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BOUNDS FOR THE DERIVATIVE OF CERTAIN MEROMORPHIC FUNCTIONS AND ON MEROMORPHIC BLOCH-TYPE FUNCTIONS

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Abstract. It is known that if f is holomorphic in the open unit disc $\mathbb{D}$ of the complex plane and if, for some $c>0$, $|f(z)|\leqslant 1/(1-|z|^2)^c$, $z\in\mathbb{D}$, then $|f'(z)|\leqslant 2(c+1)/(1-|z|^2)^{c+1}$. We consider a meromorphic analogue of this result. Furthermore, we introduce and study the class of meromorphic Bloch-type functions that possess a nonzero simple pole in $\mathbb{D}$. In particular, we obtain bounds for the modulus of the Taylor coefficients of functions in this class.

Keywords: Bloch function; meromorphic function; Landau's reduction; Taylor coefficient

MSC 2020: 30D45, 30C50, 30C99

1. Introduction

Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{R}$ and $\mathbb{C}$ be the sets of all real and complex numbers, respectively. We define $\mathbb{D}:=\{z\in\mathbb{C}\colon |z|<1\}$ to be the open unit disc, $\mathbb{D}_r:=\{z\in\mathbb{C}\colon |z|<r\}$ as the disc with center at the origin and radius r, and $\partial\mathbb{D}:=\{z\in\mathbb{C}\colon |z|=1\}$ as the boundary of $\mathbb{D}$. Let $\mathcal{O}(\mathbb{D})$ be the class of all holomorphic functions on $\mathbb{D}$. In 1965, the following problem was posed in the Bulletin of the American Mathematical Society, see, f.i., [16], Problem N.

Problem N. If $f\in\mathcal{O}(\mathbb{D})$ and

$$|f(z)|\leqslant\frac{1}{(1-r^2)},\quad z=re^{i\theta},$$

for what positive real number β it is true that

$$|f'(z)|\leqslant\frac{\beta}{(1-r^2)^2}?$$

What is the coefficient region for this class?

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Duren, Shapiro, and Shields in [9] on page 249 initially showed $\beta = 4$ by using Cauchy's integral formula. Afterwards, Robertson in [18] generalized Problem N for the class

$$\mathcal{R}_c := \left\{ f \in \mathcal{O}(\mathbb{D})\colon\ |f(z)| \leqslant \frac{1}{(1-r^2)^c},\ z = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{D},\ \text{and } c\ (>0) \in \mathbb{R} \right\}$$

as follows. If $f \in \mathcal{R}_c$, then what is the value of $\beta(c)$ such that

$$|f'(z)| \leqslant \frac{\beta(c)}{(1-r^2)^{c+1}}?$$

He obtained the value of $\beta(c) = a(c+1)$ for $1 \leqslant a \leqslant \mathrm{e}$ without using Cauchy's integral formula. For the special case $c = 1$, Robertson obtained that $3 \leqslant \beta = \beta(1) \leqslant 4$. In [14], Liu chose a very particular integral path in Cauchy's integral formula to obtain an improved range for values of β by proving $8/\mathrm{e} \leqslant \beta < 10/3$. In 1978, Wirths in [19] delivered a new method to solve Problem N for the class $\mathcal{R}_c$ and proved that $\beta(c) = 2(c+1)$, where 2 cannot be replaced by any smaller constant independent of f and c.

Motivated by these results, in this article, we allow the functions in $\mathcal{R}_c$ to possess a nonzero pole of order n inside $\mathbb{D}$ and wish to see whether an analogue of Problem N can be established by suitably defining the problem in this case. For this purpose, we consider the class $\mathcal{M}_c^n$ of all functions f holomorphic in $\mathbb{D} \setminus \{p\}$ with a pole of order $n \in \mathbb{N}$ at the point $p \in (0,1)$ and

$$|f(z)| \leqslant \frac{1}{(1-|z|^2)^c}\left|\frac{1-pz}{z-p}\right|^n, \quad z \in \mathbb{D}, \tag{1.1}$$

where c is a positive real number. We now ask that if $f \in \mathcal{M}_c^n$, then for what positive real number $\beta_n(c)$, the following estimate holds:

$$|f'(z)| \leqslant \frac{\beta_n(c)}{(1-|z|^2)^{c+1}}\left|\frac{1-pz}{z-p}\right|^{n+1}, \quad z \in \mathbb{D}. \tag{1.2}$$

In our problem, the right-hand side of (1.1) has the factor $1/|z-p|^n$ because f has a pole of order n at $z = p$. Also the reason for multiplying $|1-pz|^n$ with the factor $1/((1-|z|^2)^c|z-p|^n)$ is that if we allow $p \to 1^-$ and $c \geqslant n$, then our problem reduces to Problem N for the class $\mathcal{R}_c$. Hence, this newly defined problem can be thought of as a generalisation of Problem N.

