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PARTITIONING PLANAR GRAPH OF GIRTH 5 INTO TWO FORESTS WITH MAXIMUM DEGREE 4

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Abstract. Given a graph $G = (V, E)$, if we can partition the vertex set V into two nonempty subsets V_1 and V_2 which satisfy $\Delta(G[V_1]) \leqslant d_1$ and $\Delta(G[V_2]) \leqslant d_2$, then we say G has a $(\Delta_{d_1}, \Delta_{d_2})$-partition. And we say G admits an (F_{d_1}, F_{d_2})-partition if $G[V_1]$ and $G[V_2]$ are both forests whose maximum degree is at most d_1 and d_2, respectively. We show that every planar graph with girth at least 5 has an (F_4, F_4)-partition.

Keywords: vertex partition; girth; forest; maximum degree

MSC 2020: 05C10, 05C69

1. Introduction

Throughout the paper, we consider finite, simple and undirected graphs. We call a graph G *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Given a graph $G = (V, E)$, for $V' \subseteq V$, we use $G[V']$ to denote the *induced subgraph* of G, obtained from the vertex set V' and all of the edges connecting pairs of vertices in V'. Denote by $G_1, \ldots, G_k$ k families of graphs. If we can partition the vertex set V into k vertex subsets $V_1, \ldots, V_k$ which satisfy that for each $1 \leqslant i \leqslant k$ the obtained induced subgraph $G[V_i] \in G_i$, then we call it a $(G_1, \ldots, G_k)$-*partition.* For brevity, let I, Δ_d, F and F_d represent the independent sets, the family of graphs of maximum degree at most d, the family of forests, and the family of forests of maximum degree at most d, respectively. Notice that $\Delta_0 = F_0 = I$ and $\Delta_1 = F_1$. Denote by $g(G)$ the *girth* of G, that is, the length of a smallest cycle contained in G. If two cycles have at least one common vertex, then they are said to be *intersecting.*

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Recently, for planar graph the problem concerning vertex partitions has attracted much more attention. The well-known Four Color Theorem (see [1], [2]) confirms every planar graph admits an (I, I, I, I)-partition. This cannot be strengthened to be any (I, I, I)-partition due to the complete graph K_4. However, by working on the acyclic coloring problems of planar graphs, Borodin in [3] affirmed that every planar graph admits an (I, F, F)-partition. Neither I nor F can be left out from the above result basing on the counterexamples established by Chartrand and Kronk in [8]. Poh in [18], in 1990, proved that planar graph admits (F_2, F_2, F_2)-partition. Furthermore, about planar graphs without specifical cycles, Raspaud and Wang in [19] showed us that for a fixed $k \in \{3, 4, 5, 6\}$, if planar graph does not contain k-cycles, then it admits an (F, F)-partition. And they further conjectured in [19] that every planar graph has an (F, F)-partition if it does not contain any intersecting triangles. In 2012, Chen, Raspaud and Wang in [9] proved this conjecture positively.

Denote by $\mathcal{G}_g$ the class of planar graphs whose girth is at least g. It has been shown in [17] that there exists a graph belonging to $\mathcal{G}_4$ having no $(\Delta_{d_1}, \Delta_{d_2})$-partition for any nonnegative integers d_1 and d_2. Later, Dross, Montassier and Pinlou in [13] studied the refinement of an (F, F)-partition of $\mathcal{G}_4$. In particular, they in [13] showed every graph in $\mathcal{G}_4$ admits an (F, F_5)-partition, which has been slightly improved to have an (F, F_3)-partition by Feghali and Sámal, see [14].

In 2015, Choi and Raspaud in [12] raised an interesting question.

Question 1.1. Does every graph in $\mathcal{G}_5$ have a $(\Delta_{d_1}, \Delta_{d_2})$-partition for all $d_1 + d_2 \geqslant 8$, where $d_2 \geqslant d_1 \geqslant 1$?

For a graph in $\mathcal{G}_5$, it has been proved to have an (F, F_0)-partition by Borodin and Glebov (see [4]), a (Δ_4, Δ_4)-partition by Havet and Sereni (see [15]), a (Δ_3, Δ_5)-partition by Choi and Raspaud (see [12]), and a (Δ_2, Δ_6)-partition by Borodin and Kostochka, see [7]. Recently, Choi et al. in [11] showed us that Question 1.1 is true for the case that $d_1 = 1$ and $d_2 = 10$. It remains to consider the case that $d_1 = 1$ and $7 \leqslant d_2 \leqslant 9$. In [20], Zhang, Chen and Wang proved that every graph in $\mathcal{G}_5$ which does not contain adjacent 5-cycles has a (Δ_1, Δ_7)-partition. For the other known results on vertex partitions of planar graphs with different restrictions, the readers can refer to [5], [6], [10], [16].

In this paper, we aim to find a refinement of a forest partition for graphs in $\mathcal{G}_5$. More specifically, we showed the result as follows.

Theorem 1.2. *Every graph in $\mathcal{G}_5$ admits an (F_4, F_4)-partition.*

Theorem 1.2 is a strengthening of a result in [15] which shows that every graph in $\mathcal{G}_5$ has a (Δ_4, Δ_4)-partition.

