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ON THE IRREDUCIBLE FACTORS OF A POLYNOMIAL OVER A VALUED FIELD

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Abstract. We explicitly provide numbers d, e such that each irreducible factor of a polynomial $f(x)$ with integer coefficients has a degree greater than or equal to d and $f(x)$ can have at most e irreducible factors over the field of rational numbers. Moreover, we prove our result in a more general setup for polynomials with coefficients from the valuation ring of an arbitrary valued field.

Keywords: irreducibility; Eisenstein criterion; polynomial

MSC 2020: 12E05, 11R09, 12J10

1. Introduction

Let (K,v) be a valued field. For any given polynomial $F(x)\in K[x]$, it is natural to ask: "Is $f(x)$ irreducible over K? If it is not irreducible, then what can we say about the irreducible factors of this polynomial over K?" This problem has been of interest to several mathematicians, cf. [2], [3], [5], [7], [8], [9], [12], [13], [14], [15]. In 1850, Eisenstein in [3] gave the most popular irreducibility criterion, viz. Eisenstein irreducibility criterion. In 1906, Dumas in [2] generalized this criterion which states that if $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_0$ is a polynomial with integer coefficients and if there exists a prime p whose exact power r_i dividing a_i satisfies $r_n=0$, $nr_i\geqslant(n-i)r_0$ for $0\leqslant i\leqslant n-1$ and r_0, n are coprime, then $f(x)$ is irreducible over $\mathbb{Q}$. In 2020, we extended Dumas result and proved the following result which provides us a lower bound of the degree of each irreducible factor of a polynomial over K, cf. [7].

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Theorem 1.1. *Let v be a Krull valuation of arbitrary rank of a field K with value group G_v. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, $a_0 \neq 0$ be a polynomial having coefficients in K with $v(a_n) = 0$ and $v(a_0) > 0$. Let d be the smallest positive element such that $dv(a_0)/n \in G_v$. If $nv(a_i) \geqslant (n-i)v(a_0)$ for $0 \leqslant i \leqslant n-1$, then each irreducible factor of $f(x)$ has degree at least d over K.*

In another direction, in 2020, we also proved a result which provides us a lower bound on the degree of an irreducible factor of a polynomial, see [8]. Later, this result was generalized using a weaker hypothesis, see [9]. In this article, we extend Theorem 1.1 and prove a result which provides us a lower bound on the degrees of each irreducible factors of a polynomial over K. Further, we give an upper bound on the number of irreducible factors of a polynomial over K.

The outline of the paper is as follows. In Section 2, we state our results and we illustrate them with some examples. In Section 3, we write some preliminary results. In Section 4, we prove our main result.

2. Statement of the results

Before stating our main result, we first introduce some notations and definitions.

Let v be a Krull valuation of a field K with value group G_v and valuation ring R_v having maximal ideal M_v. For a polynomial $g(x) \in R_v[x]$, $\overline{g}(x)$ will stand for the polynomial over R_v/M_v obtained by replacing each coefficient of $g(x)$ by its v-residue. We shall denote by v^x the Gaussian valuation of the field $K(x)$ of rational functions in an indeterminate x which extends the valuation v of K defined on $K[x]$ such that

$$v^x\left(\sum_i a_i x^i\right) = \min_i\{v(a_i)\}, \quad a_i \in K. \tag{2.1}$$

Its residue field is a transcendental extension of the residue field of v, cf. [4], Corollary 2.2.2. We now define the notion of Newton polygon, see Section 6.4 of [6], Defintion 1.D of [10] for details.

Definition 2.1. Let (K, v) be a valued field with value group G_v and valuation ring R_v having maximal ideal M_v. Let $\varphi(x) \in R_v[x]$ be a monic polynomial with $\bar{\varphi}(x)$ irreducible over R_v/M_v and v^x be the Gaussian prolongation defined by (2.1). Let $f(x) \in R_v[x]$ be a polynomial not divisible by $\varphi(x)$ with φ-expansion[1] $\sum\limits_{i=0}^{n} a_i(x)\varphi(x)^i$,

[1] On dividing by successive powers of $\varphi(x)$, every polynomial $f(x) \in K[x]$ can be uniquely written as a finite sum $\sum\limits_{i\geqslant 0} f_i(x)\varphi(x)^i$ with $\deg(f_i(x)) < \deg(\varphi(x))$, called the φ-expansion of $f(x)$.