For $f \in \mathcal{M}_c^n$ we define

$$\eta_f^n(z) := \left(\frac{z-p}{1-pz}\right)^n f(z), \quad z \in \mathbb{D}.$$

Then inequality (1.2) reduces to

$$|\eta_{f'}^{n+1}(z)| \leqslant \frac{\beta_n(c)}{(1-|z|^2)^{c+1}}, \quad z \in \mathbb{D}. \tag{1.3}$$

Let us note that by considering η_f^n as a function of the class $\mathcal{R}_c$ and applying the corresponding result for the class $\mathcal{R}_c$ proved by Wirths (see [19]), we readily obtain the estimate

$$|(\eta_f^n)'(z)| \leqslant \frac{2(c+1)}{(1-|z|^2)^{c+1}}, \quad z \in \mathbb{D},$$

which implies that

$$(1-|z|^2)^{c+1}|f'(z)|\left|\frac{z-p}{1-pz}\right|^{n+1} \leqslant 2(c+1) + \frac{n(1-p^2)(1-|z|^2)}{(1-p|z|)^2}, \quad z \in \mathbb{D}.$$

The maximum value of the right-hand side of the above inequality is $2(c+1)+n$ for all $z \in \mathbb{D}$. Therefore, from a short calculation we easily get the following crude estimate:

$$|f'(z)| \leqslant \frac{2(c+1)+n}{(1-|z|^2)^{c+1}}\left|\frac{1-pz}{z-p}\right|^{n+1}, \quad z \in \mathbb{D}.$$

This last inequality shows that $\beta_n(c) \leqslant 2(c+1)+n$. In this paper, we obtain a better value for $\beta_n(c)$, see Table 1 for the comparison between this value and $2(c+1)+n$. Additionally, it is noteworthy that our estimate coincides with the result of the class $\mathcal{R}_c$ for $c \geqslant n$ when we allow p to approach 1^-. These results are the contents of Section 2 of this article.

In 1924, Andre Bloch proved a classical result in function theory, known as the Bloch's theorem (see [5], [6], [11]), which states: If $f \in \mathcal{O}(\mathbb{D})$ and $f'(0) = 1$, then $f(\mathbb{D})$ contains a univalent disc of largest radius $B_f > 0$. Here, by univalent disc Δ in $f(\mathbb{D})$ we mean that there exists a domain $\Omega \subseteq \mathbb{D}$ such that f is univalent in Ω and $f(\Omega) = \Delta$. The infimum value of B_f for all functions in $\mathcal{O}(\mathbb{D})$ with $f'(0) = 1$ is called the *Bloch constant* B. The exact value of the Bloch constant is not known, but the estimates for the Bloch constant proved hitherto are (see [1], [7], [8])

$$0.4332\ldots = \frac{\sqrt{3}}{4} + 2\times 10^{-4} < B \leqslant \frac{\Gamma(1/3)\Gamma(11/12)}{\sqrt{1+\sqrt{3}}\Gamma(1/4)} = 0.47186\ldots$$

It is an interesting problem in geometric function theory to find the exact value of the Bloch constant for holomorphic functions. While trying to find this constant, a new class of holomorphic functions has been studied, called the class of Bloch functions, which is defined as follows.

Definition 1.1. The function $f \in \mathcal{O}(\mathbb{D})$ is called a *Bloch function* if

$$\delta(f) := \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty$$

and in this case $\delta(f)$ is called the *Bloch seminorm* of f.

