Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 74 (2024), No. 2, 389–395

Persistent URL: `http://dml.cz/dmlcz/152447`

Czechoslovak Mathematical Journal, 74 (149) (2024), 389–395

MAXIMAL NON-PSEUDOVALUATION SUBRINGS OF AN INTEGRAL DOMAIN

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Received March 21, 2023. Published online June 5, 2024.

Abstract. The notion of maximal non-pseudovaluation subring of an integral domain is introduced and studied. Let $R \subset S$ be an extension of domains. Then R is called a maximal non-pseudovaluation subring of S if R is not a pseudovaluation subring of S, and for any ring T such that $R \subset T \subset S$, T is a pseudovaluation subring of S. We show that if S is not local, then there no such T exists between R and S. We also characterize maximal non-pseudovaluation subrings of a local integral domain.

Keywords: maximal non-pseudovaluation domain; pseudovaluation subring

MSC 2020: 13B02, 13G05, 3F30, 13B22

1. Introduction

All rings considered in this paper are integral domains. A domain containing a ring R and contained in the quotient field of R is called an *overring* of R. The symbol $\subseteq$ is used for inclusion, while $\subset$ is used for proper inclusion. A ring with a unique (or finitely many) maximal ideal(s) is called a *local ring* (or *semi-local ring*). Let $R \subset S$ be an extension of domains. Then R' denotes the integral closure of R in S. Our present work is motivated by [1] and [9]. Ayache and Echi in [1] generalized the concepts of valuation domains and pseudovaluation domains (see [8]) and defined new classes of rings, namely valuation subrings and pseudovaluation subrings of an integral domain. For an extension of domains $R \subset S$, R is said to be a valuation subring of S (R is a VD in S for short) (see [1]) if whenever $x \in S$, we have $x \in R$ or $x^{-1} \in R$. Note that if S is the quotient field of R, then R is a valuation domain. Moreover, R is called a *pseudovaluation subring* of S (R is a PV in S for short) if for each $x \in S \setminus R$ and for all nonunits a in R, we

The author was supported by the SERB-SURE grant scheme (No. SUR/2022/004326) from the Science and Engineering Research Board (SERB), India.

DOI: 10.21136/CMJ.2024.0122-23

have $x^{-1}a \in R$. Note that if S is the quotient field of R, then R is a pseudovaluation domain. Now, if R is not a pseudovaluation domain and if each intermediate ring (except R) between R and S is a pseudovaluation domain, then R is said to be a maximal non-pseudovaluation subring of S (R is a maximal non-PVD subring of S for short), see [9]. Motivated by this idea, we introduce the notion of maximal non-pseudovaluation subring of an integral domain, which is a generalization of the concept of maximal non-PVD subrings. A proper subring R of an integral domain S is called a *maximal non-pseudovaluation subring* of S (R is a maximal non-PV in S for short) if R is not a pseudovaluation subring of S, and for any ring T such that $R \subset T \subset S$, T is a pseudovaluation subring of S. We establish some properties and characterizations of a maximal non-PV in an integral domain.

We discuss the properties of a maximal non-PV R in an integral domain S and characterize both R and S. We prove that if R is a maximal non-PV in S and S is not local, then there is no intermediate ring between R and S (Theorem 2.2). We also prove that if R is a maximal non-PV in S such that R is not a field, then S is an overring of R (Proposition 2.1). The equivalence of an integrally closed maximal non-PV in a local domain S and a maximal non-local subring of S is established in Theorem 2.3.

2. Maximal non-pseudovaluation subrings

We begin the section by defining a maximal non-pseudovaluation subrings of an integral domain formally.

Definition 2.1. A proper subring R of an integral domain S is called a *maximal non-pseudovaluation subring* of S (R is a maximal non-PV in S for short) if R is not a pseudovaluation subring of S, and for any ring T such that $R \subset T \subset S$, T is a pseudovaluation subring of S.

Let $R \subseteq S$ be a ring extension. If for each prime ideal $\mathfrak{q}$ of S we have that $S/\mathfrak{q}$ is algebraic over $R/(\mathfrak{q} \cap R)$, then $R \subseteq S$ is called a *residually algebraic extension*, see [2], [6]. Moreover, if for any intermediate ring T between R and S we have that $R \subseteq T$ is residually algebraic, then (R, S) is called a *residually algebraic pair*, see [2]. Our first result shows that whenever we have a ring extension $R \subset S$ such that each proper subring of S properly containing R is a PV in S, then (R, S) is a residually algebraic pair. To prove this result, we need the following lemma:

Lemma 2.1. *Let $R \subset S$ be an extension of integral domains, where R is not a field. If each proper subring of S properly containing R is a PV in S, then $R \subset S$ is a residually algebraic extension.*