2. Notation

Given a planar graph $G = (V, E, F)$ drawn on the plane, for $v \in V(G)$, denote by $d(v)$ the *degree* of v, $N(v)$ represents the set of neighbours of v in G. For $f \in F(G)$, let $d(f)$ denote the *degree* of f, which is the number of edges incident with f. A *k-vertex* (*k^+-vertex* or *k^--vertex*) is a vertex of degree k (at least k or at most k). The same notation can be applied to faces.

For $x \in V(G) \cup F(G)$, denote by $n_i(x)$ the number of i-vertices adjacent or incident to x. We call a vertex v *big* if $d(v) \geqslant 6$, otherwise we call it *small*. If a neighbour of v is a big vertex, then we call it a *big neighbour*. Usually, denoted by $n_b(v)$ the number of big neighbours of v. And for a 3-vertex v in G, we say v is *heavy* if $n_b(v) \geqslant 2$, and *light* otherwise. For simplicity, denote by $n_{3^h}(u)$ and $n_{3^l}(u)$ the number of heavy 3-vertices and light 3-vertices adjacent to u, respectively.

3. Proof of Theorem 1.2

Suppose that G is the smallest counterexample to Theorem 1.2 with minimizing $|V(G)|$. We have a plane graph $G = (V, E, F)$ drawn on the plane with the face set F. Obviously, G is connected and of minimum degree at least 2. For a 5-face f in G. If $n_2(f) = 0$, $n_2(f) = 1$ and $n_2(f) = 2$, respectively, call f *good*, *weak*, and *bad*, respectively. For any subset $S \subset V(G)$, denote by $G - S$ the graph formed from G by deleting the vertices in S.

Given a (partial) (F_4, F_4)-partition $\{F_A, F_B\}$ of $G' \subset G$, if a vertex $v \in F_X$, then we call it an *F_X-vertex*, where $X \in \{A, B\}$. We call an F_X-vertex x an *F_X-neighbour* of v if $x \in N(v)$. Besides, if v is an F_X-vertex in G' which is adjacent to exactly four F_X-vertices, then we say is v *F_X-saturated*.

3.1. Structural lemmas.

Lemma 3.1. *Every 2-vertex has two big neighbours.*

P r o o f. Assume v is a 2-vertex which is adjacent to v_1 and v_2 such that $d(v_i) \leqslant 5$ for a fixed $i \in \{1, 2\}$. Let $d(v_1) \leqslant 5$ and $G' = G - \{v\}$. Then G' has an (F_4, F_4)-partition $\{F_A, F_B\}$ due to the minimality of G. Notice that if one cannot successfully extend $\{F_A, F_B\}$ to G, then we may assume that v_1 is F_A-saturated and v_2 is F_B-saturated in G'. It is easy to compute that $d(v_1) = 5$ and thus we can further change v_1 to F_B, and then add v to F_A, in such a way we obtain an (F_4, F_4)-partition of G, which is a contradiction. $\square$

Corollary 3.2. *There is no 3-vertex on any bad 5-faces.*

Lemma 3.3. *Every 3-vertex has at least one big neighbour.*

P r o o f. Assume v is a 3-vertex adjacent to v_1, v_2 and v_3 such that $d(v_i) \leqslant 5$ for all $i = 1, 2, 3$. Then $G' = G - \{v\}$ has an (F_4, F_4)-partition $\{F_A, F_B\}$ due to the minimality of G. Obviously, there exists an $X \in \{A, B\}$ such that F_X appears at most once on the set $\{v_1, v_2, v_3\}$. By symmetry, assume $X = A$ and $v_1 \in F_A$ (if exists). Then v_2 and v_3 are both in F_B. We add v to F_A firstly. If this is not an (F_4, F_4)-partition, it has to be $d(v_1) = 5$ and v_1 is F_A-saturated in G'. Likewise, it suffices to further change v_1 to F_B, and thus obtain an (F_4, F_4)-partition of G, a contradiction. $\square$

Lemma 3.4. *Let $d(v) = 6$. If $n_2(v) + n_{3^l}(v) \geqslant 1$, then v has at least one big neighbour.*

P r o o f. Let $N(v) = \{v_1, v_2, \ldots, v_6\}$ such that v_1 is either a light 3-vertex or a 2-vertex. Assume $d(v_i) \leqslant 5$ for all $i = 2, 3, \ldots, 6$. Let $G' = G - \{v_1\}$. We derive that G' has an (F_4, F_4)-partition $\{F_A, F_B\}$ due to the minimality of G. Next, we shall discuss two cases, according to the situation of v_1.

Case 1: Suppose that $d(v_1) = 2$. Denote by v_1' the other neighbour of v_1 distinct to v. If we are not able to obtain an (F_4, F_4)-partition by extending $\{F_A, F_B\}$ to G, then w.l.o.g., assume that $v \in F_A$ and $v_1' \in F_B$ with v being F_A-saturated in G'. It means that exactly four vertices among v_2, v_3, v_4, v_5 and v_6 are in F_A and the remaining one belongs to F_B. W.l.o.g., assume $v_i \in F_A$ for all $i = 2, 3, 4, 5$ and $v_6 \in F_B$. If v_6 is not F_B-saturated in G', then one may directly change v to F_B and then add v_1 to F_A. Otherwise, we deduce that $d(v_6) = 5$ and all other neighbours of v_6 different from v belong to F_B. At this point, it suffices to change v_6 to F_A, v to F_B and finally add v_1 to F_A successfully.