We denote by $\mathcal{B}$ the set of all Bloch functions on $\mathbb{D}$. The class $\mathcal{B}$ forms a Banach space with respect to the norm $\|f\|_{\mathcal{B}} = \delta(f) + |f(0)|$. The class of Bloch functions on the unit disc is well-known and it has been studied by many eminent mathematicians. One of the many reasons for which the Bloch functions are interesting is that they can be characterized in many different ways, and so, they arise in many different contexts. We refer to the articles [2], [10], [11], [12], [15] and references therein for more details. The class of α-Bloch functions, which generalises the class of Bloch functions, is defined as follows: A function $f \in \mathcal{O}(\mathbb{D})$ is called an α-Bloch function for $\alpha \in (0, \infty)$ if

$$\delta_\alpha(f) := \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

For $\alpha \in (0, 1]$, the α-Bloch class is a subclass of $\mathcal{B}$. We refer to articles [11], [13], [20] and references therein for more details for the α-Bloch functions.

After having prepared all these backgrounds related to the Bloch constant and the Bloch functions in the holomorphic case, we are now ready to describe analogous concepts for meromorphic functions. To this end, we start with the class $\mathcal{A}(p)$ of functions f holomorphic in $\mathbb{D}\backslash\{p\}$ having simple pole at $z = p \in (0, 1)$ with $f'(0) \neq 0$. Let $B_f(p)$ be the radius of the largest univalent disc contained in $f(\mathbb{D})$. Then

$$B(p) := \inf\{B_f(p)\colon\ f \in \mathcal{A}(p)\}$$

is called the Bloch constant for the class $\mathcal{A}(p)$. In [4], we proved that $B(p) \geqslant p^2 \times |f'(0)|(8 - \sqrt{63})^2$. In the same article, we improved the lower bound of $B(p)$ by considering the subclass $\mathcal{A}_1(p)$ of $\mathcal{A}(p)$ that is defined as

$$\mathcal{A}_1(p) := \{f \in \mathcal{A}(p)\colon\ N_f = \psi_f(q),\ q \in \mathbb{D} \setminus \{p\}\},$$

where

$$N_f := \sup_{z\in\mathbb{D}} \psi_f(z) \quad \text{and} \quad \psi_f(z) := \left|\frac{z-p}{1-pz}\right|^2 (1 - |z|^2)|f'(z)|, \quad z \in \mathbb{D}.$$

In [4], Theorem 4, we proved that $p^2|f'(0)|/27$ is the lower bound of the Bloch constant for the class $\mathcal{A}_1(p)$. In [3], we further improved the lower bound of $B(p)$ to $4pB|f'(0)|/(1+p)^2$, where B is the Bloch constant for the holomorphic functions in $\mathbb{D}$ and since $B > \sqrt{3}/4 + 2 \times 10^{-4}$, we obtain $B(p) > (\sqrt{3} + 8 \times 10^{-4})p|f'(0)|/(1 + p)^2$.

The construction of the class $\mathcal{A}_1(p)$ motivates us to consider the following class of functions:

$$\Omega(p) := \{f \in \mathcal{A}(p)\colon\ N_f < \infty\}.$$

This class $\Omega(p)$ is analogous to the Bloch space $\mathcal{B}$ of holomorphic functions in $\mathbb{D}$. However, it is easy to see that $\Omega(p)$ is not a vector space, since the sum of two functions in $\Omega(p)$ with opposite sign residues at p does not necessarily belong to $\Omega(p)$. We call this class $\Omega(p)$ the class of meromorphic Bloch-type functions. If we allow $p \to 1^-$, then the class $\Omega(1)$ is a subclass of the α-Bloch class for $\alpha \in [2, \infty)$.

It is clear that $\mathcal{A}_1(p)$ is a subset of $\Omega(p)$. Hence, all the function in Examples 2.1–2.4 in [4] belong to $\Omega(p)$. On the other hand, the function $f_0(z) = (z-p)^{-1} \in \Omega(p) \setminus \mathcal{A}_1(p)$. The inclusion $\Omega(p) \subset \mathcal{A}(p)$ holds. However, any function in $\mathcal{A}(p)$ with a pole on $\partial\mathbb{D}$ does not belong to $\Omega(p)$. Thus, we have

$$\mathcal{A}_1(p) \subsetneq \Omega(p) \subsetneq \mathcal{A}(p).$$

If f is a Bloch function with Bloch seminorm 1, then $f' \in \mathcal{R}_1$. In [11], [12], Kayumov and Wirths obtained estimates for the Taylor coefficients of functions in the class of Bloch functions and remarked that these estimates is equivalent to the Taylor coefficient estimates for the class $\mathcal{R}_1$. In a similar way, we find bounds for the Taylor coefficients of the meromorphic Bloch-type functions which is equivalent to obtaining the bounds for the Taylor coefficients of the functions in $\mathcal{M}_1^2$.

