |
| $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $a$ |
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |

**Tab. 1.** The uninorm $U^*$ on $[0, a]$.

| $U_{11}$ | $0$ | $b$ | $e$ | $k$ | $c$ | $a$ | $m$ | $t$ | $n$ | $l$ | $s$ | $d$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $k$ | $c$ | $a$ | $m$ | $t$ | $n$ | $l$ | $s$ | $d$ | $1$ |
| $b$ | $0$ | $b$ | $b$ | $k$ | $c$ | $a$ | $m$ | $t$ | $n$ | $l$ | $s$ | $d$ | $1$ |
| $e$ | $0$ | $b$ | $e$ | $k$ | $c$ | $a$ | $m$ | $t$ | $n$ | $l$ | $s$ | $d$ | $1$ |
| $k$ | $k$ | $k$ | $k$ | $k$ | $c$ | $a$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $a$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $m$ | $m$ | $m$ | $m$ | $1$ | $1$ | $1$ | $m$ | $t$ | $n$ | $l$ | $1$ | $1$ | $1$ |
| $t$ | $t$ | $t$ | $t$ | $1$ | $1$ | $1$ | $t$ | $t$ | $l$ | $l$ | $1$ | $1$ | $1$ |
| $n$ | $n$ | $n$ | $n$ | $1$ | $1$ | $1$ | $n$ | $l$ | $n$ | $l$ | $1$ | $1$ | $1$ |
| $l$ | $l$ | $l$ | $l$ | $1$ | $1$ | $1$ | $l$ | $l$ | $l$ | $l$ | $1$ | $1$ | $1$ |
| $s$ | $s$ | $s$ | $s$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $d$ | $d$ | $d$ | $d$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

**Tab. 2.** The uninorm $U_{11}$ on $L_1$.

**Example 3.5.** Given a bounded lattice $L_2 = \{0, e, a, b, k, m, n, 1\}$ depicted in Figure 2 and a uninorm $U^* : [0, a]^2 \to [0, a]$ shown in Table 3. Since $m \vee n = b \neq 1$ for $m, n \in I_{e,a}$ and $m \vee k = m \in I_{e,a}$ for $m, k \in I_{e,a}$, $L_2$ does not satisfy the conditions in Theorem 3.1. By using the construction method in Theorem 3.1, we can obtain a function $U_{12}$ on $L_2$, shown in Table 4. Since $U_{12}(k, U_{12}(m, n)) = U_{12}(m, b) = 1$ and $U_{12}(U_{12}(k, m), n) = U_{12}(m, n) = b$ for $k, m, n \in L_2$, the function $U_{12}$ does not satisfy associativity. Thus, the function $U_{12}$ is not a uninorm on $L_2$.

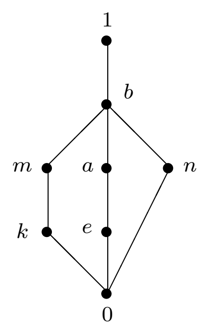


**Fig. 2:** The lattice $L_2$.

| $U^*$ | $0$ | $e$ | $a$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $a$ |
| $e$ | $0$ | $e$ | $a$ |
| $a$ | $a$ | $a$ | $a$ |

**Tab. 3.** The uninorm $U^*$ on $[0, a]$.

| $U_{12}$ | $0$ | $e$ | $a$ | $k$ | $m$ | $n$ | $b$ | $1$ |
|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $a$ | $k$ | $m$ | $n$ | $b$ | $1$ |
| $e$ | $0$ | $e$ | $a$ | $k$ | $m$ | $n$ | $b$ | $1$ |
| $a$ | $a$ | $a$ | $a$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $k$ | $k$ | $k$ | $1$ | $k$ | $m$ | $b$ | $1$ | $1$ |
| $m$ | $m$ | $m$ | $1$ | $m$ | $m$ | $b$ | $1$ | $1$ |
| $n$ | $n$ | $n$ | $1$ | $b$ | $b$ | $n$ | $1$ | $1$ |
| $b$ | $b$ | $b$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

**Tab. 4.** The function $U_{12}$ on $L_2$.

**Remark 3.6.** Let $U_1$ be a uninorm in Theorem 3.1. In this case, we can obtain the following results:

(1) If $U^* \in \mathcal{U}_\perp^*$, then $U_1 \in \mathcal{U}_\perp^*$.

(2) $U_1 \in \mathcal{U}_{\max}$ if and only if $U^* \in \mathcal{U}_{\max}$.

Next, we illustrate that the function $U$ given by (1) with $q \in I_{e,a}$ can be a uninorm on bounded lattices under some conditions.

**Theorem 3.7.** Let $a \in L \setminus \{0, 1\}$, $q \in I_{e,a}$, $U^*$ be a uninorm on $[0, a]$ with a neutral element $e$ and $U_2$ be a function given by (1).

(1) Suppose that $x \vee q = 1$ for all $x \in I_{e,a}$ with $x \neq q$.

(i) Let us assume that $U^* \in \mathcal{U}_\perp^*$. Then the function $U_2$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $x \parallel q$ for all $x \in I_e^a$.

(ii) Let us assume that $I_a^e \cup I_{e,a} \cup (a, 1) \neq \emptyset$. Then the function $U_2$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $U^* \in \mathcal{U}_\perp^*$ and $x \parallel q$ for all $x \in I_e^a$.

(2) Suppose that $x \vee y \in I_{e,a}$ for all $x, y \in I_{e,a}$.

(i) Let us assume that $U^* \in \mathcal{U}_\perp^*$. Then the function $U_2$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$.

