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PARTITIONING A PLANAR GRAPH WITHOUT CHORDAL 5-CYCLES INTO TWO FORESTS

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Abstract. It was known that the vertex set of every planar graph can be partitioned into three forests. We prove that the vertex set of a planar graph without chordal 5-cycles can be partitioned into two forests. This extends a result obtained by Raspaud and Wang in 2008.

Keywords: planar graph; vertex-arboricity; forest; vertex partition

MSC 2020: 05C15

1. Introduction

All graphs considered in this paper are finite, simple and undirected. A *plane graph* is a particular drawing of a planar graph on the Euclidean plane. For a plane graph G, let $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$ denote its vertex set, edge set, face set, maximum degree and minimum degree, respectively. The *distance* between two vertices u and v, denoted $d_G(u,v)$, is the length of the shortest path connecting them in G. For $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of G induced by S. For a cycle C of length m ($\geqslant 4$) in a graph G, an edge $xy \in E(G)$ is called a *chord* of C if $x, y \in V(C)$ and $xy \notin E(C)$. A k-cycle with a chord is called a *chordal k-cycle*.

The *vertex-arboricity* $a(G)$ of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a forest. There is an equivalent definition to the vertex-arboricity in terms of the coloring version. An *acyclic-k-coloring* of a graph G is a mapping φ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$

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such that each color class induces an acyclic graph, i.e., a forest. The vertex-arboricity $a(G)$ of G is the smallest integer k such that G has an acyclic-k-coloring.

The vertex-arboricity of a graph was first introduced by Chartrand, Kronk and Wall in 1968, see [3]. They proved that $a(G) \leqslant \lceil \frac{\Delta(G)+1}{2} \rceil$ for any graph G and $a(G) \leqslant 3$ for any planar graph G. Moreover, Chartrand and Kronk in [2] provided a planar graph of the vertex-arboricity 3. It was known (see [5]) that determining the vertex-arboricity of a graph is NP-hard. Hakimi and Schmeichel in [6] showed that determining whether $a(G) \leqslant 2$ is NP-complete for maximal planar graphs. Stein in [10] proved that a plane graph G has $a(G) = 2$ if and only if G^*, the dual of G, is Hamiltonian. This result was further strengthened by Hakimi and Schmeichel (see [6]) by showing that a plane graph G has $a(G) = 2$ if and only if G^* contains a connected Eulerian spanning subgraph.

In 2008, Raspaud and Wang in [9] obtained the following results:

(a) Every planar graph G without k-cycles for a fixed $k \in \{3, 4, 5, 6\}$ has $a(G) \leqslant 2$;

(b) Every planar graph G without triangles at distance less than 2 has $a(G) \leqslant 2$;

(c) Every planar graph G with $|V(G)| \leqslant 20$ has $a(G) \leqslant 2$ and there exists a planar graph H such that $|V(H)| = 21$ and $a(H) = 3$.

Huang, Shiu and Wang in [7] showed that every planar graph G without 7-cycles has $a(G) \leqslant 2$. Huang and Wang in [8] showed that every planar graph G having no chordal 6-cycles has $a(G) \leqslant 2$. Borodin and Ivanova in [1] proved that every planar graph with no 4-cycles adjacent to 3-cycles has the list vertex arboricity at most 2. Chen, Raspaud and Wang in [4] proved that every planar graph G without intersecting triangles has $a(G) \leqslant 2$. More recently, Yang, Wang and Lih in [11] showed that every planar graph G without chordal 4-cycles has $a(G) \leqslant 2$.

In this paper, we will prove the following result, which can be regarded as an extension of results in [1], [8], [9], [11].

Theorem 1. *Every planar graph without chordal 5-cycles can be partitioned into two forests.*

Let $\mathcal{G}$ denote the family of plane graphs without chordal 5-cycles. Let $G \in \mathcal{G}$ and C be a cycle of G. We use $V_{\rm int}(C)$ and $V_{\rm ext}(C)$ to denote the set of vertices inside C and outside C, respectively. Set $V_{\rm Int}(C) = V_{\rm int}(C) \cup V(C)$, $V_{\rm Ext}(C) = V_{\rm ext}(C) \cup V(C)$, $G_{\rm Int}(C) = G[V_{\rm Int}(C)]$, and $G_{\rm Ext}(C) = G[V_{\rm Ext}(C)]$. We say that C is *separating* if both $V_{\rm int}(C)$ and $V_{\rm ext}(C)$ are not empty. For $k \geqslant 2$, a *k-chord* of C is a path $x_1 x_2 \ldots x_{k+1}$ of G such that $x_2, \ldots, x_k \in V_{\rm int}(C)$, and $x_1, x_{k+1} \in V(C)$ which are not consecutive on C. Moreover, C is a *small cycle* if $|V(C)| \leqslant 6$.

Let φ be an acyclic-2-coloring of G using the color set $A = \{1, 2\}$. A path P of G is called an *(i)-path* if all the vertices of P are colored by the same color i. Let C be a small outer face cycle of G. We say that G is *C-extendable* if any acyclic-2-coloring of $G[V(C)]$, satisfying the property that the two ends of any chord of C are colored differently, can be extended into an acyclic-2-coloring of G such that there is no (i)-path $P = x_1x_2\ldots x_k$ such that $x_1, x_k \in V(C)$ and $x_2, \ldots, x_{k-1} \in V(G) \setminus V(C)$ for each $i = 1, 2$.