$a_n(x) \neq 0$. Let P_i stand for the pair $(i, v^x(a_{n-i}(x)))$ when $a_{n-i}(x) \neq 0$, $0 \leqslant i \leqslant n$. For distinct pairs P_i, P_j, let μ_{ij} denote the element of the divisible closure of G_v defined by

$$\mu_{ij} = \frac{v^x(a_{n-j}(x)) - v^x(a_{n-i}(x))}{j-i};$$

in case v is a real valuation, μ_{ij} is the slope of the line segment joining P_i and P_j. Let i_1 denote the largest index $0 < i_1 \leqslant n$ such that

$$\mu_{0i_1} = \min_j \{\mu_{0j} \colon\ 0 < j \leqslant n,\ a_{n-j}(x) \neq 0\}.$$

If $i_1 < n$, let i_2 be the largest index such that $i_1 < i_2 \leqslant n$ and

$$\mu_{i_1 i_2} = \min_j \{\mu_{i_1 j} \colon\ i_1 < j \leqslant n,\ a_{n-j}(x) \neq 0\}.$$

Proceeding in this way if $i_l = n$, then the φ-Newton polygon of $f(x)$ is said to have l edges whose slopes are defined to be $\mu_{0i_1}, \mu_{i_1 i_2}, \ldots, \mu_{i_{l-1} i_l}$ which are in strictly increasing order. The pairs $P_0, P_{i_1}, P_{i_2}, \ldots, P_{i_l}$ are called the *successive vertices* of the φ-Newton polygon of $f(x)$. The interval $[i_{j-1}, i_j]$ will be referred to as the interval of the horizontal projection of the jth side and the numbers $i_1 - 0, i_2 - i_1, \ldots, i_l - i_{l-1}$ will be referred to as the lengths of the horizontal projections of these sides.

We now state our main result.

Theorem 2.2. *Let v be a Krull valuation of a field K with value group G_v and valuation ring R_v having maximal ideal M_v. Let $\varphi(x) \in R_v[x]$ be a monic polynomial of degree m which is irreducible* $\pmod{M_v}$. *Let $f(x) \in R_v[x]$ be a polynomial not divisible by $\varphi(x)$ having φ-expansion $f(x) = a_n \varphi(x)^n + \sum\limits_{i=0}^{n-1} a_i(x)\varphi(x)^i$, where $a_n \in R_v \setminus M_v$ and $v^x(a_i(x)) > 0$ for $0 \leqslant i \leqslant n-1$. Assume that the φ-Newton polygon of $f(x)$ has l edges with slopes λ_j, $1 \leqslant j \leqslant l$. Let k_i's be integers such that the numbers $n - k_1, k_1 - k_2, \ldots, k_{l-2} - k_{l-1}, k_{l-1}$ denote the lengths of the horizontal projections of the successive edges of the φ-Newton polygon of $f(x)$. If $\mathfrak{d}_j$ is the smallest positive number such that $\mathfrak{d}_j \lambda_j \in G_v$ for $1 \leqslant j \leqslant l$, then the number of irreducible factors of $f(x)$ over K will be less than or equal to $(n-k_1)/\mathfrak{d}_1 + (k_1 - k_2)/\mathfrak{d}_2 + \ldots + (k_{l-2} - k_{l-1})/\mathfrak{d}_{l-1} + k_{l-1}/\mathfrak{d}_l$ and each irreducible factor of $f(x)$ has degree at least $\min\limits_{1 \leqslant j \leqslant l} \{\mathfrak{d}_j m\}$ over K.*

Remark 2.3. It may be pointed out that in the above theorem if we take $\mathfrak{d}_1 = \ldots = \mathfrak{d}_l$, then it provides its best lower bound on the degree of each irreducible factor of $f(x)$. Also, in this situation, the number of irreducible factors of $f(x)$ will be less than or equal to $n/\mathfrak{d}_1$.