(ii) Let us assume that $I_a^e \cup I_{e,a} \cup (a, 1) \neq \emptyset$. Then the function $U_2$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $U^* \in \mathcal{U}_\perp^*$ and $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$.

P r o o f. (1)(i) Necessity. Let $U_2(x, y)$ be a uninorm on $L$ with a neutral element $e$. We prove that $x \parallel q$ for $x \in I_e^a$.

Assume that there exists $x \in I_e^a$ such that $x \nparallel q$, i.e., $x < q$. Then $U_2(x, q) = 1$ and $U_2(q, q) = q \vee q \vee q = q \in I_{e,a}$. Since $q < 1$, the increasingness property of $U_2$ is violated. Thus $x \parallel q$ for $x \in I_e^a$.

Sufficiency. By the definition of $U_2$, it is easy to obtain that $U_2$ is commutative and $e$ is the neutral element of $U_2$. Thus, we only need to prove the increasingness and the associativity of $U_2$.

I. Increasingness: We prove that if $x \leq y$, then $U_2(x, z) \leq U_2(y, z)$ for all $z \in L$. Taking into account Theorem 3.1, it is enough to check only those cases that are different from the cases in Theorem 3.1.

1. $x \in [0, e], y \in I_{e,a}, z \in I_{e,a}$
   $U_2(x, z) = z \leq y \vee z \vee q = U_2(y, z)$

2. $x \in I_e^a, y \in I_{e,a}, z \in I_{e,a}$
   $U_2(x, z) = 1 = y \vee z \vee q = U_2(y, z)$

3. $x \in I_{e,a}$
   3.1. $y \in I_{e,a}, z \in I_{e,a}$
   $U_2(x, z) = x \vee z \vee q \leq y \vee z \vee q = U_2(y, z)$
   3.2. $y \in I_a^e \cup (a, 1], z \in I_{e,a}$
   $U_2(x, z) = x \vee z \vee q \leq 1 = U_2(y, z)$

II. Associativity: It can be shown that $U_2(x, U_2(y, z)) = U_2(U_2(x, y), z)$ for all $x, y, z \in L$. By Theorem 3.12 in [31] and taking into account Theorem 3.1, it is enough to check only those cases that are different from the cases in Theorem 3.1.

1. Suppose that $x, y, z \in I_{e,a}$.

1.1. If $x \neq y$, $x \neq z$ and $y \neq z$, then $U_2(x, U_2(y, z)) = U_2(x, y \vee z \vee q) = 1 = U_2(x \vee y \vee q, z) = U_2(U_2(x, y), z)$ and $U_2(y, U_2(x, z)) = U_2(y, x \vee z \vee q) = 1$.

1.2. If $x = y$ and $x, y \neq z$, then $U_2(x, U_2(y, z)) = U_2(x, U_2(x, z)) = U_2(x, x \vee z \vee q) = U_2(x, 1) = 1 = U_2(x \vee y \vee q, z) = U_2(U_2(x, x), z) = U_2(U_2(x, y), z)$ and $U_2(y, U_2(x, z)) = U_2(x, U_2(x, z)) = U_2(x, x \vee z \vee q) = 1$.

1.3. If $y = z$ and $y, z \neq x$, then we also have $U_2(x, U_2(y, z)) = U_2(U_2(x, y), z) = U_2(y, U_2(x, z)) = U_2(y, U_2(x, y))$ by the commutativity property of $U_2$.

1.4. If $x = z$ and $x, z \neq y$, then we also have $U_2(x, U_2(y, z)) = U_2(U_2(x, y), z) = U_2(y, U_2(x, z)) = U_2(y, U_2(x, y))$ by the commutativity property of $U_2$.

1.5. If $x = y = z$, then $U_2(x, U_2(y, z)) = U_2(x, U_2(x, x)) = U_2(x, x \vee x \vee q) = U_2(x, x \vee q) = U_2(x \vee q, x) = U_2(U_2(x, x), x) = U_2(U_2(x, y), z)$ and $U_2(y, U_2(x, z)) = U_2(x, U_2(x, x)) = U_2(x, x \vee q)$.

2. If $x, y \in I_{e,a}$ and $z \in (a, 1]$, then $U_2(x, U_2(y, z)) = U_2(x, 1) = 1 = U_2(x \vee y \vee q, z) = U_2(U_2(x, y), z)$.

3. If $x \in [0,e]$ and $y, z \in I_{e,a}$, then $U_2(x, U_2(y,z)) = U_2(x, y \vee z \vee q) = y \vee z \vee q = U_2(y,z) = U_2(U_2(x,y),z)$.

4. If $x \in I_e^a \cup (e,a] \cup I_a^e$ and $y, z \in I_{e,a}$, then $U_2(x, U_2(y,z)) = U_2(x, y \vee z \vee q) = 1 = U_2(1,z) = U_2(U_2(x,y),z)$.

Therefore, $U_2$ is a uninorm on $L$ with the neutral element $e$.

(1)(ii) It can be proved with the proof of Theorem 3.1(2) in a similar way.

(2)(i) Necessity. Let $U_2(x,y)$ be a uninorm on $L$ with a neutral element $e$. We prove that $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$.

Assume that there exist $x \in I_{e,a}$ and $y \in I_e^a$ such that $x \nparallel y$, i. e., $y < x$. Then $U_2(x,y) = 1$ and $U_2(x,x) = x \vee x \vee q = x \vee q \in I_{e,a}$. Since $x \vee q < 1$, the increasingness property of $U_2$ is violated. Thus $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$.

Sufficiency. By the definition of $U_2$, it is easy to obtain that $U_2$ is commutative and $e$ is the neutral element of $U_2$. Thus, we only need to prove the increasingness and the associativity of $U_2$.