Instead of showing Theorem 1, we actually prove a stronger result:

Theorem 2. *Let $G \in \mathcal{G}$. If D, the outer face cycle of G, is small, then G is D-extendable.*

Assuming Theorem 2, we can easily derive Theorem 1:

Proof of Theorem 1. Let $G \in \mathcal{G}$. We proceed by induction on $|V(G)|$. If $|V(G)| \leqslant 3$, the result holds trivially. Suppose that $G \in \mathcal{G}$ is connected and $|V(G)| \geqslant 4$. If there is a 3^--vertex u, then $G - u \in \mathcal{G}$ and $|V(G-u)| < |V(G)|$. By the induction hypothesis, $G - u$ admits an acyclic-2-coloring φ. We color u with a color that appeared at most once in the neighbors of u to extend φ to G. Assume that $\delta(G) \geqslant 4$. It is well-known that G contains a face, f^*, of degree at most five. Re-embed G in the plane such that f^* is its outer face, and the boundary of f^* forms a small cycle D. It is easy to give an acyclic-2-coloring φ of $G[V(D)]$ satisfying the property that the two ends of any chord of D are colored differently. By Theorem 2, φ is extended to G. $\square$

2. Notation and preliminary results

Let $G \in \mathcal{G}$. For $x \in V(G) \cup F(G)$, the *degree* of x is denoted by $d_G(x)$, or simply $d(x)$, when no confusion can arise. A vertex v is called a *k-vertex*, *k^+-vertex*, or *k^--vertex* if $d(v) = k$, $d(v) \geqslant k$, or $d(v) \leqslant k$. Similarly, we can define a *k-face*, *k^+-face*, and *k^--face*. Let $N_G(v)$ denote the set of the neighbours of v. For a face $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2\ldots u_k]$ if $u_1, u_2, \ldots, u_k$ are the vertices of $b(f)$ in cyclic order. Set $V(f) = V(b(f))$ and $E(f) = E(b(f))$. For a face f and an edge $e \in E(f)$, we use f_e to denote the face adjacent to f such that $e \in E(f) \cap E(f_e)$.

Let f_0 denote the outer face of G and write $D = b(f_0)$. Set $F^0 = F(G) \setminus \{f_0\}$ and $V^0 = V(G) \setminus V(D)$. A face in F^0 is called an *internal face* of G, and a vertex in V^0 is called an *internal vertex* of G. For $v \in V(G)$ and $i \geqslant 0$, let $m_i(v)$ denote the number of internal i-faces that are incident with v.

Suppose that H is a subgraph of G with an acyclic-2-coloring φ using $A = \{1, 2\}$. We say that a vertex $v \in V(G) \setminus V(H)$ admits a *legal coloring* if it can be colored with a color in A so that the modified coloring does not produce a monochromatic cycle.

For $S \subseteq V(G)$, an acyclic-2-coloring φ of $G-S$ is called a *partial acyclic-2-coloring* of G. For $v \in S$ and $i \in A$, let $\mu_i(v)$ denote the number of times that i appears in the colored neighbors of v in G. Let $\mu(v) = \min\{\mu_1(v), \mu_2(v)\}$. The vertex v is *free* if φ can be extended into a partial acyclic-2-coloring of G by coloring v with a color $i \in \{1,2\}$ such that $\mu_i(v) = \mu(v)$.

Lemma 1. *Suppose that $G \in \mathcal{G}$ and $S \subseteq V(G)$. Let φ be an acyclic-2-coloring of $G-S$ using $A = \{1,2\}$. Let $v \in S$. If one of the following conditions holds, then v is free.*

(1) $d(v) \leqslant 3$.

(2) *At most three of the neighbors of v are colored.*

(3) *Some color appears at least $d(v)-1$ times in the neighbors of v.*

(4) *v is a 4-vertex with neighbors v_0, v_1, v_2, v_3 in cyclic order such that v_0 and v_2 have the same color.*

Proof. If one of (1)–(3) holds, then there is a color $j \in \{1,2\}$ such that $\mu_j(v) \leqslant 1$. Color v with j so that φ can be extended into a partial acyclic-2-coloring of G. Hence, v is free. Assume that (4) holds, say $\varphi(v_0) = \varphi(v_2) = 1$ and $\varphi(v_1) = \varphi(v_3) = 2$. If no (1)-path connects v_0 and v_2, then we color v with 1. Otherwise, by the planarity of G, no (2)-path connects v_1 and v_3, so we can color v with 2. Hence v is free. $\square$

3. Proof of Theorem 2

Proof. Assume that G is a counterexample to Theorem 2 that is vertex-minimal. Then $G \in \mathcal{G}$ has a small outer face cycle D such that G is not D-extendable, but any graph $H \in \mathcal{G}$ with a small outer face cycle D' and $|V(H)| < |V(G)|$ is D'-extendable. Then G is connected and $V^0 \neq \emptyset$. Moreover, the following statements (P1) and (P2) are true:

(P1) G has no chordal 5-cycles. This implies the following facts:

- (P1.1) No 3-face is adjacent to a 4-face.
- (P1.2) Every 4^+-vertex v is not adjacent to three successive 3-faces, that is, $m_3(v) \leqslant \lfloor \frac{2}{3} d(v) \rfloor$.