Remark 2.4. Theorem 1.1 is the special case of Theorem 2.2 when $\varphi(x) = x$ and $l = 1$. We wish to point out that the above theorem can also be obtained from Theorem 1.2 of [10]. However, the proof given here is new and simpler.

When $K = \mathbb{Q}$, $G_v = \mathbb{Z}$ and $\varphi(x) = x$, then Theorem 2.2 quickly yields the following simple result.

Corollary 2.5. *Let p be a prime number and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, $a_0 \neq 0$ be a polynomial of degree n with rational coefficients such that $p \nmid a_n$ and p divides a_i for $0 \leqslant i \leqslant n-1$. If $k_0, k_1, \ldots, k_{l-1}, k_l$ are integers such that the successive vertices of the Newton polygon of $f(x)$ with respect to the p-adic valuation v_p are given by the set $\{(0,0), (n-k_1, v_p(a_{k_1})), \ldots, (n-k_{l-1}, v_p(a_{k_{l-1}})), (n, v_p(a_0))\}$ with $k_0 = n$, $k_l = 0$, then the number of irreducible factors of $f(x)$ will be less than or equal to $\sum\limits_{j=1}^{l} \gcd(k_{j-1}-k_j, v_p(a_{k_j}) - v_p(a_{k_{j-1}}))$ and each irreducible factor of $f(x)$ has degree at least $\min\limits_{1\leqslant j\leqslant l}\{(k_{j-1}-k_j)/\gcd(k_{j-1}-k_j, v_p(a_{k_j}) - v_p(a_{k_{j-1}}))\}$ over $\mathbb{Q}$.*

We now illustrate our main result with an example. In this example, Theorem 1.1 is not applicable.

Example 2.6. Let k be a positive integer and p be a prime number of the form $4k+3$. Let r, s with $s > 2r$ be positive integers such that $p \nmid r(r-s)$. Consider a polynomial $f(x) = (x^2+1)^{2p} + (ax+b)(x^2+1)^p + cx + d$ with $a, b, c, d \in \mathbb{Z}$. Let p^r divide $(ax+b)$ with $p^{r+1} \nmid (ax+b)$ and p^s divide $(cx+d)$ with $p^{s+1} \nmid (cx+d)$. Then in view of Theorem 2.2, $f(x)$ can have at most two irreducible factors over $\mathbb{Q}$ and each irreducible factor of $f(x)$ has degree at least $2p$ over $\mathbb{Q}$. Therefore, either $f(x)$ is irreducible over $\mathbb{Q}$ or $f(x)$ has two irreducible factors of degree $2p$ over $\mathbb{Q}$.

3. Preliminary results

Let v be a Krull valuation of arbitrary rank of a field K with value group G_v and valuation ring R_v having maximal ideal M_v. We fix a prolongation $\tilde{v}$ of v to an algebraic closure $\widetilde{K}$ of K. For an element β belonging to the valuation ring $R_{\tilde{v}}$ of $\tilde{v}$, $\overline{\beta}$ will denote its $\tilde{v}$-residue, i.e., the image of β under the canonical homomorphism from $R_{\tilde{v}}$ onto its residue field. For a polynomial $h(x)$ belonging to $R_{\tilde{v}}$, $\bar{h}(x)$ will stand for the polynomial over the residue field of $\tilde{v}$ obtained on replacing each coefficient of $h(x)$ by its $\tilde{v}$-residue.

We now define a (K, v)-minimal pair.

Definition 3.1. Let (K, v) be a henselian valued field of arbitrary rank and $\tilde{v}$ be the unique prolongation of v to the algebraic closure $\widetilde{K}$ of K with value group $G_{\tilde{v}}$.

A pair $(\alpha, \delta) \in \widetilde{K} \times G_{\tilde{v}}$ is called a *minimal pair* (more precisely a (K, v)-minimal pair) if whenever β belongs to $\widetilde{K}$ with $[K(\beta) : K] < [K(\alpha) : K]$, then $\tilde{v}(\alpha - \beta) < \delta$.