P r o o f. Consider a prime ideal $\mathfrak{q}$ of S and let $\mathfrak{p}$ be the contraction of $\mathfrak{q}$ in R. If possible, assume that $S/\mathfrak{q}$ is not algebraic over $R/\mathfrak{p}$. Then we have an element $\bar{s} \in S/\mathfrak{q}$ which is transcendental over $R/\mathfrak{p}$. Now, set $T := (R/\mathfrak{p})[\bar{s^2}]$. Then T is a proper subring of $S/\mathfrak{q}$ properly containing $R/\mathfrak{p}$. Note that there is a one-to-one order-preserving correspondence between subrings of S which contain $\mathfrak{q}$, and subrings of $S/\mathfrak{q}$. Therefore, there exists an intermediate ring U such that $R \subseteq R+\mathfrak{q} \subset U \subset S$ and $T = U/\mathfrak{q}$. Since each proper subring of S properly containing R is PV in S, U is a PV in S. Thus, either $\bar{s} \in T$ or $\bar{s}^{-1}\bar{a} \in T$ for all nonunits a in U. Note that $\bar{s}^2$ is a nonunit in T. It follows that $\bar{s} \in T$, which is a contradiction. Thus, $S/\mathfrak{q}$ is algebraic over $R/\mathfrak{p}$ and hence $R \subset S$ is a residually algebraic extension. $\square$

Theorem 2.1. *Let $R \subset S$ be an extension of integral domains, where R is not a field. If each proper subring of S properly containing R is a PV in S, then (R, S) is a residually algebraic pair.*

P r o o f. Let T be a subring of S properly containing R. Then either R is not a PV in T but each proper subring of T properly containing R is a PV in T or (R, T) is a PV pair. If former condition holds, then $R \subset T$ is a residually algebraic extension, by Lemma 2.1. Now, assume that (R, T) is a PV pair. Let $\mathfrak{q}$ be a prime ideal of T and let $\mathfrak{p}$ be the contraction of $\mathfrak{q}$ in R. If possible, assume that $T/\mathfrak{q}$ is not algebraic over $R/\mathfrak{p}$. Then we have an element $\bar{t} \in T/\mathfrak{q}$ which is transcendental over $R/\mathfrak{p}$. Now, set $U := (R/\mathfrak{p})[\bar{t}^2]$. Then, as we discussed in the proof of Lemma 2.1, we have $R \subseteq R+\mathfrak{q} \subset V \subset T$, where $U = V/\mathfrak{q}$. Since (R, T) is a PV pair, V is a PV in T. Consequently, $\bar{t} = \bar{t}^{-1}\bar{t}^2 \in U$, which is a contradiction. Therefore, $T/\mathfrak{q}$ is algebraic over $R/\mathfrak{p}$. Thus, $R \subset T$ is a residually algebraic extension. Hence, (R, S) is a residually algebraic pair. $\square$

Recall from [1], Remark 1.1 (3) that if R is a PV in S, then R and S have the same quotient field. Clearly, this may not be true if R is not a PV in S. However, if R is a maximal non-PV in S, where R is not a field, then R and S have the same quotient field. The proof is similar to Lemma 4 (ii) of [9] and thus we omit the proof.

Proposition 2.1. *Let $R \subset S$ be an extension of integral domains where R is not a field. If R is a maximal non-PV in S, then the following hold true:*
(i) *R and S have the same quotient field.*
(ii) *If S is a field, then S is the quotient field of R.*

A ring extension is called a minimal ring extension if there is no intermediate ring between them, see [5], [7], [11]. Now we characterize maximal non-PV R in an integral domain S. We start with the case when S is not local.

Theorem 2.2. *Let $R \subset S$ be an extension of integral domains. If S is not local, then the following are equivalent:*

(i) *R is a maximal non-PV in S.*

(ii) *$R \subset S$ is a minimal ring extension.*

Proof. (ii) $\Rightarrow$ (i) is trivial. Assume that (i) holds. If possible, let T be a proper subring of S properly containing R. Then each proper subring of S containing T is a PV in S. It follows that T is local, by Proposition 1.7 of [1]. Consequently, either S is local or T' is a VD in S, by Theorem 6.6 of [1]. Since S is not local, T' is a VD in S. Thus, by Corollary 1.6 of [1], we have that S is local, which is a contradiction. Hence, (ii) holds. $\square$

Now, assume that S is a local domain and R be a proper subring of S. Then R is said to be a maximal non-local subring of S if R is not local but each subring of S properly containing R is local, see [9]. In our next result, we discuss maximal non-PV R in S when $R' = R$.