(P2) $d(v) \geqslant 4$ for all $v \in V^0$.

In the following, we always assume that $D = t_0 t_1 \ldots t_{p-1} t_0$, where $p = |V(D)|$. We first investigate the structures of G, then use Euler's formula and discharging analysis to derive a contradiction. $\square$

Claim 1. *G has no separating small cycles.*

Proof. Suppose that G contains a separating small cycle. Choose D^* to be a separating small cycle such that $|V_{\rm Int}(D^*)|$ is minimal. Then D^* has no chord. It is easy to check that $G_{\rm Ext}(D^*) \in \mathcal{G}$, $|V(G_{\rm Ext}(D^*))| < |V(G)|$, and D is the small outer face cycle of $G_{\rm Ext}(D^*)$. By the minimality of G, $G_{\rm Ext}(D^*)$ is D-extendable and hence it admits an acyclic-2-coloring φ satisfying the required conditions. By restricting φ on $V(D^*)$, we get an acyclic-2-coloring of D^*. Note that $G_{\rm Int}(D^*) \in \mathcal{G}$, $|V(G_{\rm Int}(D^*))| < |V(G)|$, and D^* is the small outer cycle of $G_{\rm Int}(D^*)$. By the minimality of G, $G_{\rm Int}(D^*)$ is D^*-extendable and therefore admits an acyclic-2-coloring π. Consequently, the combination of φ and π yields an acyclic-2-coloring of G, which is a contradiction. □

The following Remark 1 can be derived from Claim 1:

Remark 1.

(1) G is 2-connected.

(2) Assume that $P = x_0x_1 \dots x_m$ is a path of G with $2 \leqslant m \leqslant 6$. If there is an $i \in \{1, 2, \dots, m-1\}$ such that the neighbors of x_i are $x_{i-1}, y_1, \dots, y_p, x_{i+1}, z_1, \dots, z_q$ in cyclic order and $p, q \geqslant 1$, then $x_0 \neq x_m$.

Claim 2. *D has no chord or 2-chord.*

Proof. If D has a chord, then G would contain a separating small cycle since $V^0 \neq \emptyset$, contradicting Claim 1.

Suppose that D has a 2-chord $P = t_0yz$, where $y \in V^0$ and $z \in V(D)$. By (P2), $d(y) \geqslant 4$. Then it is easy to observe by the definition that $4 \leqslant |V(D)| \leqslant 6$. Let P_1 and P_2 be the two paths between t_0 and z on D, and set $D_i = P_i \cup P$ for $i = 1, 2$. Then D_i is a 6^--cycle. By Claim 1, $V_{\rm int}(D_i) = \emptyset$. This implies that all the neighbors of y must belong to $V(D)$. By symmetry, we may consider two possibilities as follows.

Assume that $z = t_2$. If $|V(D)| = 4$, then $yt_1, yt_3 \in E(G)$ since $d(y) \geqslant 4$. It follows that $yt_0t_1t_2t_3y$ is a chordal 5-cycle of G, contradicting (P1). If $|V(D)| = 5$, then at least one of t_3 and t_4 is adjacent to y because $d(y) \geqslant 4$, which implies that $t_0yt_2t_3t_4t_0$ is a chordal 5-cycle of G, contradicting (P1). If $|V(D)| = 6$, then at least one of t_3, t_4 and t_5 is adjacent to y, and we can similarly find a chordal 5-cycle of G, which is impossible.

Assume that $z = t_3$ and $|V(D)| = 6$. Then both D_1 and D_2 are 5-cycles and hence have no chords by (P1). This implies that $d(y) = 2$, which contradicts (P2). □

Claim 3. *Let $v \in V(D)$.*

(1) *If $d(v) = 2$, then v is incident with an internal 5^+-face. Moreover, when v is incident with an internal 5-face, then the neighbors of v in $V(D)$ are of degree at least 3.*

(2) *If $d(v) = 3$, then $m_3(v) \leqslant 1$.*

Proof. (1) Assume that $d(t_0) = 2$ and f is the internal face of G incident with t_0. Suppose to the contrary that $d(f) \leqslant 4$. Then $|V(D)| = 3$ for otherwise D has a chord or a 2-chord, contradicting Claim 2. Let $D = t_0t_1t_2$, then $d(f) = 4$ since $V^0 \neq \emptyset$, say $f = [t_0t_1xt_2]$, where $x \in V^0$. Let $C' = xt_1t_2x$. By Claim 1, $V_{\text{int}}(C') = \emptyset$. This implies that x is an internal vertex of degree two, contradicting (P2).

Assume that $d(f) = 5$. Let $f = [t_0t_1x_1x_2t_{p-1}]$. If at least one of t_1 and t_{p-1} is a 2-vertex, then it is easy to check that either G contains a chordal 5-cycle, or at least one of x_1 and x_2 is an internal 3^--vertex. We always get a contradiction.

(2) Suppose that $d(t_0) = 3$ and $m_3(t_0) = 2$. Let t_1, z, t_{p-1} be the neighbors of t_0 in cyclic order, where $z \in V^0$. Then $zt_1, zt_{p-1} \in E(G)$. By Claim 2, $|V(D)| = 3$. Let $C' = zt_1t_2z$. By Claim 1, $V_{\text{int}}(C') = \emptyset$. It follows that $d(z) = 3$, contradicting (P2). □

For a subgraph H of G with $u, v \in V(H)$ and $uv \notin E(H)$, we use H_{uv} to denote the graph obtained from H by identifying u and v and deleting parallel edges (if existing). Moreover, let D_{uv} denote the outer face cycle of H_{uv}.