We now give an example of a (K, v)-minimal pair, which we shall use in the sequel.

Example 3.2. Let $\varphi(x)$ belonging to $R_v[x]$ be a monic polynomial of degree $m \geqslant 1$ with $\bar{\varphi}(x)$ irreducible over the residue field of v and α be a root of $\varphi(x)$. Then for each positive $\delta \in G_{\tilde{v}}$, it is easy to see that (α, δ) is a (K, v)-minimal pair. Because whenever $\beta \in \widetilde{K}$ with degree $[K(\beta) : K] < m$, then $\tilde{v}(\alpha - \beta) \leqslant 0$, for otherwise $\overline{\alpha} = \overline{\beta}$, which in view of the fundamental inequality (see [4], Theorem 3.3.4) would imply that $[K(\beta) : K] \geqslant [\overline{K}(\overline{\beta}) : \overline{K}] = [\overline{K}(\overline{\alpha}) : \overline{K}] = m$, leading to a contradiction.

Let (K, v) be a henselian valued field and (α, δ) be a (K, v)-minimal pair. The valuation $\widetilde{w}_{\alpha,\delta}$ of a simple transcendental extension $\widetilde{K}(x)$ of $\widetilde{K}$ defined on $\widetilde{K}[x]$ by

$$\widetilde{w}_{\alpha,\delta}\left(\sum_i c_i(x-\alpha)^i\right) = \min_i\{\tilde{v}(c_i) + i\delta\}, \quad c_i \in \widetilde{K}, \tag{3.1}$$

will be referred to as the valuation with respect to the minimal pair (α, δ). The restriction of $\widetilde{w}_{\alpha,\delta}$ to $K(x)$ will be denoted by $w_{\alpha,\delta}$. It is proved in Theorem 2.1 of [1], that a prolongation w of v to $K(x)$ is residually transcendental if and only if $w = w_{\alpha,\delta}$ for a (K, v)-minimal pair (α, δ).

With (α, δ) as above, if $\varphi(x)$ is the minimal polynomial of α over K, then it is well-known (see [1], Theorem 2.1) that for any polynomial $f(x) \in K[x]$ with φ-expansion $\sum_i a_i(x)\varphi(x)^i$, one has

$$w_{\alpha,\delta}(f(x)) = \min_i\{\tilde{v}(a_i(\alpha)) + i w_{\alpha,\delta}(\varphi(x))\}. \tag{3.2}$$

It is easy to show that if (α, δ) is a minimal pair of the type described in Example 3.2 with $\varphi(x)$ as the minimal polynomial of α over K, then for any polynomial $h(x) = \sum_{i=0}^{m-1} a_i x^i \in K[x]$ having degree less than $m = \deg \varphi(x)$, one has

$$\tilde{v}(h(\alpha)) = v^x(h(x)). \tag{3.3}$$

For the verification of (3.3), it is clear that we need to check (3.3) for $m > 1$. Keeping in view that $\bar{\varphi}(x)$ is irreducible over R_v/M_v of degree $m > 1$, it follows that $\tilde{v}(\alpha) = 0$. Suppose to the contrary that (3.3) does not hold. In this case, the triangle inequality would imply that $\tilde{v}(h(\alpha)) > \min_i\{\tilde{v}(a_i\alpha^i)\} = v(a_j)$ (say), which gives $\sum_{i=0}^{m-1} (\overline{a_i/a_j})(\overline{\alpha})^i = \bar{0}$. This is impossible as $\overline{\alpha}$ is a root of $\bar{\varphi}(x)$. Hence, (3.3) holds.

Notation 3.3. Let (α,δ), $w_{\alpha,\delta}$ be as above and $\varphi(x)$ be the minimal polynomial of α having degree m over K. Let $f(x)$ belonging to $K[x]$ be a nonzero polynomial with φ-expansion $\sum_i a_i(x)\varphi(x)^i$. We shall denote by $I_{\alpha,\delta}(f)$, $S_{\alpha,\delta}(f)$, respectively, the minimum and the maximum integers belonging to the following set $\{i\colon\ w_{\alpha,\delta}(f(x)) = \tilde{v}(a_i(\alpha)) + iw_{\alpha,\delta}(\varphi(x))\}$.