Theorem 2.3. *Let $R \subset S$ be an extension of integral domains such that R is integrally closed in S. If S is local, then the following are equivalent:*

(i) *R is a maximal non-PV in S.*

(ii) *R is a maximal non-local subring of S.*

Proof. Let R be a maximal non-PV in S. Then (R, S) is a residually algebraic pair, by Theorem 2.1. Now, if R is local, then R is a VD in T, by Theorem 2.5 of [2]. It follows that R is a PV in T, which is a contradiction. Therefore, R is not local. Now, assume that T is a subring of S which contains R properly. Then T is a PV in S. Thus, T is local, by [1], Proposition 1.7. Hence, R is a maximal non-local subring of S.

Now, let R be a maximal non-local subring of S. Then R is not a PV in S, by Proposition 1.7 of [1]. Also, by Theorem 2.2 of [10], R is a maximal non-VD in S. Since each VD in S is also a PV in S, R is a maximal non-PV in S. $\square$

Let $\mathfrak{p} \subset \mathfrak{q}$ be any two prime ideals of R. Then $[\mathfrak{p}, \mathfrak{q}[$ is the set of all prime ideals of R containing $\mathfrak{p}$ but properly contained in $\mathfrak{q}$. Let $R \subset S$ be a ring extension. Then (R, S) is called a normal pair if each intermediate ring between R and S is integrally closed in S, see [4]. The next corollary is a direct consequence of [9], Theorem 1 and 2.3.

Corollary 2.1. *Let $R \subset S$ be an extension of integral domains. If R is integrally closed in S and S is local, then the following are equivalent:*

(i) *R is a maximal non-PV in S.*

(ii) (R, S) *is a normal pair, R is semi-local with exactly two maximal ideals* $\mathfrak{n}_1$ *and* $\mathfrak{n}_2$ *and either*

(a) $S = R_{\mathfrak{n}_1}$ *and* $[(0), \mathfrak{n}_2[\subseteq [(0), \mathfrak{n}_1[$,

(b) $S = R_{\mathfrak{n}_2}$ *and* $[(0), \mathfrak{n}_1[\subseteq [(0), \mathfrak{n}_2[$ *or*

(c) *there exists a prime ideal* $\mathfrak{q}$ *of* R *such that* $\mathfrak{q} \subset \mathfrak{n}_1 \cap \mathfrak{n}_2$, $S = R_{\mathfrak{q}}$ *and* $[(0), \mathfrak{n}_1[= [(0), \mathfrak{n}_2[$.

Let $R \subset S$ be an extension of integral domains and let V be a subring of S properly containing R. Then V is called a (unique) minimal S-overring of R (or R has a minimal S-overring V) if for every intermediate ring T between R and S we have either $T = R$ or $V \subseteq T$. Now, we discuss the case $R' = S$.

Theorem 2.4. *Let $R \subset S$ be an extension of integral domains such that S is integral over R. If S is local, then the following are equivalent:*

(i) *R is a maximal non-PV in S.*

(ii) *R is not a PV in S and either $R \subset S$ is a minimal ring extension or R has a unique minimal S-overring V which is a PV in S.*

Proof. Let $\mathfrak{n}$ be the maximal ideal of S. Since $R \subset S$ is an integral extension, R is local. Let $\mathfrak{m}$ be the maximal ideal of R. Then clearly $\mathfrak{m}S \subseteq \mathfrak{n} \subset S$. Now, assume that R is a maximal non-PV in S. It follows that R is not PV in S. Consequently, R is not a field as if R is a field, then R is a PV in S by the definition of PV subring. Therefore, S is an overring of R, by Proposition 2.1. Note that if $R \subset S$ is a minimal ring extension, then we are done. Therefore, assume that $R \subset S$ is not minimal. Now, if $\mathfrak{n} \subset R$, then $\mathfrak{n} = \mathfrak{m}$. Consequently, by [1], Corollary 2.2, R is a PV in S, which is a contradiction. It follows that $R \subset V := R + \mathfrak{n}$. Now, let T be a proper subring of S properly containing R. Then T is a PV in S and S is integral over T. It follows that $\mathfrak{n}$ is the maximal ideal of T, by Corollary 2.2 of [1]. Therefore, $V \subseteq T$. Thus, V is a unique minimal S-overring of R which is a PV in S.

Now, assume that (ii) holds. If $R \subset S$ is a minimal ring extension, then we are done. Assume that $R \subset S$ is not minimal. Then there is a unique minimal S-overring V of R which is a PV in S. Now, let T be a proper subring of S properly containing R. Then $V \subseteq T$. Since V is PV in S, $\mathfrak{n}$ is the maximal ideal of V, by [1], Corollary 2.2. It follows that $\mathfrak{n}$ is the maximal ideal of T. Thus, T is a PV in S, by [1], Corollary 2.2. Hence, R is a maximal non-PV in S. $\square$

Let R be a PV in S. Then R is local, by [1], Proposition 1.7. Assume that $\mathfrak{m}$ is the maximal ideal of R. Now, let V be a proper subring of S containing R. Then V is called an associated valuation overring of R in S if V is a VD in S with the maximal ideal $\mathfrak{m}$. Finally, we discuss the case $R \subset R' \subset S$.