Claim 4. *Let $u, v \in V(G)$, $S \subseteq V^0 \setminus \{u, v\}$, and $H = G - S$. If $d_H(u, v) \geqslant 5$, H_{uv} is planar, and $|\{u, v\} \cap V(D)| \leqslant 1$, then $H_{uv} \in \mathcal{G}$ and D_{uv} is small.*

Proof. Note that $|\{u, v\} \cap V(D)| \leqslant 1$ implies that $|V(D_{uv})| = |V(D)|$. Thus, D_{uv} is a small outer face cycle of H_{uv}. Now we are going to prove that $H_{uv} \in \mathcal{G}$. Let z denote the vertex of H_{uv} obtained by identifying u and v in H. Suppose that $H_{uv} \notin \mathcal{G}$. Then H_{uv} contains a chordal 5-cycle $C = x_1x_2x_3x_4x_5x_1$ with $x_1x_3 \in E(H_{uv})$ and $z \in \{x_1, x_2, x_3, x_4, x_5\}$. By symmetry, let x_1x_4 be another chord of C if it exists. We have to consider two cases as follows. Note that $H \in \mathcal{G}$.

Case 1: $x_1x_4 \notin E(H_{uv})$, assuming $z \in \{x_1, x_2, x_5\}$ by symmetry.

▷ $z = x_1$. Let $X = \{x_2, x_3, x_5\}$, then $X \subseteq N_H(u) \cup N_H(v)$. Since $|X| = 3$, assume that $|X \cap N_H(u)| \geqslant 2$ by symmetry. Furthermore, we consider three subcases by symmetry. If $x_2, x_3, x_5 \in N_H(u)$, then H contains a chordal 5-cycle, which is a contradiction. If $x_2, x_3 \in N_H(u)$ and $x_5 \in N_H(v)$, then it follows that $d_H(u, v) \leqslant 4$, contradicting the assumption that $d_H(u, v) \geqslant 5$. If $x_2, x_5 \in N_H(u)$ and $x_3 \in N_H(v)$, or $x_3, x_5 \in N_H(u)$ and $x_2 \in N_H(v)$, we can similarly derive that $d_H(u, v) \leqslant 3$, which is a contradiction.

- $\triangleright$ $z = x_2$. Then $x_1, x_3 \in N_H(u) \cup N_H(v)$. If $x_1, x_3 \in N_H(u)$ or $x_1, x_3 \in N_H(v)$, then H contains a chordal 5-cycle, which is a contradiction. Otherwise, assume that $x_1 \in N_H(u)$ and $x_3 \in N_H(v)$ by symmetry, which implies that $d_H(u, v) \leqslant 3$, also a contradiction.
- $\triangleright$ $z = x_5$. Then $x_1, x_4 \in N_H(u) \cup N_H(v)$. If $x_1, x_4 \in N_H(u)$ or $x_1, x_4 \in N_H(v)$, then H contains a chordal 5-cycle, which is a contradiction. Otherwise, assume that $x_1 \in N_H(u)$ and $x_4 \in N_H(v)$ by symmetry, it follows that $d_H(u, v) \leqslant 4$, also a contradiction.

Case 2: $x_1x_4 \in E(H_{uv})$, assuming $z \in \{x_1, x_2, x_3\}$ by symmetry. If $z = x_2$ or $z = x_3$, then we have a similar argument as in Case 1. Assume that $z = x_1$. Let $Y = \{x_2, x_3, x_4, x_5\}$, then $Y \subseteq N_H(u) \cup N_H(v)$. Furthermore, we assume that $|Y \cap N_H(u)| \geqslant 2$ by symmetry. If $Y \subseteq N_H(u)$, then H contains a chordal 5-cycle. Otherwise, by symmetry and the planarity of H, we need to deal with several subcases. If $x_2, x_3, x_4 \in N_H(u)$ and $x_5 \in N_H(v)$, or $x_2, x_3, x_5 \in N_H(u)$ and $x_4 \in N_H(v)$, then it is easy to check that $d_H(u, v) \leqslant 3$. If $x_2, x_3 \in N_H(u)$ and $x_4, x_5 \in N_H(v)$, or $x_2, x_5 \in N_H(u)$ and $x_3, x_4 \in N_H(v)$, then we have $d_H(u, v) \leqslant 3$ similarly. We always get a contradiction. □

Claim 5. *Let u be a 4-vertex with neighbors u_1, u_2, u_3, u_4 in cyclic order. If $u, u_1, u_2, u_3 \in V^0$, then $d(u_1) \geqslant 5$ or $d(u_3) \geqslant 5$.*

Proof. Suppose that $d(u_1) = d(u_3) = 4$ by (P2). Let $H = G - \{u, u_1, u_3\}$. We assert that $d_H(u_2, u_4) \geqslant 5$, otherwise there is a path Q of length at most 4 connecting u_2 and u_4 in H so that $C := Q \cup \{uu_2, uu_4\}$ forms a separating small cycle of G, contradicting Claim 1. By Claim 4, $H_{u_2u_4} \in \mathcal{G}$ and $D_{u_2u_4}$ is a small cycle. By the minimality of G, $H_{u_2u_4}$ is $D_{u_2u_4}$-extendable and admits an acyclic-2-coloring φ. Let z denote the vertex of $H_{u_2u_4}$ formed by identifying u_2 and u_4 in H. Based on φ, we color u_2 and u_4 with $\varphi(z)$. By Lemma 1 (2) and (4), u_1, u_3, u are free in their order. This establishes a contradiction. □