With the above notation, the following already known result will be used in the sequel, cf. [11], Lemma 2.1.

Theorem 3.4. *For any nonzero polynomials $g(x)$, $h(x)$ in $K[x]$, one has*
(i) $I_{\alpha,\delta}(g(x)h(x)) = I_{\alpha,\delta}(g(x)) + I_{\alpha,\delta}(h(x))$,
(ii) $S_{\alpha,\delta}(g(x)h(x)) = S_{\alpha,\delta}(g(x)) + S_{\alpha,\delta}(h(x))$

4. Proof of Theorem 2.2

Denote the constant a_n by a constant polynomial $a_n(x)$. Let the set

$$\{(n-k_0, v^x(a_{k_0}(x))), (n-k_1, v^x(a_{k_1}(x))), \ldots, (n-k_l, v^x(a_{k_l}(x)))\}$$

with $k_0 = n$, $k_l = 0$ denote the successive vertices of the φ-Newton polygon of $f(x)$. Note that $k_j > k_{j+1}$ for $0 \leqslant j \leqslant l-1$. For any fixed j, $1 \leqslant j \leqslant l$, the slope λ_j of the jth edge of the Newton polygon of $f(x)$ with respect to v is given by

$$\lambda_j = \frac{v^x(a_{k_j}(x)) - v^x(a_{k_{j-1}}(x))}{k_{j-1} - k_j}. \tag{4.1}$$

Keeping in mind the hypothesis that $f(x) \equiv a_n\varphi(x)^n \pmod{M_v}$, it follows that $\lambda_j > 0$. Since the value group and the residue field remain the same on replacing (K,v) by its henselization, we may assume that (K,v) is henselian. Let $\tilde{v}$ denote the unique prolongation of v to the algebraic closure $\widetilde{K}$ of K. Let α be a root of $\varphi(x)$ in $\widetilde{K}$. Write $\varphi(x) = \sum\limits_{i=1}^{m} c_i(x-\alpha)^i$, $c_i \in \widetilde{K}$, $c_m = 1$. Corresponding to λ_j, define a positive element δ_j of the divisible closure $G_{\tilde{v}}$ of G_v by

$$\delta_j = \max_{1\leqslant i\leqslant m}\Big\{\frac{\lambda_j - \tilde{v}(c_i)}{i}\Big\}.$$

Hence, the pair (α,δ_j) is a (K,v)-minimal pair. Let $\widetilde{w}_{\alpha,\delta_j}$ denote the valuation of $\widetilde{K}(x)$ defined by (3.1). Then by the choice of δ_j, we have

$$\widetilde{w}_{\alpha,\delta_j}(\varphi(x)) = \min_i\{\tilde{v}(c_i) + i\delta_j\} = \lambda_j.$$

Keeping in mind (3.2) and (3.3), we see that

$$w_{\alpha,\delta_j}(f(x)) = \min_{0\leqslant i\leqslant n}\{v^x(a_i(x)) + i\lambda_j\}. \tag{4.2}$$

Let $I_{\alpha,\delta_j}(f)$ and $S_{\alpha,\delta_j}(f)$ be as in Notation 3.3. We claim that

$$I_{\alpha,\delta_j}(f) = k_j, \quad S_{\alpha,\delta_j}(f) = k_{j-1}. \tag{4.3}$$

Recall that λ_j is the positive slope of the φ-Newton polygon of $f(x)$ connecting the vertices $(n-k_{j-1}, v^x(a_{k_{j-1}}(x)))$, $(n-k_j, v^x(a_{k_j}(x)))$. By virtue of Definition 2.1, we see that

$$\min_{0\leqslant i<k_{j-1}}\left\{\frac{v^x(a_i(x))-v^x(a_{k_{j-1}}(x))}{k_{j-1}-i}\right\} \geqslant \frac{v^x(a_{k_j}(x))-v^x(a_{k_{j-1}}(x))}{k_{j-1}-k_j} = \lambda_j, \tag{4.4}$$

$$\min_{k_{j-1}\leqslant i\leqslant n}\left\{\frac{v^x(a_{k_j}(x))-v^x(a_i(x))}{i-k_j}\right\} \leqslant \frac{v^x(a_{k_j}(x))-v^x(a_{k_{j-1}}(x))}{k_{j-1}-k_j} = \lambda_j. \tag{4.5}$$