I. Increasingness: We prove that if $x \leq y$, then $U_2(x,z) \leq U_2(y,z)$ for all $z \in L$. Taking into account Theorem 3.1, it is enough to check only those cases that are different from the cases in Theorem 3.1.

1. $x \in [0,e], y \in I_{e,a}, z \in I_{e,a}$
   $U_2(x,z) = z \leq y \vee z \vee q = U_2(y,z)$
2. $x \in I_{e,a}$
   2.1. $y \in I_{e,a}, z \in I_{e,a}$
   $U_2(x,z) = x \vee z \vee q \leq y \vee z \vee q = U_2(y,z)$
   2.2. $y \in I_a^e \cup (a,1], z \in I_{e,a}$
   $U_2(x,z) = x \vee z \vee q \leq 1 = U_2(y,z)$

II. Associativity: It can be shown that $U_2(x, U_2(y,z)) = U_2(U_2(x,y),z)$ for all $x, y, z \in L$. By Theorem 3.12 in [31] and taking into account Theorem 3.1, it is enough to check only those cases that are different from the cases in Theorem 3.1.

1. If $x, y, z \in I_{e,a}$, then $U_2(x, U_2(y,z)) = U_2(x, y \vee z \vee q) = x \vee y \vee z \vee q = U_2(x \vee y \vee q, z) = U_2(U_2(x,y),z)$.

2. If $x, y \in I_{e,a}$ and $z \in (a,1]$, then $U_2(x, U_2(y,z)) = U_2(x,1) = 1 = U_2(x \vee y \vee q, z) = U_2(U_2(x,y),z)$.

3. If $x \in [0,e]$ and $y, z \in I_{e,a}$, then $U_2(x, U_2(y,z)) = U_2(x, y \vee z \vee q) = y \vee z \vee q = U_2(y,z) = U_2(U_2(x,y),z)$.

4. If $x \in I_e^a \cup (e,a] \cup I_a^e$ and $y, z \in I_{e,a}$, then $U_2(x, U_2(y,z)) = U_2(x, y \vee z \vee q) = 1 = U_2(1,z) = U_2(U_2(x,y),z)$.

(2)(ii) It can be proved with the proof of Theorem 3.1(2) in a similar way. □

**Remark 3.8.** If we take $e = 0$ in Theorem 3.7, then we obtain the $t$-conorm $S$ in Theorem 1 of [8].

The next example illustrates the construction method of uninorms on bounded lattices in Theorem 3.7.

**Example 3.9.** Given a bounded lattice $L_1 = \{0, b, e, k, c, a, m, t, n, l, s, d, 1\}$ depicted in Figure 1, $q = n$ and a uninorm $U^* : [0, a]^2 \to [0, a]$ shown in Table 5. It is easy to see that $L_1$ and $U^*$ satisfy the conditions in Theorem 3.7(2). By using the construction method in Theorem 3.7, the uninorm $U_{21} : L_1^2 \to L_1$ with the neutral element $e$ is defined as in Table 6.

| $U^*$ | $0$ | $b$ | $e$ | $k$ | $c$ | $a$ |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $k$ | $c$ | $a$ |
| $b$ | $0$ | $b$ | $b$ | $k$ | $c$ | $a$ |
| $e$ | $0$ | $b$ | $e$ | $k$ | $c$ | $a$ |
| $k$ | $k$ | $k$ | $k$ | $k$ | $c$ | $a$ |
| $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $a$ |
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |

**Tab. 5.** The uninorm $U^*$ on $[0, a]$.

| $U_{21}$ | $0$ | $b$ | $e$ | $k$ | $c$ | $a$ | $m$ | $t$ | $n$ | $l$ | $s$ | $d$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $k$ | $c$ | $a$ | $m$ | $t$ | $n$ | $l$ | $s$ | $d$ | $1$ |
| $b$ | $0$ | $b$ | $b$ | $k$ | $c$ | $a$ | $m$ | $t$ | $n$ | $l$ | $s$ | $d$ | $1$ |
| $e$ | $0$ | $b$ | $e$ | $k$ | $c$ | $a$ | $m$ | $t$ | $n$ | $l$ | $s$ | $d$ | $1$ |
| $k$ | $k$ | $k$ | $k$ | $k$ | $c$ | $a$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $a$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $m$ | $m$ | $m$ | $m$ | $1$ | $1$ | $1$ | $n$ | $l$ | $n$ | $l$ | $1$ | $1$ | $1$ |
| $t$ | $t$ | $t$ | $t$ | $1$ | $1$ | $1$ | $l$ | $l$ | $l$ | $l$ | $1$ | $1$ | $1$ |
| $n$ | $n$ | $n$ | $n$ | $1$ | $1$ | $1$ | $n$ | $l$ | $n$ | $l$ | $1$ | $1$ | $1$ |
| $l$ | $l$ | $l$ | $l$ | $1$ | $1$ | $1$ | $l$ | $l$ | $l$ | $l$ | $1$ | $1$ | $1$ |
| $s$ | $s$ | $s$ | $s$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $d$ | $d$ | $d$ | $d$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

**Tab. 6.** The uninorm $U_{21}$ on $L_1$.

**Remark 3.10.**

(1) In Theorem 3.7(1), we observe that the condition $x \vee q = 1$ for all $x \in I_{e,a}$ with $x \neq q$ can not be omitted, in general.

(2) In Theorem 3.7(2), we observe that the condition $x \vee y \in I_{e,a}$ for all $x, y \in I_{e,a}$ can not be omitted, in general.