Claim 6. *Let $v_1v_2v_3v_4v_1$ be a 4-cycle of G with $v_1, v_2, v_3 \in V^0$ and $v_1v_3 \in E(G)$. If $d(v_1) = 4$, then $d(v_3) \geqslant 6$.*

Proof. Suppose that $4 \leqslant d(v_3) \leqslant 5$ by (P2). Let $H = G - \{v_1, v_3\}$. We assert that $d_H(v_2, v_4) \geqslant 5$, otherwise there is a path Q of length at most 4 connecting v_2 and v_4 in H so that $C_1 := Q \cup \{v_1v_2, v_1v_4\}$ or $C_2 := Q \cup \{v_3v_2, v_3v_4\}$ forms a separating small cycle of G, contradicting Claim 1. By Claim 4, $H_{v_2v_4} \in \mathcal{G}$ and $D_{v_2v_4}$ is a small cycle. By the minimality of G, $H_{v_2v_4}$ is $D_{v_2v_4}$-extendable and admits an acyclic-2-coloring φ. Let z denote the vertex of $H_{v_2v_4}$ formed by identifying v_2

and v_4 in H and $\varphi(z) = 1$. Based on φ, we color v_2 and v_4 with 1. If v_3 has at most one neighbor of the color 2, then we color v_3 with 2. Otherwise, we color v_3 with 1. Since H contains no (1)-path connecting v_2 and v_4, such coloring is legal. By Lemma 1 (4), v_1 is free. This leads to a contradiction. $\square$

Claim 7. *Let $v_1v_2v_3v_4v_1$ be a 4-cycle of G with $v_1, v_2, v_3, v_4 \in V^0$ and $v_2v_4 \in E(G)$. If $d(v_1) = d(v_3) = 4$ and $d(v_2) = 5$, then $d(v_4) \geqslant 6$.*

Proof. Suppose that $4 \leqslant d(v_4) \leqslant 5$ by (P2). By Claim 1, $[v_1v_2v_4]$ and $[v_2v_3v_4]$ are 3-faces. Let the neighbors of v_1 be v_2, v_4, u_1, w_1 in cyclic order, the neighbors of v_2 be v_3, v_4, v_1, u_2, w_2 in cyclic order, the neighbors of v_3 be v_4, v_2, u_3, w_3 in cyclic order, and the neighbors of v_4 be v_1, v_2, v_3, u_4 and w_4 (if exists) in cyclic order. Let $H = G - \{v_1, v_2, v_3\}$. We claim hat $d_H(u_2, v_4) \geqslant 5$, otherwise there is a path Q of length at most 4 connecting u_2 and v_4 in H so that $C := Q \cup \{v_2v_4, v_2u_2\}$ forms a separating small cycle of G, contradicting Claim 1. Moreover, $|\{u_2, v_4\} \cap V(D)| \leqslant 1$ since $v_4 \in V^0$. By Claim 4, $H_{u_2v_4} \in \mathcal{G}$ and $D_{u_2v_4}$ is a small cycle. By the minimality of G, $H_{u_2v_4}$ is $D_{u_2v_4}$-extendable and admits an acyclic-2-coloring φ. Let z denote the vertex of $H_{u_2v_4}$ formed by identifying u_2 and v_4 in H and assume that $\varphi(z) = 1$. Based on φ, we color u_2 and v_4 with 1. Obviously, v_2 can be legally colored with 2. If v_1 and v_3 are free, we are done. Otherwise, by symmetry, we assume that v_1 is not free and therefore it holds that $\varphi(u_1) = 1$ and $\varphi(w_1) = 2$ by Lemma 1 (4). There are two possibilities as follows:

- ▷ $\varphi(w_2) = 1$. Color v_1 with 2. If v_3 is free, then we are done. Otherwise, $\varphi(u_3) = 2$ and $\varphi(w_3) = 1$ by Lemma 1(4). Remove the color of v_1. If v_4 can be recolored with 2, then we color v_1 and v_3 with 1. Otherwise, w_4 exists and $\varphi(u_4) = \varphi(w_4) = 2$. We color v_1 with 1 and v_3 with 2.
- ▷ $\varphi(w_2) = 2$. Recolor v_2 with 1 so that v_1 is free. Since H contains no (1)-path connecting u_2 and v_4, such recoloring is legal. If v_3 is free, then we are done. Otherwise, $\varphi(u_3) = \varphi(w_3) = 2$ by Lemma 1 (4). If v_4 can be recolored with 2, then v_1 and v_3 are free. Otherwise, w_4 exists and $\varphi(u_4) = \varphi(w_4) = 2$. In this case, we color or recolor v_2 with 2 and both v_1 and v_3 with 1. $\square$