Case 2: Assume v_1 is a light 3-vertex. Let x_1 and x_2 denote the other two neighbours of v_1 distinct to v. By definition, we see that both x_1 and x_2 are 5^--vertices. Obviously, there exists an $X \in \{A, B\}$ such that F_X appears at most once on the set $\{v, x_1, x_2\}$. W.l.o.g., assume $X = A$. Firstly, we add v_1 to F_A. If the obtained partition is not an (F_4, F_4)-partition of G, then there exists a vertex belonging to $\{v, x_1, x_2\}$ being F_A-saturated in G'. If $x_i \in F_A$ for some $i \in \{1, 2\}$, say x_1, then we only need to further change x_1 to F_B. Otherwise, assume $v \in F_A$. Similarly, we deduce that exactly one neighbour of v, say v_6, belongs to F_B, and v_6 is F_B-saturated in G'. This makes us sure that $d(v_6) = 5$. Thus, it suffices to change v_6 to F_A and v to F_B.

In any case, we have an (F_4, F_4)-partition of G, a contradiction. $\square$

Lemma 3.5. *Let $d(v) = 7$. If $n_{4^+}(v) = 0$, then $n_{3^h}(v) \geqslant 2$.*

P r o o f. Let $N(v) = \{v_1, v_2, \ldots, v_7\}$ and v_i be 3^--vertices for all $i = 1, 2, \ldots, 7$. Denote by v_i' the other neighbour of v_i if $d(v_i) = 2$, and v_{i1}, v_{i2} the other two neighbours of v_i if $d(v_i) = 3$. Assume that among all v's neighbours there is at most one heavy 3-vertex. W.l.o.g., let v_1 be the unique heavy 3-vertex if it exists. This implies that both v_{j1} and v_{j2} are 5^--vertices for each $j \in \{2, 3, \ldots, 7\}$ if v_{j1} and v_{j2} exist. Let $G' = G - \{v, v_2, \ldots, v_7\}$. Then G' has an (F_4, F_4)-partition $\{F_A, F_B\}$ due to the minimality of G.

For $2 \leqslant j \leqslant 7$, if v_j is a 2-vertex or a 3-vertex with $v_{j1}, v_{j2} \in F_X$ for some $X \in \{A, B\}$, then we add v_j to the different part from v_j' or v_{j1}, respectively. Now look at the 3-vertex v_j whose neighbours v_{j1} and v_{j2} belong to the different parts. W.l.o.g., assume that $v_{j1} \in F_A$ and $v_{j2} \in F_B$. Let $S = \{v_1, v_2, \ldots, v_7\}$. Denote by S_{F_A} (or S_{F_B}) the vertex set of F_A-vertices (or F_B-vertices) being in S. Since $|S| = 7$, we have that $|S_{F_A} \cup S_{F_B}| \leqslant 7$, and thus, there is an $X \in \{A, B\}$ which appears at most three times on $S_{F_A} \cup S_{F_B}$. W.l.o.g., suppose $X = A$. Thus, one may put all remaining un-added vertices v_j to F_B, and then change v_{j2} to F_A if it is F_B-saturated in G'. Afterwards, we add v to F_A. In any case, it is easy to get the (F_4, F_4)-partition of G, which is a contradiction. □

3.2. Structure properties of 5-faces.

Lemma 3.6. *Suppose that $f = [v_1 v_2 \ldots v_5]$ is a bad 5-face such that $d(v_1) = d(v_4) = 2$ and $d(v_2) = 6$. Then*
(1) *$n_b(v_2) \geqslant 2$ if $d(v_5) = 6$;*
(2) *$n_b(v_i) \geqslant 2$ for each $i \in \{2, 3\}$ if $d(v_3) = 6$.*

P r o o f. Let $N(v_2) = \{v_1, v_3, u_1, \ldots, u_4\}$. We will make use of contradictions to show (1)–(2).

(1) Let $N(v_5) = \{v_1, v_4, w_1, \ldots, w_4\}$. Suppose to the contrary that v_2 has at most one big neighbour. Let $G' = G - \{v_1\}$. Then G' admits an (F_4, F_4)-partition $\{F_A, F_B\}$ due to the minimality of G. Obviously, if one fails to obtain an (F_4, F_4)-partition by adding v_1 to F_A or F_B, then we may suppose that v_2 is F_A-saturated and v_5 is F_B-saturated in G'. Since $d(v_3) \geqslant 6$ by Lemma 3.1, we assert that $d(u_j) \leqslant 5$ for all $j = 1, 2, 3, 4$. If $v_4 \in F_A$, then each $w_i \in F_B$, and thus we can change v_5 to F_A and then add v_1 to F_B. Otherwise, assume that $v_4 \in F_B$. If $v_3 \in F_B$, then we change v_4 to F_A and then add v_1 to F_B. So now assume that $v_3 \in F_A$. Recall that v_2 is F_A-saturated in G'. This guarantees us that there exists some $i \in \{1, 2, 3, 4\}$, say $i = 1$, so that $u_1 \in F_B$. At this moment, it remains us to further change v_2 to F_B, u_1 to F_A if it is F_B-saturated in G', and afterwards add v_1 to F_A.