Theorem 2.5. *Let $R \subset S$ be an extension of integral domains such that $R \subset R' \subset S$. Then the following are equivalent:*

(i) *R is a maximal non-PV in S.*

(ii) *R is not a PV in S and R has a unique minimal S-overring V which is a PV in S with associated valuation overring R' in S.*

Proof. Assume that (i) holds. Then R' is a PV in S. It follows that R' is local, by [1], Proposition 1.7. Also, (R, S) is a residually algebraic pair, by Theorem 2.1. It follows that (R', S) is a residually algebraic pair. Consequently, R' is a VD in S, by [2], Theorem 2.5. Now, assume that $\mathfrak{m}$ is the maximal ideal of R'. Then $\mathfrak{m}S = S$ as R' is a VD in S. It follows that $R' = (\mathfrak{m} : \mathfrak{m})_S = \{x \in S \colon x\mathfrak{m} \subseteq \mathfrak{m}\}$, by [1], Corollary 3.4. Set $V \colon R + \mathfrak{m}$. Now, we claim that $R \subset V$. If possible, assume that $V = R$. Then $\mathfrak{m}$ is the maximal ideal of R. Consequently, by [1], Proposition 3.3, R is a PV in S, which is a contradiction. Thus, $R \subset V$. Since $\mathfrak{m} \cap R$ is the maximal ideal of R, $\mathfrak{m}$ is the maximal ideal of V. Therefore, V is a PV in S with associated valuation overring R' in S. Now, we claim that $R \subset V$ is a minimal ring extension. If possible, assume that $R \subset U \subset V$. Then U is a PV in S and hence U is PV in R'. Since $U \subset R'$ is an integral extension, $\mathfrak{m}$ is the maximal ideal of U, by [1], Corollary 2.2. It follows that $V = R + \mathfrak{m} \subseteq U$, which is a contradiction. Therefore, $R \subset V$ is a minimal extension. Finally, we claim that V is a minimal S-overring of R. For this, let T be a proper subring of S properly containing R. Then T is a PV in S. If T is integrally closed in S, then $R' \subseteq T$. It follows that $V \subseteq T$. Now, assume that $T \subset T'$. Then T is a PV in T'. Note that $R' \subseteq T'$. If $R' = T'$, then $\mathfrak{m}$ is the maximal ideal of T, by [1], Corollary 2.2. It follows that $V \subseteq T$. Now, only one case is remaining, that is, $R' \subset T'$. Since R' is a VD in S, R' is a VD in T'. Consequently, by [1], Theorem 1.5, there is a non-maximal prime ideal $\mathfrak{q}$ of R' such that $T' = R'_{\mathfrak{q}}$. Also, $\mathfrak{q}$ is the unique maximal ideal of T', by the proof of [1], Theorem 1.5. Since T is a PV in T', $\mathfrak{q}$ is the unique maximal ideal of T, by [1], Corollary 2.2. Now, by [11], Theorem 1, we have $(R : V) = \mathfrak{m} \cap R$ as $R \subset V$ is a minimal integral ring extension. Also, $\mathfrak{q} \cap R \subset \mathfrak{m} \cap R$ as $\mathfrak{q} \subset \mathfrak{m}$ and $R \subset R'$ is an integral extension. Consequently, $(R : V) = \mathfrak{m} \cap R \not\subseteq \mathfrak{q}$. Since $V \subseteq R'$, either $V = R'$ or V is a PV in R'. It follows that $\operatorname{Spec}(V) = \operatorname{Spec}(R')$, by [1], Corollary 2.2. Consequently, $\mathfrak{q}$ is a prime ideal of V. Therefore, by [3], Proposition 0, $V_{\mathfrak{q}} = R_{\mathfrak{q} \cap R}$. Now, $V_{\mathfrak{q}}$ is a VD in $S_{\mathfrak{q}}$, by [1], Corollary 1.9. It follows that $R_{\mathfrak{q} \cap R}$ is a VD in $S_{\mathfrak{q}}$. Consequently, $T = T_{\mathfrak{q}}$ is a VD in $S_{\mathfrak{q}}$. Therefore, T is integrally closed in $S \subseteq S_{\mathfrak{q}}$. Thus, we have $V \subseteq R' \subseteq R'_{\mathfrak{q}} = T' = T$. Hence, (ii) holds.

Now, assume that (ii) holds. Then V is local, by [1], Proposition 1.7. Let $\mathfrak{m}$ be the maximal ideal of V. Then $\mathfrak{m}$ is the maximal ideal of R' as R' is associated valuation overring of V in S. Now, let T be a proper subring of S properly containing R. Then $V \subseteq T$. Then T is a PV in S, by [1], Theorem 6.6. Hence, (i) holds. $\square$

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