To derive a contradiction by discharging analysis, we first define an initial charge ω as $\omega(f_0) = d(f_0) + 6$, $\omega(x) = 2d(x) - 6$ for every $x \in V(G)$ and $\omega(x) = d(x) - 6$ for every $x \in F(G) \setminus \{f_0\}$. By Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, the total sum of charges of the vertices and faces satisfies the following identity:

$$\sum_{x \in V(G) \cup F(G)} \omega(x) = d(f_0) + 6 + \sum_{x \in V(G)} (2d(x) - 6) + \sum_{x \in F(G) \setminus \{f_0\}} (d(x) - 6) = 0.$$

Next, we design appropriate discharging rules, and redistribute charges accordingly. Once the discharging is finished, a new charge ω^* is produced. Note that the discharging process preserves the total sum of charges of G. However, we will show that $\omega^*(x) \geqslant 0$ for all $x \in V(G) \cup (F(G) \setminus \{f_0\})$ and $\omega^*(f_0) > 0$, which leads to an obvious contradiction

$$0 < \sum_{x \in V(G) \cup F(G)} \omega^*(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) = 0,$$

and henceforth the proof is complete.

An internal 4-vertex v is called *bad* if $m_3(v) = 2$ and *good* otherwise. Let v be an internal 5-vertex with neighbors $v_0, v_1, \ldots, v_4$ in cyclic order. For $i = 0, 1, \ldots, 4$, let f_i be the incident face of v with $vv_i, vv_{i+1} \in E(f_i)$, where indices are taken modulo 5. Assume that f_0, f_1, f_3 are 3-faces. If $v_0 \in V(D)$ or v_0 is an internal 5^+-vertex, v_1 is an internal 5^+-vertex, and v_2 is an internal 4-vertex, then v is called *poor* and moreover, f_0 is called *poor* on v. Otherwise, v is called *weak* and moreover, both f_0 and f_1 are called *weak* on v.

Remark 2.

(1) If v is a poor or weak 5-vertex, then v is not incident with any 4-face.

(2) If $f = [x_1 x_2 x_3]$ is a poor 3-face on a poor 5-vertex x_1, and $d(f_{x_1 x_3}) = 3$, then f is not a poor or weak 3-face on x_2.

Our discharging rules are defined as follows. For two elements $x, y \in V(G) \cup F(G)$, we use $\tau(x \to y)$ to denote the amount of charge that x sends to y in the defined rules.

(R1) f_0 sends 2 to each incident 2-vertex, and 1.9 to each incident 3^+-vertex.

(R2) Every 3^+-vertex $v \in V(D)$ sends 1.4 to each incident internal 3-face in F^0, and 0.5 to each incident internal 4-face and 5-face.

(R3) Every internal vertex sends 0.5 to each incident 4-face and 0.2 to each incident 5-face.

(R4) Every good 4-vertex sends 1.1 to each incident 3-face in F^0.

(R5) Let v be an internal 5^+-vertex and f be a 3-face incident with v.

(R5.1) Let v be a poor 5-vertex. Then $\tau(v \to f) = 0.8$ if f is poor on v, and $\tau(v \to f) = 1.4$ otherwise.

(R5.2) Let v be a weak 5-vertex. Then $\tau(v \to f) = 1.1$ if f is weak on v, and $\tau(v \to f) = 1.4$ otherwise.

(R5.3) For other cases, $\tau(v \to f) = (2d(v) - 6 - 0.5m_4(v) - 0.2m_5(v))/m_3(v)$.

Let $\alpha(f)$ denote the resultant charge of an internal 3-face f after (R1)–(R5) were carried out.

(R6) Every bad 4-vertex sends $|\alpha(f)|/t$ to each incident 3-face f with $\alpha(f) < 0$, where t is the number of bad 4-vertices in $V(f)$.

Claim 8. *Let f be an internal 3-face incident with an internal 5^+-vertex v. If $d(v) \geqslant 6$, or $d(v) = 5$ and $m_3(v) \leqslant 2$, then $\tau(v \to f) \geqslant 1.4$.*

P r o o f. Set $\sigma = 2d(v) - 6 - 0.2m_5(v) - 0.5m_4(v) - 1.4m_3(v)$. By (R5), it is enough to show that $\sigma \geqslant 0$. In fact, it follows from (P1.2) that $\sigma \geqslant 2d(v) - 6 - 0.5(m_4(v) + m_5(v)) - 1.4m_3(v) \geqslant 2d(v) - 6 - 0.5(d(v) - m_3(v)) - 1.4m_3(v) = 1.5d(v) - 6 - 0.9m_3(v) \geqslant 1.5d(v) - 6 - 0.9\lfloor\frac{2}{3}d(v)\rfloor \geqslant 0.9d(v) - 6$. If $d(v) \geqslant 7$, then it holds automatically that $\sigma \geqslant 0.3$.

Assume that $d(v) = 6$. Then $2d(v) - 6 = 6$. By (P1.2), $m_3(v) \leqslant 4$. If $m_3(v) \leqslant 3$, then $\sigma \geqslant 6 - 3 \times 1.4 - 3 \times 0.5 = 0.3$. If $m_3(v) = 4$, then (P1.1) claims that $m_4(v) = 0$ and hence $\sigma \geqslant 6 - 4 \times 1.4 - 2 \times 0.2 = 0$.