(2) Let $N(v_3) = \{v_2, v_4, w_1, \ldots, w_4\}$. By symmetry, suppose to the contrary that v_2 has at most one big neighbour. Obviously, $G' = G - \{v_1\}$ has an (F_4, F_4)-partition $\{F_A, F_B\}$ due to the minimality of G. Likewise, suppose that v_2 is F_A-saturated and v_5 is F_B-saturated in G'. Since v_3 is already a big vertex, we deduce that $d(u_i) \leqslant 5$ for all $i = 1, 2, 3, 4$. If $v_3 \in F_A$, then exactly one of $u_1, \ldots, u_4$, say u_1, belongs to F_B. Thus, we can change v_2 to F_B, u_1 to F_A if it is F_B-saturated in G', and finally add v_1 to F_A. Otherwise, assume that $v_3 \in F_B$. If there are at most three vertices among $v_4, w_1, \ldots, w_4$ belonging to F_B, then one can directly change v_2 to F_B and then add v_1 to F_A. So now assume that exactly four of them belong to F_B and the unique one is in F_A. If $v_4 \in F_A$, that is, $w_i \in F_B$ for all $i = 1, 2, 3, 4$, then we switch the partition of v_2 and v_3, and then add v_1 to F_A. Or else, assume that $v_4 \in F_B$. In this case, we only need to further change v_4 to F_A, v_2 to F_B, and finally add v_1 to F_A.

In each of the above cases, we can easily get an (F_4, F_4)-partition of G, which is a contradiction. □

Lemma 3.7. *Let $f = [v_1v_2v_3v_4v_5]$ and $g = [v_1v_5v_6v_7v_8]$ be adjacent weak 5-faces, $d(v_2) = d(v_8) = 2$, $d(v_4) = d(v_5) = d(v_6) = 3$ and $d(v_1) = 6$. Then v_4 and v_6 are both heavy 3-vertices.*

P r o o f. W.l.o.g., assume v_4 is a light 3-vertex. Let v_4' denote another neighbour of v_4 except v_3 and v_5. By definition, $d(v_4') \leqslant 5$ because $d(v_3) \geqslant 6$ by Lemma 3.1. Let $N(v_1) = \{v_2, v_5, v_8, w_1, w_2, w_3\}$. Denote $G' = G - \{v_5\}$. Clearly, G' has an (F_4, F_4)-partition $\{F_A, F_B\}$ due to the minimality of G.

It is obvious that there exists an $X \in \{A, B\}$ such that F_X appears at most once on the set $\{v_1, v_4, v_6\}$. By symmetry, let $X = A$. Firstly, add v_5 to F_A. If the obtained partition is not our wanted, then it must be the case that $v_1 \in F_A$ and $v_4, v_6 \in F_B$. Furthermore, v_1 is F_A-saturated in G'. If $v_3 \in F_B$, then we can change v_4 to F_A, v_4' to F_B if it is F_A-saturated in G', and finally change v_5 to F_B. Otherwise, assume that $v_3 \in F_A$. If $v_2 \in F_A$, then change v_2 to F_B directly, and thus reach an (F_4, F_4)-partition of G, which is a contradiction. So now assume $v_2 \in F_B$. Since v_1 is F_A-saturated in G', we are sure that w_1, w_2, w_3 and v_8 are all in F_A. At this point, it is easy to change v_1 to F_B to attain an (F_4, F_4)-partition of G, which is a contradiction. □

Lemma 3.8. *Let $f = [vv_1v_2v_3v_4]$ and $g = [v_1vv_4v_5v_6]$ be adjacent bad 5-faces, $d(v) = d(v_6) = 2$ and $d(v_1) = d(v_4) = 6$. Then*

(1) *$n_b(v_4) \geqslant 3$ if $d(v_2) = 2$;*

(2) *$n_b(v_i) \geqslant 2$ for each $i \in \{1, 4\}$ if $d(v_3) = 2$.*

P r o o f. Let $N(v_1) = \{v, v_2, v_6, w_1, w_2, w_3\}$ and $N(v_4) = \{v, v_3, v_5, u_1, u_2, u_3\}$. Since $d(v_6) = 2$, we see that v_5 is a big vertex by Lemma 3.1. In each of the following two cases, we always let $G' = G - \{v\}$. Then G' admits an (F_4, F_4)-partition $\{F_A, F_B\}$ due to the minimality of G. Obviously, if one fails to extend $\{F_A, F_B\}$ to G, we may suppose that v_4 is F_A-saturated and v_1 is F_B-saturated in G'. We will prove (1)–(2) by contradiction.

(1) Suppose v_4 has at most two big neighbours. Since $d(v_2) = 2$, we know that $d(v_3) \geqslant 6$ by Lemma 3.1. Then $d(u_i) \leqslant 5$ for all $i = 1, 2, 3$. If there exists some $u_i \in F_B$, say u_1, then we deduce that v_3, v_5, u_2, u_3 are all in F_A. Thus, it suffices to change v_4 to F_B, u_1 to F_A if it is F_B-saturated in G', and finally add v to F_A. Otherwise assume $v_3 \in F_B$. If $v_2 \in F_B$, then one can change v_2 to F_A and then add v to F_B easily. Or else, assume that $v_2 \in F_A$. Then v_6, w_1, w_2, w_3 all belong to F_B. At this moment, one can first change v_1 to F_A, and then add v to F_B successfully.