The next example illustrates the facts in Remark 3.10. That is, if the conditions in Theorem 3.7 do not hold, then the associativity of $U_2$ is violated.

**Example 3.11.** Given a bounded lattice $L_2 = \{0, e, a, b, k, m, n, 1\}$ depicted in Figure 2., $q = m$ and a uninorm $U^* : [0, a]^2 \to [0, a]$ shown in Table 7. It is easy to see that $U^* \in \mathcal{U}^*_\perp$ and $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I^a_e$. Since $m \vee n = b \notin I_{e,a}$ for $m, n \in I_{e,a}$ and $m \vee k = k \in I_{e,a}$ for $m, k \in I_{e,a}$, the condition $x \vee q = 1$ for all $x \in I_{e,a}$ with $x \neq q$ in Theorem 3.7(1) and the condition $x \vee y \in I_{e,a}$ for all $x, y \in I_{e,a}$ in Theorem 3.7(2) do not hold. By using the construction method in Theorem 3.7, we can obtain a function $U_{22}$ on $L_2$, shown in Table 8. Since $U_{22}(k, U_{22}(k, n)) = U_{22}(k, b) = 1$ and $U_{22}(U_{22}(k, k), n) = U_{22}(m, n) = b$ for $k, n \in L_2$, the function $U_{22}$ does not satisfy associativity. Thus, the function $U_{22}$ is not a uninorm on $L_2$.

| $U^*$ | $0$ | $e$ | $a$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $a$ |
| $e$ | $0$ | $e$ | $a$ |
| $a$ | $a$ | $a$ | $a$ |

**Tab. 7.** The uninorm $U^*$ on $[0, a]$.

| $U_{22}$ | $0$ | $e$ | $a$ | $k$ | $m$ | $n$ | $b$ | $1$ |
|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $a$ | $k$ | $m$ | $n$ | $b$ | $1$ |
| $e$ | $0$ | $e$ | $a$ | $k$ | $m$ | $n$ | $b$ | $1$ |
| $a$ | $a$ | $a$ | $a$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $k$ | $k$ | $k$ | $1$ | $m$ | $m$ | $b$ | $1$ | $1$ |
| $m$ | $m$ | $m$ | $1$ | $m$ | $m$ | $b$ | $1$ | $1$ |
| $n$ | $n$ | $n$ | $1$ | $b$ | $b$ | $n$ | $1$ | $1$ |
| $b$ | $b$ | $b$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

**Tab. 8.** The function $U_{22}$ on $L_2$.

**Remark 3.12.** Let $U_2$ be a uninorm in Theorem 3.7. In this case, we can obtain the following results:

(1) If $U^* \in \mathcal{U}^*_\perp$, then $U_2 \in \mathcal{U}^*_\perp$.

(2) $U_2 \in \mathcal{U}_{\max}$ if and only if $U^* \in \mathcal{U}_{\max}$.

Next, we take an example to show that Theorem 3.7 is not a result of Theorem 3.1 and vice versa.

**Example 3.13.** Given a bounded lattice $L_3 = \{0, e, k, m, n, f, s, u, v, t, r, a, 1\}$ depicted in Figure 3. It is easy to see that the bounded lattice $L_3$ satisfies the conditions $x \vee y \in I_{e,a}$ for all $x, y \in I_{e,a}$ and $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I^a_e$ in Theorems 3.1 and 3.7.

(1) By using the construction method for uninorms in Theorem 3.1, we can obtain a uninorm $U_1$, satisfying $U_1(m,n) = m \vee n = f$ and $U_1(u,v) = u \vee v = t$.

(2) By using the construction method for uninorms in Theorem 3.7, we can obtain different uninorms $U_2$ when $q \in I_{e,a}$ as follows.

($i$) If $q = k \in I_{e,a}$, then $U_2(m,n) = m \vee n \vee k = f$ and $U_2(u,v) = u \vee v \vee k = r$.

($ii$) If $q = m \in I_{e,a}$, then $U_2(m,n) = m \vee n \vee m = f$ and $U_2(u,v) = u \vee v \vee m = r$.

($iii$) If $q = n \in I_{e,a}$, then $U_2(m,n) = m \vee n \vee n = f$ and $U_2(u,v) = u \vee v \vee n = r$.

($iv$) If $q = f \in I_{e,a}$, then $U_2(m,n) = m \vee n \vee f = f$ and $U_2(u,v) = u \vee v \vee f = r$.

($v$) If $q = s \in I_{e,a}$, then $U_2(m,n) = m \vee n \vee s = r$ and $U_2(u,v) = u \vee v \vee s = t$.

($vi$) If $q = u \in I_{e,a}$, then $U_2(m,n) = m \vee n \vee u = r$ and $U_2(u,v) = u \vee v \vee u = t$.

($vii$) If $q = v \in I_{e,a}$, then $U_2(m,n) = m \vee n \vee v = r$ and $U_2(u,v) = u \vee v \vee v = t$.

($viii$) If $q = t \in I_{e,a}$, then $U_2(m,n) = m \vee n \vee t = r$ and $U_2(u,v) = u \vee v \vee t = t$.

($ix$) If $q = r \in I_{e,a}$, then $U_2(m,n) = m \vee n \vee r = r$ and $U_2(u,v) = u \vee v \vee r = r$.

We can see that the uninorm $U_1$ in (1) differs from uninorms $U_2$ in (2). This show that Theorem 3.7 is not a result of Theorem 3.1 and vice versa.