Note that the smallest index i for which equality in (4.4) holds is $i=k_j$. On the other hand, $i=k_{j-1}$ is the largest index such that equality holds in (4.5). Therefore, keeping in mind (4.2), it follows that

$$w_{\alpha,\delta_j}(f(x)) = \min_{0\leqslant i\leqslant n}\{v^x(a_i(x))+i\lambda_j\} = v^x(a_{k_j}(x))+k_j\lambda_j = v^x(a_{k_{j-1}}(x))+k_{j-1}\lambda_j \tag{4.6}$$

and $I_{\alpha,\delta_j}(f)=k_j$, $S_{\alpha,\delta_j}(f)=k_{j-1}$.

Let $f(x) = f_1(x)f_2(x)\dots f_t(x)$ be the factorization of $f(x)$ into irreducible factors over K. Since $f(x)\in R_v[x]$, in view of Remark 4.1.2 in [4], there are $f_1(x), f_2(x), \dots, f_t(x)$ belonging to $R_v[x]$, irreducible in $K[x]$ such that $f(x) = f_1(x)f_2(x)\dots f_t(x)$. Since $\varphi(x)$ is irreducible $\pmod{M_v}$ and $f(x)\equiv a_n\varphi(x)^n \pmod{M_v}$ with $a_n\in R_v\setminus M_v$, we can write the φ-expansion of $f_s(x)$ as $f_s(x) = b_{sd_s}\varphi(x)^{d_s}+\sum\limits_{u=0}^{d_s-1} b_{su}(x)\varphi(x)^u$ for $1\leqslant s\leqslant t$ with $d_s\geqslant 1$ and $b_{sd_s}\in R_v\setminus M_v$, $v^x(b_{su}(x))>0$ for $0\leqslant u\leqslant d_s-1$. Therefore, we see that

$$f(x) = f_1(x)f_2(x)\dots f_t(x) \equiv b_{1d_1}b_{2d_2}\dots b_{td_t}\varphi(x)^{d_1+d_2+\ldots+d_t} \equiv a_n\varphi(x)^n \pmod{M_v}.$$

This forces the condition

$$n = d_1+d_2+\ldots+d_t. \tag{4.7}$$

Denote $I_{\alpha,\delta_j}(f_s)$ by $k_j^{(s)}$, $S_{\alpha,\delta_j}(f_s)$ by $k_{j-1}^{(s)}$ and b_{sd_s} by a constant polynomial $b_{sd_s}(x)$ for $1\leqslant s\leqslant t$. So, we have

$$w_{\alpha,\delta_j}(f_s(x)) = v^x(b_{sk_{j-1}^{(s)}}(x))+k_{j-1}^{(s)}\lambda_j = v^x(b_{sk_j^{(s)}}(x))+k_j^{(s)}\lambda_j. \tag{4.8}$$

Applying Theorem 3.4 together with (4.3), we see that

$$k_j = k_j^{(1)}+\ldots+k_j^{(t)}, \quad k_{j-1} = k_{j-1}^{(1)}+\ldots+k_{j-1}^{(t)}. \tag{4.9}$$

Since $k_l=0$, in view of the above equation, we have $k_l^{(s)}=0$ for each s, $1\leqslant s\leqslant t$. Note that $w_{\alpha,\delta_0}(f_s(x)) = \min\limits_{0\leqslant u\leqslant d_s}\{v^x(b_{su}(x))+u\lambda_0\} \leqslant d_s\lambda_0$ as $v^x(b_{sd_s}(x))=0$.