(2) W.l.o.g., assume v_1 has at most one big neighbour. Since $d(v_3) = 2$, we affirm that v_2 is a big vertex by Lemma 3.1. It follows that w_1, w_2 and w_3 are all 5^--vertices. There are two cases below.

- ▷ Assume that $v_6 \in F_A$. Then $v_2 \in F_B$ and each $w_i \in F_B$. So we can change v_1 to F_A, and then add v to F_B. We get an (F_4, F_4)-partition of G, which is a contradiction.
- ▷ Assume that $v_6 \in F_B$. If $v_3 \in F_B$, then we can change v_4 to F_B, and then add v to F_A to get an (F_4, F_4)-partition of G. Thus, we assume $v_3 \in F_A$. If $v_5 \in F_B$, then we can first change v_6 to F_A and then add v to F_B. Otherwise, assume $v_5 \in F_A$. Similarly, we deduce that $v_2 \in F_B$. So exactly one of w_1, w_2, w_3 belongs to F_A, say w_1. At this moment, we can change v_1 to F_A, w_1 to F_B if it is F_A-saturated in G', and afterwards add v to F_B. □

3.3. Discharging procedure. In this section, we will use the discharging technique to get a contradiction. Define the *initial charge* ω on $V(G) \cup F(G)$ as follows: $\omega(x) = d(x) - 4$ for every $x \in V(G) \cup F(G)$. According to the relation $\sum\limits_{v \in V(G)} d(v) = \sum\limits_{f \in F(G)} d(f) = 2|E(G)|$ and Euler's formula, the initial total charge of the vertices and faces satisfies the following equation:

$$\sum_{x \in V(G) \cup F(G)} \omega(x) = \sum_{x \in V(G) \cup F(G)} (d(x) - 4) = -4|V(G)| + 4|E(G)| - 4|F(G)| = -8.$$

Since any discharging procedure does not change the total charge of G, after applying appropriate discharging rules to change the initial charge ω to the final charge ω^*

on $V(G) \cup F(G)$ such that $\sum\limits_{x \in V(G) \cup F(G)} \omega^*(x) \geqslant 0$, we can have the contradiction below:

$$0 \leqslant \sum_{x \in V(G) \cup F(G)} \omega^*(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = -8,$$

and this ends the proof of Theorem 1.2.

We call a 2-vertex v which satisfies that $\omega^*(v) \geqslant 0$ after carrying out rules (R1)–(R4) *rich*, and *poor* otherwise. For $x, y \in V(G) \cup F(G)$, let $\tau(x \to y)$ denote the charge transferring from x to y. Below are our needed discharging rules:

(R1) Suppose f is a 5-face which is incident to a k-vertex v_k, $k \in \{2, 3\}$. Then

- (R1.1) $\tau(f \to v_2) = \frac{1}{2}$ if f is bad;
- (R1.2) $\tau(f \to v_3) = 1/n_3(f)$ if f is good;
- (R1.3) $\tau(f \to v_2) = \frac{2}{3}$ and $\tau(f \to v_3) = 1/3n_3(f)$ if f is weak.

(R2) Every 6^+-face f sends $\frac{2}{3}$ to each incident 2-vertex, $\frac{1}{3}$ to each incident 3-vertex.

(R3) Suppose $6 \leqslant d(v) \leqslant 7$ and v is adjacent to a k-vertex v_k, $k \in \{2, 3\}$.

- (R3.1) If v has degree 6, then
  - (R3.1a) $\tau(v \to v_{3^-}) = 2/n_{3^-}(v)$ if $n_2(v) + n_{3^l}(v) \geqslant 1$;
  - (R3.1b) $\tau(v \to v_3) = \frac{1}{4}$ otherwise.
- (R3.2) Otherwise, suppose v has degree 7, then
  - (R3.2a) $\tau(v \to v_{3^-}) = 3/n_{3^-}(v)$ if $n_{3^-}(v) \leqslant 6$;
  - (R3.2b) $\tau(v \to v_2) = \tau(v \to v_{3^l}) = \frac{1}{2}$ and $\tau(v \to v_{3^h}) = \frac{1}{4}$ if $n_{3^-}(v) = 7$.

(R4) Every 8^+-vertex v sends $(d(v) - 4)/n_{3^-}(v)$ to each adjacent 3^--vertex.

(R5) Every rich 2-vertex v assigns the extra charge evenly to each poor 2-vertex u when u is incident to two bad 5-faces and u, v are lying on the same bad 5-face.

At first, we observe from (R3.1a) that the following fact is true by applying Lemma 3.4.

Fact 3.9. Suppose v has degree 6 and $n_2(v) + n_{3^l}(v) \geqslant 1$. Then $\tau(v \to v_{3^-}) \geqslant \frac{2}{5}$.