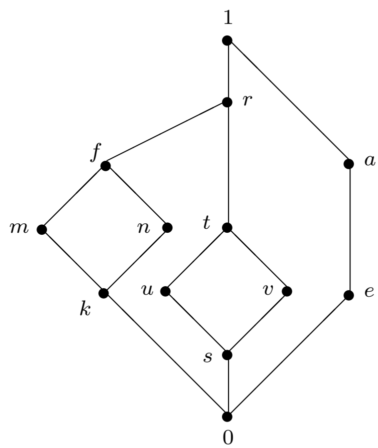


**Fig. 3:** The lattice $L_3$.

Similarly, let $b \in L \setminus \{0,1\}$, $p \in I_{e,b}$ and $U^*$ be a uninorm on $[b,1]$ with a neutral element $e$. Then we define a function $U : L^2 \to L$ by

$$U(x,y) = \begin{cases} U^*(x,y) & \text{if } (x,y) \in [b,1]^2, \\ x & \text{if } (x,y) \in (L \setminus [b,1]) \times [e,1], \\ y & \text{if } (x,y) \in [e,1] \times (L \setminus [b,1]), \\ x \wedge y \wedge p & \text{if } (x,y) \in I_{e,b} \times I_{e,b}, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In the following, we discuss how the function $U$ given by (2) with $p = 1$ or $p \in I_{e,b}$ can be a uninorm.

First, we illustrate that the function $U$ given by (2) with $p = 1$ can be a uninorm on bounded lattices under some conditions. Meanwhile, the dual result of Theorem 3.1 can be given.

**Theorem 3.14.** Let $b \in L \setminus \{0, 1\}$, $p = 1$, $U^*$ be a uninorm on $[b, 1]$ with a neutral element $e$ and $U_3$ be a function given by (2). Suppose that either $x \wedge y = 0$ for all $x, y \in I_{e,b}$ with $x \neq y$ or $x \wedge y \in I_{e,b}$ for all $x, y \in I_{e,b}$.

(1) Let us assume that $U^* \in \mathcal{U}^*_\top$. Then $U_3$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $x \parallel y$ for all $x \in I_{e,b}$ and $y \in I_e^b$.

(2) Let us assume that $I_b^e \cup I_{e,b} \cup (0, b) \neq \emptyset$. Then $U_3$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $U^* \in \mathcal{U}^*_\top$ and $x \parallel y$ for all $x \in I_{e,b}$ and $y \in I_e^b$.

Proof. It can be proved with the proof of Theorem 3.1 in a similar way. □

If we take $e = 1$ or $b$ in Theorem 3.14, then we can obtain some existing results in the literature.

**Remark 3.15.**
(1) If we take $e = 1$ in Theorem 3.14, then we obtain the $t$-norm $T$ in Theorem 1 of [8].

(2) If we take $e = b$ in Theorem 3.14, then we obtain the uninorm $U^e_{int}$ in Corollary 4.5 of [27].

**Remark 3.16.** Similarly, in Theorem 3.14, we observe that the condition either $x \wedge y = 0$ for all $x, y \in I_{e,b}$ with $x \neq y$ or $x \wedge y \in I_{e,b}$ for all $x, y \in I_{e,b}$ can not be omitted, in general.

**Remark 3.17.** Let $U_3$ be a uninorm in Theorem 3.14. In this case, we can obtain the following results:

(1) If $U^* \in \mathcal{U}^*_\top$, then $U_3 \in \mathcal{U}^*_\top$.

(2) $U_3 \in \mathcal{U}_{\min}$ if and only if $U^* \in \mathcal{U}_{\min}$.

Next, we show that the function $U$ given by (2) with $p \in I_{e,b}$ can be a uninorm on bounded lattices under some conditions. Meanwhile, the dual result of Theorem 3.7 can be given.

**Theorem 3.18.** Let $b \in L \setminus \{0, 1\}$, $p \in I_{e,b}$, $U^*$ be a uninorm on $[b, 1]$ with a neutral element $e$ and $U_4$ be a function given by (2).

(1) Suppose that either $x \wedge p = 0$ for all $x \in I_{e,b}$ with $x \neq p$.

(i) Let us assume that $U^* \in \mathcal{U}^*_\top$. Then $U_4$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $x \parallel p$ for $x \in I_e^b$.

(ii) Let us assume that $I_b^e \cup I_{e,b} \cup (0,b) \neq \emptyset$. Then $U_4$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $U^* \in \mathcal{U}_\top^*$ and $x \parallel p$ for $x \in I_e^b$.

(2) Suppose that $x \wedge y \in I_{e,b}$ for all $x, y \in I_{e,b}$.

(i) Let us assume that $U^* \in \mathcal{U}_\top^*$. Then $U_4$ is a uninorm on $L$ with the neutral element $e \in L$ and $x \parallel y$ for all $x \in I_{e,b}$ and $y \in I_e^b$.

(ii) Let us assume that $I_b^e \cup I_{e,b} \cup (0,b) \neq \emptyset$. Then $U_4$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $U^* \in \mathcal{U}_\top^*$ and $x \parallel y$ for all $x \in I_{e,b}$ and $y \in I_e^b$.

Proof. It can be proved with the proof of Theorem 3.7 in a similar way. □

**Remark 3.19.** If we take $e = 1$ in Theorem 3.18, then we obtain the $t$-norm $T$ in Theorem 1 of [8].

**Remark 3.20.**
(1) In Theorem 3.18(1), we observe that the condition $x \wedge p = 0$ for all $x \in I_{e,b}$ with $x \neq p$ can not be omitted, in general.

(2) In Theorem 3.18(2), we observe that the condition $x \wedge y \in I_{e,b}$ for all $x, y \in I_{e,b}$ can not be omitted, in general.