In this article, we organize the obtained results as follows. In Section 2, we study the meromorphic analogue of Problem N. In Section 3, we show that the Bloch constants for the classes $\mathcal{A}(p)$ and $\Omega(p)$ are equal. Next, we present a meromorphic analogue of the Landau reduction, see [10], Theorem 4.2.2. Finally, we establish sharp estimates for the Taylor coefficients of the meromorphic Bloch-type functions.

2. Meromorphic analogue of Problem N

We are now ready to state and prove our main theorem of this section. We will use the following result by Raupach, see [17], Chapter IV, Sections 2–5.

Theorem A. *Let the function α be defined on $\mathbb{D}$ and satisfy the following:*

(i) *There is $K_1 \in \mathbb{R}$ such that $-\infty \leqslant \alpha(z) \leqslant K_1$ for all $z \in \mathbb{D}$.*

(ii) *For every $z_0 \in \mathbb{D}$ there exists a neighbourhood U_{z_0} such that in $(U_{z_0} \cap \mathbb{D}) \setminus \{z_0\}$, α has the representation*

$$\alpha(z) = \log(1 - |z|^2) + v(z_0) \log|z - z_0| + \psi(z),$$

where $v(z_0)$ is a real number and ψ is a harmonic function in $\mathbb{D}$.

(iii) *There is a real number $\mu > 0$ such that for all $z_0 \in \mathbb{D}$, $\alpha(z_0) = -\infty$ implies $v(z_0) \geqslant \mu$. Then the estimate*

$$(1 - |z|^2)\Big|\frac{\partial \alpha}{\partial z}\Big| \leqslant g_1(\alpha(z), \mu, K_1)$$

holds for all $z \in D^* := \mathbb{D} \setminus \{z\colon \alpha(z) = -\infty\}$. *Here* $g_1(\alpha(z), \mu, K_1) = G(h^{-1}(\alpha(z)))$ *is defined by the following rules:*

$$a = \frac{1}{1+\mu}, \quad b = K_1 + \frac{a+1}{2a}\log(1+a) + \frac{a-1}{2a}\log(1-a);$$
$$G(t) = \frac{1}{a}\sinh(a(t+b)) - \cosh(a(t+b)), \quad -b + \frac{1}{2a}\log\Bigl(\frac{1+a}{1-a}\Bigr) \leqslant t < \infty;$$
$$\alpha(z) = h(t) = \log\Bigl(\frac{\mathrm{e}^{-t}}{a}\sinh(a(t+b))\Bigr), \quad -b + \frac{1}{2a}\log\Bigl(\frac{1+a}{1-a}\Bigr) \leqslant t < \infty.$$

Now for $f \in \mathcal{M}_c^n$, $c > 0$, $n \in \mathbb{N}$, we prove the following theorem using the method of Wirths, see [19], Theorem 1.

Theorem 2.1. *If* $f \in \mathcal{M}_c^n$, *then for* $z \in \mathbb{D}$ *with* $|z| = r < 1$

(2.1)
$$|\eta_{f'}^{n+1}(z)|(1-r^2)^{c+1} \leqslant \frac{(2c+1)^{c+1/2}}{(2c)^c}(1-\varrho^2)^c\Bigl(1-\varrho^2+2c\varrho(r-\varrho)+\frac{n\varrho(1-p^2)(1-r^2)}{(1-pr)^2}\Bigr),$$

where $\varrho \in (0, (2c+1)^{-1/2})$ *satisfies the equation*

$$(2c+1)(c+1)\varrho^3 - c(2c+1)R(r)\varrho^2 - (3c+1)\varrho + cR(r) = 0 \tag{2.2}$$

with

$$R(r) = r + \frac{n(1-p^2)(1-r^2)}{2c(1-pr)^2}. \tag{2.3}$$

P r o o f. For $f \in \mathcal{M}_c^n$, let us define

$$\alpha(z) := \frac{\log((1-|z|^2)^c|\eta_f^n(z)|)}{c}, \quad z \in \mathbb{D}. \tag{2.4}$$

Now we check that α satisfies all the hypothesis of Theorem A.

(i) Since $f \in \mathcal{M}_c^n$, we easily see that

$$\alpha(z) \leqslant \frac{\log 1}{c} = 0.$$

Also $\alpha(z) = -\infty$ for $\eta_f^n(z) = 0$. Therefore $-\infty \leqslant \alpha(z) \leqslant 0$, $z \in \mathbb{D}$.