Claim 3.10. *For every face $f \in F(G)$ we have that $\omega^*(f) \geqslant 0$.*

Proof. Obviously, since $G \in \mathcal{G}_5$, we have $d(f) \geqslant 5$. First suppose f is a 6^+-face, then we can affirm that $n_3(f) \leqslant d(f) - 2n_2(f)$ by Lemma 3.1. Thus, $\omega^*(f) \geqslant d(f) - 4 - \frac{2}{3}n_2(f) - \frac{1}{3}n_3(f) \geqslant d(f) - 4 - \frac{2}{3}n_2(f) - \frac{1}{3}(d(f) - 2n_2(f)) = \frac{2}{3}d(f) - 4 \geqslant \frac{2}{3} \times 6 - 4 = 0$ by (R2). Now assume that f is a 5-face. If f is bad, then $n_3(f) = 0$ by Corollary 3.2. So by (R1.1), $\omega^*(f) \geqslant 1 - 2 \times \frac{1}{2} = 0$. Suppose now f is good, then $n_2(f) = 0$ by definition, and so by (R1.2), $\omega^*(f) \geqslant 1 - (n_3(f))^{-1} \times n_3(f) = 0$. Otherwise, we can suppose f is weak. That is, $n_2(f) = 1$. By (R1.3), $\omega^*(f) \geqslant 1 - \frac{2}{3} - (3n_3(f))^{-1} \times n_3(f) = 0$. $\square$

Claim 3.11. *For every vertex $v \in V(G)$ we have that $\omega^*(v) \geqslant 0$.*

P r o o f. Obviously, $d(v) \geqslant 2$. Since neither 4-vertex nor 5-vertex is involved in the discharging procedure, we obtain that $\omega^*(v) = \omega(v) = d(v) - 4 \geqslant 0$. It remains to discuss the following five cases.

Case 1: If $d(v) \geqslant 8$, then $\omega^*(v) \geqslant d(v) - 4 - (d(v) - 4/n_{3^-}(v)) \times n_{3^-}(v) = 0$ by (R4).

Case 2: Assume $d(v) = 7$. If $n_{3^-}(v) \leqslant 6$, then by (R3.2a), $\omega^*(v) \geqslant 3 - (3/n_{3^-}(v)) \times n_{3^-}(v) = 0$. Suppose $n_{3^-}(v) = 7$. Namely, $n_{4^+}(v) = 0$, we can derive $n_{3^h}(v) \geqslant 2$ by Lemma 3.5, and hence $\omega^*(v) \geqslant 3 - 2 \times \frac{1}{4} - 5 \times \frac{1}{2} = 0$ by (R3.2b).

Case 3: Assume $d(v) = 6$. By (R3.1a), if v is adjacent to a light 3-vertex or a 2-vertex, then $\omega^*(v) \geqslant 2 - (2/n_{3^-}(v)) \times n_{3^-}(v) = 0$. Otherwise, by (R3.1b) $\omega^*(v) \geqslant 2 - 6 \times \frac{1}{4} = \frac{1}{2}$.

Case 4: Assume $d(v) = 3$. Let $N(v) = \{v_1, v_2, v_3\}$. Denote by f_i the face that is incident to v such that vv_i and vv_{i+1} are its two boundary edges (indices modulo 3). Clearly, for all $i = 1, 2, 3$, $d(f_i) \geqslant 5$. By Corollary 3.2, none of f_1, f_2 and f_3 can be any bad 5-face.

Case 4.1: Assume that v is heavy. Namely, $n_b(v) \geqslant 2$. We may suppose v_1 and v_2 are both 6^+-vertices. For each v_i, $i = 1, 2$, if $d(v_i) = 6$, then $\tau(v_i \to v) \geqslant \min\{\frac{1}{4}, \frac{1}{3}\}$ by (R3.1). If $d(v_i) = 7$, then $\tau(v_i \to v) \geqslant \min\{\frac{1}{4}, \frac{1}{2}\}$ by (R3.2). If $d(v_i) \geqslant 8$, then $\tau(v_i \to v) \geqslant (d(v_i) - 4)/n_{3^-}(v_i) \geqslant \frac{1}{2}$ by (R4). Therefore, both v_1 and v_2 will send charge at least $\frac{1}{4}$ to v. Moreover, for some $i \in \{1, 2, 3\}$, if $d(f_i) \geqslant 6$, we have $\tau(f_i \to v) = \frac{1}{3}$ by (R2), and if $d(f_i) = 5$, $\tau(f_i \to v) \geqslant \min\{\frac{1}{5}, \frac{1}{6}\}$ by (R1.2) and (R1.3). Hence, $\omega^*(v) \geqslant -1 + 2 \times \frac{1}{4} + 3 \times \frac{1}{6} = 0$.

Case 4.2: Suppose now v is light. We may suppose $d(v_1) \geqslant 6$ and $d(v_i) \leqslant 5$ for both $i = 2, 3$ by Lemma 3.3. By a similar discussion as Case 4.1, we know that if there exists some f_i being a 6^+-face, then v gets charge at least in total $\frac{1}{3} + \frac{1}{6} \times 2 = \frac{2}{3}$ by (R1.2), (R1.3) and (R2). At this moment, by Fact 3.9 and (R3.2), it is evident that $\tau(v_1 \to v) \geqslant \frac{2}{5}$. Thus, $\omega^*(v) \geqslant -1 + \frac{2}{5} + \frac{2}{3} = \frac{1}{15}$. So in what follows, suppose $d(f_1) = d(f_2) = d(f_3) = 5$. Notice that f_2 must be good by Lemma 3.1, implying that $\tau(f_2 \to v) \geqslant \frac{1}{5}$ by (R1.2). Moreover, by (R1.2) and (R1.3), $\tau(f_i \to v) \geqslant \frac{1}{6}$ for each $i = 1, 3$.