**Remark 3.21.** Let $U_4$ be a uninorm in Theorem 3.18. In this case, we can obtain the following results:

(1) If $U^* \in \mathcal{U}_\top^*$, then $U_4 \in \mathcal{U}_\top^*$.

(2) $U_4 \in \mathcal{U}_{\min}$ if and only if $U^* \in \mathcal{U}_{\min}$.

Next, we present a new construction method for uninorms on bounded lattices with the uninorms on $[0,a]$ and the $t$-conorms on $[a,1]$.

**Theorem 3.22.** Let $a \in L \setminus \{0,1\}$, $U^*$ be a uninorm on $[0,a]$ with a neutral element $e$ and $S$ be a $t$-conorm on $[a,1]$. Let $U_5 : L^2 \to L$ be a function as follows

$$U_5(x,y) = \begin{cases} U^*(x,y) & \text{if } (x,y) \in [0,a]^2, \\ x & \text{if } (x,y) \in (L \setminus [0,a]) \times [0,e], \\ y & \text{if } (x,y) \in [0,e] \times (L \setminus [0,a]), \\ S(x \vee a, y \vee a) & \text{if } (x,y) \in I_{e,a} \times I_{e,a}, \\ 1 & \text{otherwise,} \end{cases}$$

Suppose that $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$.

(1) If $U^* \in \mathcal{U}_\perp^*$, then $U_5$ is a uninorm on $L$ with the neutral element $e \in L$.

(2) Let us assume that $I_a^e \cup I_{e,a} \cup (a,1) \neq \emptyset$. Then $U_5$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $U^* \in \mathcal{U}_\perp^*$.

P r o o f . (1) By the definition of $U_5$, it is easy to obtain that $U_5$ is commutative and $e$ is the neutral element of $U_5$. Thus, we only need to prove the increasingness and the associativity of $U_5$.

I. Increasingness: We prove that if $x \le y$, then $U_5(x,z) \le U_5(y,z)$ for all $z \in L$. Taking into account Theorem 3.1, it is enough to check only those cases that are different from the cases in Theorem 3.1.

1. $x \in [0,e], y \in I_{e,a}, z \in I_{e,a}$
   $U_5(x,z) = z < S(y \vee a, z \vee a) = U_5(y,z)$
2. $x \in I_{e,a}$
   - 2.1. $y \in I_{e,a}, z \in I_{e,a}$
     $U_5(x,z) = S(x \vee a, z \vee a) \le S(y \vee a, z \vee a) = U_5(y,z)$
   - 2.2. $y \in I_a^e \cup (a,1], z \in I_{e,a}$
     $U_5(x,z) = S(x \vee a, z \vee a) \le 1 = U_5(y,z)$

II. Associativity: It can be shown that $U_5(x, U_5(y,z)) = U_5(U_5(x,y),z)$ for all $x,y,z \in L$. By Theorem 3.12 in [31] and taking into account Theorem 3.1, it is enough to check only those cases that are different from the cases in Theorem 3.1.

1. If $x,y,z \in I_{e,a}$, then $U_5(x,U_5(y,z)) = U_5(x, S(y \vee a, z \vee a)) = 1 = U_5(S(x \vee a, z \vee a), z) = U_5(U_5(x,y),z)$.

2. If $x,y \in I_{e,a}$ and $z \in (a,1]$, then $U_5(x,U_5(y,z)) = U_5(x,1) = 1 = U_5(S(x \vee a, z \vee a), z) = U_5(U_5(x,y),z)$.

3. If $x \in [0,e]$ and $y,z \in I_{e,a}$, then $U_5(x,U_5(y,z)) = U_5(x,S(y \vee a, z \vee a)) = S(y \vee a, z \vee a) = U_5(y,z) = U_5(U_5(x,y),z)$.

4. If $x \in I_e^a \cup (e,a] \cup I_a^e$ and $y,z \in I_{e,a}$, then $U_5(x,U_5(y,z)) = U_5(x,S(y \vee a, z \vee a)) = 1 = U_5(1,z) = U_5(U_5(x,y),z)$.

Therefore, $U_5$ is a uninorm on $L$ with the neutral element $e$.

(2) It can be proved with the proof of Theorem 3.1(2) in a similar way. □

If we take $e = 0$ or $a$ in Theorem 3.22, then we can obtain some existing results in the literature.

**Remark 3.23.**
(1) If we take $e = 0$ in Theorem 3.22, then we obtain the $t$-conorm $S$ in Theorem 1 of[8].

(2) If we take $e = a$ in Theorem 3.22, then we obtain the uninorm $U_{1,e}$ in Theorem 7 of [13].

Next, we show that if we take different $t$-conorms $S$ on $[a,1]$ in Theorem 3.22, then the uninorm is just the function given by (1) with some $q \in L$. In other words, we obtain how the function given by (1) can be a uninorm with $q \in [e,a]$ or $q \in (a,1]$.

**Remark 3.24.** (1) Suppose that $S(x,y) = x \vee y$ for $x, y \in [a,1]$ in Theorem 3.22. Then $U_5$ is just the function given by (1) with $q \in [e,a]$, since $U_5(x,y) = S(x \vee a, y \vee a) = x \vee y \vee a = x \vee y \vee q = U(x,y)$ for $x, y \in I_{e,a}$.

(2) Suppose that $S(x,y) = \begin{cases} x \vee y \vee q & \text{if } (x,y) \in (a,1]^2, \\ x \vee y & \text{otherwise} \end{cases}$ in Theorem 3.22, where $q \in (a,1]$. Then $U_5$ is just the function given by (1) with $q \in (a,1]$, since $U_5(x,y) = S(x \vee a, y \vee a) = x \vee y \vee a \vee q = x \vee y \vee q = U(x,y)$ for $x, y \in I_{e,a}$.