(ii) Let $z_0 \in \mathbb{D}$. If $\eta_f^n(z_0) \neq 0$, then we get a neighbourhood U_{z_0} of z_0 such that for all $z \in U'_{z_0} := (U_{z_0} \setminus \{z_0\}) \cap \mathbb{D}$,

$$\alpha(z) = \log(1-|z|^2) + \psi(z),$$

where $\psi(z) = (1/c)\log|\eta_f^n(z)|$. But if $\eta_f^n(z_0) = 0$, then $z - z_0$ is a factor of η_f^n. Thus, for all $z \in U'_{z_0}$, α has the form

$$\alpha(z) = \log(1 - |z|^2) + v(z_0)\log|z - z_0| + \psi(z),$$

where

$$v(z_0) = \frac{m}{c} \quad \text{and} \quad \psi(z) = \frac{1}{c}\log\frac{|\eta_f^n(z)|}{|z - z_0|^m},$$

where m is the order of the zero of η_f^n at $z = z_0$. In both cases ψ is harmonic.

(iii) If $\alpha(z_0) = -\infty$ for $z_0 \in \mathbb{D}$, then $\eta_f^n(z_0) = 0$. Thus, in this case, we take $\mu = 1/c \leqslant v(z_0)$ as $v(z_0) = m/c$, $m \in \mathbb{N}$. Therefore, we get $K_1 = 0$ and $\mu = 1/c$ as per notations used in Theorem A. Thus, applying Theorem A, we have

$$(1 - |z|^2)\left|\frac{\partial\alpha}{\partial z}\right| \leqslant g_1\Big(\alpha(z), \frac{1}{c}, 0\Big), \quad z \in D^* := \mathbb{D} \setminus \{z\colon \eta_f^n(z) = 0\}, \tag{2.5}$$

where

$$a = \frac{c}{c+1}, \quad b = \Big(1 + \frac{1}{2c}\Big)\log(2c+1) - \log(c+1).$$

Substituting $\sigma = \mathrm{e}^{-a(t+b)} \in (0, (2c+1)^{-1/2}]$ in Theorem A, g_1 is obtained from the representation $g_1(\alpha(z), 1/c, 0) = G(h^{-1}(\alpha(z)))$ such that

$$G(\sigma) = \frac{1}{2c\sigma} - \frac{(2c+1)\sigma}{2c}, \tag{2.6}$$

and

$$\mathrm{e}^{\alpha(z)} = \left|\frac{z-p}{1-pz}\right|^{n/c}|f(z)|^{1/c}(1-|z|^2) = \frac{(2c+1)^{1+1/(2c)}}{2c}\sigma^{1/c}(1-\sigma^2) = \mathrm{e}^{h(\sigma)}. \tag{2.7}$$

Now from equation (2.4) we have

$$\begin{aligned}\frac{\partial\alpha}{\partial z} &= \frac{\partial}{\partial z}\Big(\frac{1}{c}\log((1-|z|^2)^c|\eta_f^n(z)|)\Big) = \frac{\partial}{\partial z}\Big(\log(1 - z\overline{z}) + \frac{1}{2c}\log(\eta_f^n(z)\overline{\eta_f^n(z)})\Big)\\ &= -\frac{\overline{z}}{1-|z|^2} + \frac{1}{2c}\Big(\frac{(\eta_f^n)'(z)}{\eta_f^n(z)}\Big) = -\frac{\overline{z}}{1-|z|^2} + \frac{n(1-p^2)}{2c(z-p)(1-pz)} + \frac{f'(z)}{2cf(z)}.\end{aligned}$$

Therefore, from inequality (2.5) and by virtue of the above computation, we get

$$(1-|z|^2)\left|-\frac{\overline{z}}{1-|z|^2} + \frac{n(1-p^2)}{2c(z-p)(1-pz)} + \frac{f'(z)}{2cf(z)}\right| \leqslant G(h^{-1}(\alpha(z))),$$

which implies that for $|z| = r$ and $\sigma = h^{-1}(\alpha(z))$

$$(1-r^2)\Big(-\frac{r}{1-r^2} - \frac{n(1-p^2)}{2c|z-p||1-pz|} + \frac{1}{2c}\frac{|f'(z)|}{|f(z)|}\Big) \leqslant G(\sigma).$$