Assume that $d(v) = 5$. Then $2d(v) - 6 = 4$. By (P1.2), $m_3(v) \leqslant 3$. If $m_3(v) \leqslant 1$, then $\sigma \geqslant 4 - 1.4 - 4 \times 0.5 = 0.6$. If $m_3(v) = 2$, then it is easy to examine by (P1.1) that $m_4(v) \leqslant 1$ and henceforth $\sigma \geqslant 4 - 2 \times 1.4 - 0.5 - 2 \times 0.2 = 0.3$. □

Claim 9. *Let $f = [x_1x_2x_3]$ be an internal 3-face and x_1 be a bad 4-vertex. If $V(f)$ contains three bad 4-vertices, or two bad 4-vertices and one good 4-vertex, then $\tau(x_1 \to f) \leqslant 1$. Otherwise, $\tau(x_1 \to f) \leqslant 0.8$.*

P r o o f. The definition implies that f cannot be poor. If x_2 and x_3 are bad 4-vertices, then $\alpha(f) = -3$ and therefore $\tau(x_1 \to f) = 1$ by (R6). If x_3 is a good 4-vertex, then $\tau(x_3 \to f) = 1.1$ by (R4), which implies that $\alpha(f) = -1.9$. Thus, $\tau(x_1 \to f) = \frac{1.9}{2} = 0.95$ by (R6). Assume that x_3 is an internal 5-vertex with $m_3(x_3) = 3$. Since $x_1, x_2, x_3 \in V^0$, by Claim 6 and (P1.1), $d(f_{x_1x_3}) \geqslant 5$ and $d(f_{x_2x_3}) \geqslant 5$. Thus, f is not weak on x_3, then $\tau(x_3 \to f) \geqslant 1.4$ by (R5.1) and (R5.2). Hence, $\alpha(f) \geqslant -1.6$ and $\tau(x_1 \to f) \leqslant 0.8$ by (R6). Otherwise, $\tau(x_3 \to f) \geqslant 1.4$ by Claim 8 and hence $\alpha(f) \geqslant -1.6$. By (R6), $\tau(x_1 \to f) \leqslant \frac{1.6}{2} = 0.8$. Finally, assume that x_2 and x_3 are not bad 4-vertices. By (R2), (R4), (R5) and Claim 8, $\tau(x_i \to f) \geqslant 1.1$ for $i = 2, 3$. It follows that $\alpha(f) \geqslant -0.8$ and hence $\tau(x_1 \to f) \leqslant 0.8$ by (R6). □

Lemma 2. *Let $x \in V(G) \cup F(G)$. If $x = f_0$, then $\omega^*(x) \geqslant 0.1$; otherwise $\omega^*(x) \geqslant 0$.*

P r o o f. Assume that $x = f_0$. Since $V^0 \neq \emptyset$, $V(D)$ contains a 3^+-vertex u. Using (R1) and noting that $d(f_0) \leqslant 6$, we conclude that $\omega^*(f_0) \geqslant d(f_0) + 6 - 2(d(f_0) - 1) - 1.9 = 6.1 - d(f_0) \geqslant 0.1$.

Assume that $x \neq f_0$. We have to consider the following three cases.

(1) x is an internal face f. If $d(f) \geqslant 6$, then $\omega^*(f) = \omega(f) = d(f) - 6 \geqslant 0$.

Assume that $d(f) = 5$. Then $\omega(f) = -1$. If f is not incident with a 2-vertex, then $\omega^*(f) \geqslant -1 + 5 \times 0.2 = 0$ by (R2) and (R3). Otherwise, f is incident with a 2-vertex u. By (P2), $u \in V(D)$. Then f is incident with at least two 3^+-vertices in $V(D)$ by Claim 3 (1). Thus, $\omega^*(f) \geqslant -1 + 2 \times 0.5 = 0$ by (R2).

Assume that $d(f) = 4$. Then $\omega(f) = -2$. By Claim 3 (1), f is not incident with any 2-vertex. By (R2) and (R3), $\omega^*(f) \geqslant -2 + 4 \times 0.5 = 0$.

Assume that $d(f) = 3$, say $f = [v_1v_2v_3]$, then $\omega(f) = -3$. By Claim 3 (1), f is not incident with a 2-vertex. If f is incident with a bad 4-vertex, then $\omega^*(f) \geqslant 0$ by (R6). Otherwise, f is not incident with any bad 4-vertex. If f is not poor on any vertex, then $\tau(v_i \to f) \geqslant 1.1$ for $i = 1, 2, 3$ by (R2), (R4) and (R5). Consequently, $\omega^*(f) \geqslant -3 + 1.1 \times 3 = 0.3$. Otherwise, f is poor on a poor 5-vertex, say v_1. By the definition, we may assume that $v_2 \in V(D)$ or v_2 is an internal 5^+-vertex and $f_{v_1v_3} = [v_1v_3z]$ is a 3-face such that z is an internal 4-vertex and v_3 is an internal 5^+-vertex. By (R5.1), $\tau(v_1 \to f) = 0.8$. By Remark 2(2), f is not poor or weak on v_2. So $\tau(v_2 \to f) \geqslant 1.4$ and $\tau(v_3 \to f) \geqslant 0.8$ by (R5) and Claim 8. Consequently, $\omega^*(f) \geqslant -3 + 1.4 + 2 \times 0.8 = 0$.