The next example illustrates the construction method of uninorms on bounded lattices in Theorem 3.22.

**Example 3.25.** Given a bounded lattice $L_4 = \{0, b, e, c, a, d, l, m, n, s, 1\}$ depicted in Figure 4 and a uninorm $U^* : [0,a]^2 \to [0,a]$ shown in Table 9. It is easy to see that $L_4$ and $U^*$ satisfy the conditions in Theorem 3.22. By using the construction method in Theorem 3.22 and taking $S(x,y) = x \vee y$ on $[a,1]$, the uninorm $U_{51} : L_3^2 \to L_3$ with the neutral element $e$ is defined as in Table 10.

**Remark 3.26.** In Theorem 3.22, we observe that the condition $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$ can not be omitted, in general.

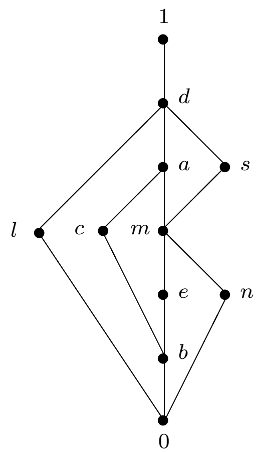


**Fig. 4:** The lattice $L_4$.

The next example illustrates the fact that if the condition in Remark 3.26 does not hold, then the increasingness of $U_5$ is violated.

**Example 3.27.** Given a bounded lattice $L_5 = \{0, e, a, b, k, m, n, 1\}$ depicted in Figure 5 and a uninorm $U^* : [0,a]^2 \to [0,a]$ shown in Table 11. Since $k < m$ for $k \in I_e^a$ and $m \in I_{e,a}$, $L_5$ does not satisfy the condition in Theorem 3.22. By using the construction method in Theorem 3.22 and taking $S(x,y) = x \vee y$ on $[a,1]$, we can obtain a function $U_{52}$ on $L_5$, shown in Table 12. Then $U_{52}(k,m) = 1$ and $U_{52}(m,m) = S(m \vee a, m \vee a) = S(b,b) = b \vee b = b$ for $k, m \in L_5$. Since $k < m$, the function $U_{52}$ does not satisfy increasingness. Thus, the function $U_{52}$ is not a uninorm on $L_5$.

| $U^*$ | $0$ | $b$ | $e$ | $n$ | $c$ | $m$ | $a$ |
|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $n$ | $c$ | $m$ | $a$ |
| $b$ | $0$ | $b$ | $b$ | $n$ | $c$ | $m$ | $a$ |
| $e$ | $0$ | $b$ | $e$ | $n$ | $c$ | $m$ | $a$ |
| $n$ | $n$ | $n$ | $n$ | $n$ | $c$ | $a$ | $a$ |
| $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $a$ | $a$ |
| $m$ | $m$ | $m$ | $m$ | $a$ | $a$ | $m$ | $a$ |
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ |

**Tab. 9.** The uninorm $U^*$ on $[0, a]$.

| $U_{51}$ | $0$ | $b$ | $e$ | $n$ | $c$ | $m$ | $a$ | $l$ | $s$ | $d$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $n$ | $c$ | $m$ | $a$ | $l$ | $s$ | $d$ | $1$ |
| $b$ | $0$ | $b$ | $b$ | $n$ | $c$ | $m$ | $a$ | $l$ | $s$ | $d$ | $1$ |
| $e$ | $0$ | $b$ | $e$ | $n$ | $c$ | $m$ | $a$ | $l$ | $s$ | $d$ | $1$ |
| $n$ | $n$ | $n$ | $n$ | $n$ | $c$ | $a$ | $a$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $c$ | $c$ | $c$ | $c$ | $c$ | $a$ | $a$ | $1$ | $1$ | $1$ | $1$ |
| $m$ | $m$ | $m$ | $m$ | $a$ | $a$ | $m$ | $a$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $a$ | $1$ | $1$ | $1$ | $1$ |
| $l$ | $l$ | $l$ | $l$ | $1$ | $1$ | $1$ | $1$ | $d$ | $1$ | $1$ | $1$ |
| $s$ | $s$ | $s$ | $s$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $d$ | $d$ | $d$ | $d$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

**Tab. 10.** The uninorm $U_{51}$ on $L_3$.

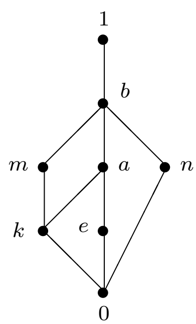


**Fig. 5** The lattice $L_5$.

**Remark 3.28.** Let $U_5$ be a uninorm in Theorem 3.22. In this case, we can obtain the following results:

(1) If $U^* \in \mathcal{U}_{\perp}^*$, then $U_5 \in \mathcal{U}_{\perp}^*$.

(2) $U_5 \in \mathcal{U}_{\max}$ if and only if $U^* \in \mathcal{U}_{\max}$.