Therefore, keeping in mind that $n = d_1 + d_2 + \ldots + d_t$, $w_{\alpha,\delta_0}(f(x)) = n\lambda_0$ and $w_{\alpha,\delta_0}(f(x)) = w_{\alpha,\delta_0}(f_1(x)) + \ldots + w_{\alpha,\delta_0}(f_t(x))$, it follows that $w_{\alpha,\delta_0}(f_s(x)) = d_s\lambda_0$. Hence, we have $k_0^{(s)} = d_s$.

Now we claim that $k_{j-1}^{(s)} > k_j^{(s)}$ for some j, $1 \leqslant j \leqslant l$. Note that if $k_0^{(s)} = k_1^{(s)} = \ldots = k_l^{(s)}$, then in view of the fact that $k_l^{(s)} = 0$ and $k_0^{(s)} = d_s$, we see that $d_s = 0$. This contradiction proves the claim. For such j, in view of equation (4.8), we see that $(k_{j-1}^{(s)} - k_j^{(s)})\lambda_j \in G_v$. Since $\mathfrak{d}_j$ is the smallest positive element such that $\mathfrak{d}_j\lambda_j \in G_v$, it follows that $(k_{j-1}^{(s)} - k_j^{(s)}) \geqslant \mathfrak{d}_j$. Therefore, we have

$$\deg(f_s(x)) = d_s m \geqslant (d_s - k_j^{(s)})m \geqslant (k_{j-1}^{(s)} - k_j^{(s)})m \geqslant \mathfrak{d}_j m. \tag{4.10}$$

Since s is arbitrary, it follows that each irreducible factor of $f(x)$ has degree at least $\min\limits_{1\leqslant j\leqslant l}\{\mathfrak{d}_j m\}$ over K.

Now for proving that $f(x)$ can have atmost $(n-k_1)/\mathfrak{d}_1 + (k_1-k_2)/\mathfrak{d}_2 + \ldots + (k_{l-2}-k_{l-1})/\mathfrak{d}_{l-1} + k_{l-1}/\mathfrak{d}_l$ irreducible factors over K, we shall show that

$$t \leqslant \frac{n-k_1}{\mathfrak{d}_1} + \frac{k_1-k_2}{\mathfrak{d}_2} + \ldots + \frac{k_{l-2}-k_{l-1}}{\mathfrak{d}_{l-1}} + \frac{k_{l-1}}{\mathfrak{d}_l}. \tag{4.11}$$

Recall that $k_l = 0 = k_l^{(s)}$ for $1 \leqslant s \leqslant t$. Using equations (4.7) and (4.9), we see that

$$\begin{aligned}\frac{n-k_1}{\mathfrak{d}_1} &+ \frac{k_1-k_2}{\mathfrak{d}_2} + \ldots + \frac{k_{l-2}-k_{l-1}}{\mathfrak{d}_{l-1}} + \frac{k_{l-1}-k_l}{\mathfrak{d}_l} \\ &= \Big(\frac{d_1-k_1^{(1)}}{\mathfrak{d}_1} + \frac{k_1^{(1)}-k_2^{(1)}}{\mathfrak{d}_2} + \ldots + \frac{k_{l-1}^{(1)}-k_l^{(1)}}{\mathfrak{d}_l}\Big) \\ &\quad + \ldots + \Big(\frac{d_t-k_1^{(t)}}{\mathfrak{d}_1} + \frac{k_1^{(t)}-k_2^{(t)}}{\mathfrak{d}_2} + \ldots + \frac{k_{l-1}^{(t)}-k_l^{(t)}}{\mathfrak{d}_l}\Big).\end{aligned}$$

Therefore, keeping in mind the fact that for each s, $1 \leqslant s \leqslant t$, we have $(k_{j-1}^{(s)} - k_j^{(s)}) \geqslant \mathfrak{d}_j$ for some j, $1 \leqslant j \leqslant l$, it now follows that $(n-k_1)/\mathfrak{d}_1 + (k_1-k_2)/\mathfrak{d}_2 + \ldots + (k_{l-2}-k_{l-1})/\mathfrak{d}_{l-1} + k_{l-1}/\mathfrak{d}_l \geqslant t$.

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