Case 4.2.1: Assume $d(v_1) \geqslant 7$. We have $\tau(v_1 \to v) \geqslant \frac{1}{2}$ by (R3.2) and (R4). Hence, $\omega^*(v) \geqslant -1 + \frac{1}{2} + 2 \times \frac{1}{6} + \frac{1}{5} = \frac{1}{30}$.

Case 4.2.2: Assume $d(v_1) = 6$. Recall that v is not incident to any bad 5-face by Corollary 3.2. By Fact 3.9, $\tau(v_1 \to v) \geqslant \frac{2}{5}$. If f_1 is good, then $\tau(f_1 \to v) \geqslant \frac{1}{4}$ by (R1.2), implying that $\omega^*(v) \geqslant -1 + \frac{2}{5} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{1}{60}$. Next, by symmetry, assume that f_1 and f_3 are both weak. If $d(v_2) = d(v_3) = 3$, then by Lemma 3.7, we are sure that $n_3(f_2) = 3$ since both v_2 and v_3 are heavy 3-vertices. So by (R1.2),

$\tau(f_2 \to v) = \frac{1}{3}$, which leads to $\omega^*(v) \geqslant -1 + \frac{2}{5} + 2 \times \frac{1}{6} + \frac{1}{3} = \frac{1}{15}$. Otherwise, we may suppose v_2 is either a 4-vertex of a 5-vertex. We can easily calculate that $\tau(f_1 \to v) = \frac{1}{3}$ and $\tau(f_2 \to v) \geqslant \frac{1}{4}$ by (R1.3) and (R1.2), respectively. Therefore, $\omega^*(v) \geqslant -1 + \frac{2}{5} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{3}{20}$.

Case 5: Let $N(v) = \{v_1, v_2\}$. Then each v_i is a 6^+-vertex by Lemma 3.1. By Fact 3.9 and (R3.2), $\tau(v_i \to v) \geqslant \frac{2}{5}$ for each $i = 1, 2$. In the following discussion, we will use this fact without any mention.

Case 5.1: First suppose v is not in any bad 5-face, then by (R1.3) and (R2), $\omega^*(v) \geqslant -2 + 2 \times \frac{2}{5} + 2 \times \frac{2}{3} = \frac{2}{15}$.

Case 5.2: Next assume v is in exactly one bad 5-face, denoted by $f = [v_1 v v_2 v_3 v_4]$, so that $d(v_4) = 2$. Again, $d(v_3) \geqslant 6$ by Lemma 3.1. If $d(v_1) \geqslant 7$ or $d(v_2) \geqslant 7$, say v_1, then by (R3.2) and (R4), it will send at least $\frac{1}{2}$ to v, and thus $\omega^*(v) \geqslant -2 + \frac{1}{2} + \frac{2}{5} + \frac{2}{3} + \frac{1}{2} = \frac{1}{15}$ by (R1.1), (R1.3) and (R2). Otherwise, suppose $d(v_1) = d(v_2) = 6$. By Lemma 3.6 (1), we are sure that both $n_b(v_2) \geqslant 2$, and thus by (R3.1a), $\tau(v_2 \to v) \geqslant \frac{1}{2}$. Hence, $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{1}{2} + \frac{1}{2} + \frac{2}{3} = \frac{1}{15}$ by (R1.1), (R1.3) and (R2).

Case 5.3: Now consider the case that v is incident to two bad 5-faces. Denote $f = [v_1 v v_2 u_2 u_1]$ and $g = [v_1 v v_2 u_3 u_4]$. Since no 4^--cycles can exist in G, we can make sure that $\{u_1, u_2\} \cap \{u_3, u_4\} = \emptyset$. Observe that each of f and g sends charge exactly $\frac{1}{2}$ to v by (R1.1). This fact will be applied directly without special explanation later. Next, we need to discuss two subcases, depending on the location of all these incident 2-vertices.

Case 5.3.1: $d(u_1) = d(u_4) = 2$. By Lemma 3.1, both u_2 and u_3 are 6^+-vertices, meaning that $n_b(v_2) \geqslant 2$, and so $\tau(v_2 \to v) \geqslant \frac{1}{2}$ by (R3.1a), (R3.2a) and (R4). Clearly, if $d(v_1) \geqslant 7$, then by (R3.2) and (R4), $\omega^*(v) \geqslant -2 + 4 \times \frac{1}{2} = 0$. In what follows, assume $d(v_1) = 6$. First let $d(v_2) \geqslant 7$, then $\tau(v_2 \to v) \geqslant \frac{3}{7-2} = \frac{3}{5}$ if $d(v_2) = 7$ by (R3.2a) and $\tau(v_2 \to v) \geqslant (d(v_2) - 4)/(d(v_2) - 2) \geqslant \frac{2}{3}$ if $d(v_2) \geqslant 8$ by (R4), implying that $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{3}{5} + 2 \times \frac{1}{2} = 0$ by applying Fact 3.9. Otherwise, assume that $d(v_2) = 6$. By Lemma 3.8 (1), $n_b(v_2) \geqslant 3$. Then $\tau(v_2 \to v) \geqslant \frac{2}{3}$ by (R3.1a). Therefore, $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{2}{3} + 2 \times \frac{1}{2} = \frac{1}{15}$.