(2) $x \in V(D)$. Then $d(x) \geqslant 2$. If $d(x) = 2$, then $\tau(f_0 \to x) = 2$ by (R1), and hence $\omega^*(x) \geqslant 2 \times 2 - 6 + 2 = 0$. Otherwise, $d(x) \geqslant 3$. By (R1), $\tau(f_0 \to x) = 1.9$. Note that $m_3(x) + m_4(x) + m_5(x) \leqslant d(x) - 1$. If $d(x) = 3$, then $m_3(x) \leqslant 1$ by Claim 3 (2) and hence $\omega^*(x) \geqslant 2 \times 3 - 6 + 1.9 - 1.4 - 0.5 = 0$ by (R2). If $d(x) = 4$, then $m_3(x) \leqslant 2$ by (P1), and $\omega^*(x) \geqslant 2 \times 4 - 6 + 1.9 - 2 \times 1.4 - 0.5 = 0.6$ by (R2). If $d(x) \geqslant 5$, then $\omega^*(x) \geqslant 2d(x) - 6 + 1.9 - 1.4m_3(x) - 0.5(m_4(x) + m_5(x)) \geqslant 2d(x) - 4.1 - 1.4(d(x) - 1) = 0.6d(x) - 2.7 \geqslant 0.3$ by (R2).

(3) x is an internal vertex v. Then $d(v) \geqslant 4$ by (P2). If $m_3(v) = 0$, then $\omega^*(v) = 2d(v) - 6 - 0.5m_4(v) - 0.2m_5(v) \geqslant 2d(v) - 6 - 0.5d(v) = 1.5d(v) - 6 \geqslant 0$ by (R3). Otherwise, assume that $m_3(v) \geqslant 1$. If $d(v) \geqslant 6$, or $d(v) = 5$ and v is not poor or weak, then it is easy to show that $\omega^*(v) = 0$ by (R3) and (R5.3). If v is a poor or weak 5-vertex, then $m_4(v) = 0$ by Remark 2 (1), and hence $\omega^*(v) \geqslant 2 \times 5 - 6 - 2 \times 0.2 - \max\{1.4 + 2 \times 1.1, 0.8 + 2 \times 1.4\} = 0$ by (R3), (R5.1) and (R5.2).

Assume that $d(v) = 4$. Then $\omega(v) = 2$. Let the neighbors of v be v_0, v_1, v_2, v_3 in cyclic order and the incident faces of v be f_0, f_1, f_2, f_3 in cyclic order such that $vv_i, vv_{i+1} \in E(f_i)$ for $i = 0, 1, 2, 3$, where indices are taken modulo 4. By (P1), $1 \leqslant m_3(v) \leqslant 2$. If $m_3(v) = 1$, say $f_0 = [vv_0v_1]$, then v is good. By (P1.1), $d(f_1), d(f_3) \geqslant 5$, and therefore $\omega^*(v) \geqslant 2 - 1.1 - 2 \times 0.2 - 0.5 = 0$ by (R3) and (R4). Suppose that $m_3(v) = 2$, i.e., v is bad. By symmetry, assume that f_0 and $g \in \{f_1, f_2\}$ are 3-faces. By (P1.1), the other two incident faces of v other than f_0 and g are of degree at least 5. If $\tau(v \to f_0) \leqslant 0.8$ and $\tau(v \to g) \leqslant 0.8$, then $\omega^*(v) \geqslant 2 - 2 \times 0.8 - 2 \times 0.2 = 0$ by (R3). Otherwise, assume that $\tau(v \to f_0) > 0.8$ by symmetry. By Claim 9, v_0, v_1 are internal 4-vertices and $\tau(v \to f_0) \leqslant 1$. By Claim 6, $d(f_1), d(f_3) \neq 3$ and hence $g = f_2$. By Claim 5, $v_i \in V(D)$ or v_i is an internal 5^+-vertex for $i = 2, 3$. Note that f_2 is not poor since v is an internal 4-vertex. If f_2 is not weak, then $\tau(v_i \to f_2) \geqslant 1.4$ for $i = 2, 3$ by (R2), (R5.2) and Claim 8. Hence, $\alpha(f_2) \geqslant -3 + 2 \times 1.4 = -0.2$ and $\tau(v \to f_2) \leqslant 0.2$ by (R6). In this case, $\omega^*(v) \geqslant 2 - 2 \times 0.2 - 1 - 0.2 = 0.4$. Otherwise, by symmetry, assume that f_2 is a weak 3-face on v_2, i.e., v_2 is a weak 5-vertex. By (R5.2), $\tau(v_2 \to f_2) = 1.1$. Since $d(f_1), d(f_3) \geqslant 5$ by (P1.1), $(f_2)_{v_2v_3} = [v_3v_2y]$ is a weak 3-face. If $v_3 \in V(D)$ or v_3 is an internal 6^+-vertex, then $\tau(v_3 \to f_2) \geqslant 1.4$ by (R2) and Claim 8. Thus, $\alpha(f_2) \geqslant -3 + 1.1 + 1.4 = -0.5$, which implies that $\tau(v \to f_2) \leqslant 0.5$ by (R6). It follows that $\omega^*(v) \geqslant 2 - 2 \times 0.2 - 1 - 0.5 = 0.1$. Otherwise, v_3 is an internal 5-vertex. It follows from Claim 7 that y is not an internal 4-vertex. Thus, $y \in V(D)$ or y is an internal 5^+-vertex, which implies that v_2 is poor, contradicting the assumption that v_2 is weak. $\square$

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