Also, we give the dual result of Theorem 3.22.

| $U^*$ | $0$ | $e$ | $k$ | $a$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $k$ | $a$ |
| $e$ | $0$ | $e$ | $k$ | $a$ |
| $k$ | $k$ | $k$ | $k$ | $a$ |
| $a$ | $a$ | $a$ | $a$ | $a$ |

**Tab. 11.** The uninorm $U^*$ on $[0, a]$.

| $U_{52}$ | $0$ | $e$ | $k$ | $a$ | $m$ | $n$ | $b$ | $1$ |
|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $k$ | $a$ | $m$ | $n$ | $b$ | $1$ |
| $e$ | $0$ | $e$ | $k$ | $a$ | $m$ | $n$ | $b$ | $1$ |
| $k$ | $k$ | $k$ | $k$ | $a$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $a$ | $a$ | $a$ | $a$ | $1$ | $1$ | $1$ | $1$ |
| $m$ | $m$ | $m$ | $1$ | $1$ | $b$ | $b$ | $1$ | $1$ |
| $n$ | $n$ | $n$ | $1$ | $1$ | $b$ | $b$ | $1$ | $1$ |
| $b$ | $b$ | $b$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |

**Tab. 12.** The function $U_{52}$ on $L_4$.

**Theorem 3.29.** Let $b \in L \setminus \{0, 1\}$, $U^*$ be a uninorm on $[b, 1]$ with a neutral element $e$ and $T$ be a $t$-norm on $[0, b]$. Let $U_6 : L^2 \to L$ be a function as follows

$$U_6(x, y) = \begin{cases} U^*(x, y) & \text{if } (x, y) \in [b, 1]^2, \\ x & \text{if } (x, y) \in (L \setminus [b, 1]) \times [e, 1], \\ y & \text{if } (x, y) \in [e, 1] \times (L \setminus [b, 1]), \\ T(x \wedge b, y \wedge b) & \text{if } (x, y) \in I_{e,b} \times I_{e,b}, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $x \parallel y$ for all $x \in I_{e,b}$ and $y \in I_e^b$.

(1) If $U^* \in \mathcal{U}_\top^*$, then $U_6(x, y)$ is a uninorm on $L$ with the neutral element $e \in L$.

(2) Let us assume that $I_b^e \cup I_{e,b} \cup (0, b) \neq \emptyset$. Then $U_6(x, y)$ is a uninorm on $L$ with the neutral element $e \in L$ if and only if $U^* \in \mathcal{U}_\top^*$.

Proof. It can be proved with the proof of Theorem 3.22 in a similar way. □

If we take $e = 1$ or $b$ in Theorem 3.29, then we can obtain some existing results in the literature.

**Remark 3.30.**

(1) If we take $e = 1$ in Theorem 3.29, then we obtain the $t$-norm $T$ in Theorem 1 of [8].

(2) If we take $e = b$ in Theorem 3.29, then we obtain the uninorm $U_{2,e}$ in Theorem 8 of[13].

Next, we show that if we take different $t$-norms $T$ on $[0,b]$ in Theorem 3.29, then the uninorm is just the function given by (2) with some $p \in L$. In other words, we obtain how the function given by (2) can be a uninorm with $p \in [b,e]$ or $p \in [0,b)$.

**Remark 3.31.** (1) Suppose that $T(x,y) = x \wedge y$ for $x, y \in [0,b]$ in Theorem 3.29. Then $U_6$ is just the function given by (2) with $p \in [b,e]$, since $U_6(x,y) = T(x \wedge b, y \wedge b) = x \wedge y \wedge b = x \wedge y \wedge p = U(x,y)$ for $x, y \in I_{e,b}$.

(2) Suppose that $T(x,y) = \begin{cases} x \wedge y \wedge p & \text{if } (x,y) \in [0,b)^2, \\ x \wedge y & \text{otherwise.} \end{cases}$ in Theorem 3.29, where $p \in [0,b)$. Then $U_6$ is just the function given by (2) with $p \in [0,b)$, since $U_6(x,y) = T(x \wedge b, y \wedge b) = x \wedge y \wedge b \wedge p = x \wedge y \wedge p = U(x,y)$ for $x, y \in I_{e,b}$.

**Remark 3.32.** Similarly, in Theorem 3.29, we observe that the condition $x \parallel y$ for all $x \in I_{e,b}$ and $y \in I_e^b$ can not be omitted, in general.

**Remark 3.33.** Let $U_6$ be a uninorm in Theorem 3.29. In this case, we can obtain the following results:

(1) If $U^* \in \mathcal{U}_\top^*$, then $U_6 \in \mathcal{U}_\top^*$.

(2) $U_6 \in \mathcal{U}_{\min}$ if and only if $U^* \in \mathcal{U}_{\min}$.

## 4. CONCLUSION

In this article, we have proposed a new approach to construct uninorms on a bounded lattice $L$ with some additional constraints. The important point is that our methods are based on the given uninorms on the subinterval $[0,a]$ (or $[b,1]$). So our methods are more general than those based on $t$-norms and $t$-conorms and then can generalize some known construction methods for uninorms on a bounded lattice in the literature. Specifically, about the results in this paper, we give some remarks as follows.

(1) We present a function $U$ given by (1) with $q \in L$ and investigate how this function is a uninorm when $q = 0$, $q \in I_{e,a}$, $q \in [e,a]$ or $q \in (a,1]$. See Theorem 3.1, Theorem 3.7 and Remark 3.24. Similarly, we dually present a function $U$ given by (2) with $p \in L$ and investigate how this function is a uninorm when $p = 1$, $p \in I_{e,b}$, $p \in [b,e]$ or $p \in [0,b)$. See Theorem 3.14, Theorem 3.18 and Remark 3.31.

(2) Based on Theorem 3.22, by taking different $t$-conorms, we can obtain some uninorms, which are just the function given by (1) with $q \in [e,a]$ and $q \in (a,1]$, respectively. See Remark 3.24. Similarly, based on Theorem 3.29, by taking different $t$-norms, we can obtain two uninorms, which are just the function given by (2) with $p \in [b,e]$ and $p \in [0,b)$, respectively. See Remark 3.31.

In the future work, we still investigate the methods to construct uninorms via given uninorms more comprehensively and use this idea to construct other aggregative operators.

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