Case 5.3.2: $d(u_2) = d(u_4) = 2$. Similarly, $d(u_i) \geqslant 6$ for each $i \in \{1, 3\}$ by Lemma 3.1. If v_1 and v_2 are both 7^+-vertices, then by (R3.2a) and (R4), each of them will send at least $\frac{1}{2}$ to v. So in this case, one may obtain that $\omega^*(v) \geqslant -2 + 4 \times \frac{1}{2} = 0$. Next we may assume $d(v_1) = 6$. If $d(v_2) = 6$, then by Lemma 3.8 (2), $n_b(v_i) \geqslant 2$ for both $i = 1, 2$, and thus by (R3.1a), $\omega^*(v) \geqslant -2 + 4 \times \frac{1}{2} = 0$. Otherwise suppose $d(v_2) \geqslant 7$. Apparently, if $n_b(v_1) \geqslant 2$, then similarly by (R3.1a), (R3.2) and (R4), each of v_1 and v_2 will send charge at least $\frac{1}{2}$ to v, which convinces us that $\omega^*(v) \geqslant -2 + 4 \times \frac{1}{2} = 0$. Likewise, if $n_b(v_2) \geqslant 2$, then by (R3.2a) and (R4), v_2 will send charge at least $\frac{3}{5}$ to v, thus $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{3}{5} + 2 \times \frac{1}{2} = 0$. So now, we only need to argue the case that $n_b(v_1) = n_b(v_2) = 1$. This enables us to believe that $d(u_1) \geqslant 7$ by Lemma 3.6 (2). By (R4), $\tau(v_2 \to v) \geqslant \frac{5}{8}$ if $d(v_2) \geqslant 9$, and thus $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{5}{8} + 2 \times \frac{1}{2} = \frac{1}{40}$. Next suppose $7 \leqslant d(v_2) \leqslant 8$. Denote by f' the face which is adjacent to f with u_1u_2 and u_2v_2 as boundary edges.

▷ Assume that f' is not a bad 5-face. By (R1.3) and (R2), $\tau(f' \to u_2) = \frac{2}{3}$, and by (R3.2a) and (R4), $\tau(u_1 \to u_2) \geqslant \frac{1}{2}$, which leads to $\omega^*(u_2) \geqslant -2 + 2 \times \frac{1}{2} + \frac{1}{2} + \frac{2}{3} = \frac{1}{6}$. It means that u_2 is a rich 2-vertex. As v is the only poor 2-vertex that u_2 needs to send charge to by (R5) since f' is not a bad 5-face, u_2 sends all extra charge $\frac{1}{6}$ to v. Therefore, $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{1}{2} + 2 \times \frac{1}{2} + \frac{1}{6} = \frac{1}{15}$.

▷ Suppose now f' is a bad 5-face. Denote by $f' = [u_1u_2v_2w_1w_1']$. Since $n_b(v_2) = 1$, we have that $d(w_1) = 2$, which implies that $d(w_1') \geqslant 6$ by Lemma 3.1. Let f'' denote the face adjacent to f' with $w_1'w_1$ and w_1v_2 as boundary edges. Similarly, if f'' is not bad, then $\tau(f'' \to w_1) = \frac{2}{3}$ by (R1.3) and (R2), and thus $\omega^*(w_1) \geqslant -2 + \frac{2}{5} + \frac{1}{2} + \frac{2}{3} + \frac{1}{2} = \frac{1}{15}$ by Fact 3.9. Otherwise, assume that f'' is a bad 5-face. Let $f'' = [v_2w_1w_1'w_2'w_2]$. Again, we have that $d(w_2) = 2$ and $d(w_2') \geqslant 6$. At this moment, we notice that $n_b(w_1') \geqslant 2$, meaning that $\tau(w_1' \to w_1) \geqslant \frac{1}{2}$ by (R3.1a), (R3.2a) and (R4). Hence, $\omega^*(w_1) \geqslant -2 + 4 \times \frac{1}{2} = 0$. In each of above two cases, one can ensure that w_1 is not poor. This fact guarantees us that u_2 (if it is rich) does not need to send any extra charge to w_1. Now let us check the value of $\omega^*(u_2)$. Since $d(u_1) \geqslant 7$ and $n_b(u_1) \geqslant 2$, we see that $\tau(u_1 \to u_2) \geqslant \frac{3}{5}$ by (R3.2a) and (R4), which implies that $\omega^*(u_2) \geqslant -2 + \frac{3}{5} + \frac{1}{2} + 2 \times \frac{1}{2} = \frac{1}{10}$. By (R5), all this extra charge $\frac{1}{10}$ will be sent to v. Consequently, $\omega^*(v) \geqslant -2 + \frac{2}{5} + \frac{1}{2} + 2 \times \frac{1}{2} + \frac{1}{10} = 0$. □

Up to now, we have proved the total charge of the vertices and faces is nonnegative, and thus this contradiction ends the proof of Theorem 1.2. □

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