This last inequality and equation (2.6) imply that for $0 < \sigma \leqslant (2c+1)^{-1/2}$ and $z \in D^*$

$$\frac{1}{2c}\frac{|f'(z)|}{|f(z)|}(1-r^2) \leqslant \frac{1-(2c+1)\sigma^2+2cr\sigma}{2c\sigma} + \frac{n(1-p^2)(1-r^2)}{2c|z-p||1-pz|}. \tag{2.8}$$

Thus, from (2.7) and (2.8) we get for $0 < \sigma \leqslant (2c+1)^{-1/2}$ and $z \in D^*$,

$$\begin{aligned}
&|\eta_{f'}^{n+1}(z)|(1-r^2)^{c+1} \\
&\quad \leqslant |f(z)|(1-r^2)^c\Big|\frac{z-p}{1-pz}\Big|^{n+1}\Big(\frac{1-\sigma^2+2c\sigma(r-\sigma)}{\sigma} + \frac{n(1-p^2)(1-r^2)}{|z-p||1-pz|}\Big) \\
&\quad = \frac{(2c+1)^{c+1/2}}{(2c)^c}(1-\sigma^2)^c\Big|\frac{z-p}{1-pz}\Big|\Big(1-\sigma^2+2c\sigma(r-\sigma) + \frac{n\sigma(1-p^2)(1-r^2)}{|z-p||1-pz|}\Big) \\
&\quad \leqslant \frac{(2c+1)^{c+1/2}}{(2c)^c}(1-\sigma^2)^c\Big(1-\sigma^2+2c\sigma(r-\sigma) + \frac{n\sigma(1-p^2)(1-r^2)}{(1-pr)^2}\Big).
\end{aligned}$$

Let us define

$$H_r(\sigma) := (1-\sigma^2)^c\Big(1-\sigma^2+2c\sigma(r-\sigma) + \frac{n\sigma(1-p^2)(1-r^2)}{(1-pr)^2}\Big).$$

Therefore, we have

$$|\eta_{f'}^{n+1}(z)|(1-r^2)^{c+1} \leqslant \frac{(2c+1)^{c+1/2}}{(2c)^c}H_r(\sigma). \tag{2.9}$$

Also

$$H_r'(\sigma) = 2(1-\sigma^2)^{c-1}(cR(r)-(3c+1)\sigma - c(2c+1)R(r)\sigma^2 + (2c+1)(c+1)\sigma^3),$$

where $R(r)$ is defined by (2.3). Let

$$H_r(\varrho) = \max\{H_r(\sigma)\colon\ \sigma \in [0,(2c+1)^{-1/2}]\}.$$

To prove that ϱ satisfies (2.2), let

$$k(\sigma) = (c+1)(2c+1)\sigma^3 - c(2c+1)R(r)\sigma^2 - (3c+1)\sigma + cR(r),$$

where $\sigma \in [0,(2c+1)^{-1/2}]$. Notice that $k(0) = cR(r) > 0$ and $K((2c+1)^{-1/2}) = -2c(2c+1)^{-1/2} < 0$. Therefore, k has an odd number of roots in $(0,(2c+1)^{-1/2})$. Since the variation of the sign of k is two in this interval, k has at most two positive real roots. This shows that H_r has a unique critical point ϱ, which is, indeed, a maximum, and the solution to (2.2) in $(0,(2c+1)^{-1/2})$. Therefore, (2.1) holds in the nonempty set $D^* \cap \{z\colon\ |z| = r\}$.

In order to show that (2.1) is also valid in $(\mathbb{D}\backslash D^*)\cap\{z\colon |z|=r\}$, let $z_0\in\mathbb{D}\backslash D^* := \{z\colon \eta_f^n(z)=0\}$. Then $z_0\neq p\in(0,1)$. Since the zeros of $\eta_f^n(z)$ are isolated, there exists a neighbourhood of z_0 in $\mathbb{D}$, where $\eta_{f'}^{n+1}$ is continuous and (2.1) holds for all points in the deleted neighbourhood of $z_0\in D^*$. Since $\eta_{f'}^{n+1}$ is continuous at z_0, (2.1) is also valid at this point. This ends the proof. □

We now proceed to establish the bound of $\beta_n(c)$ in (1.3). That is, we estimate the value of $\beta_n(c)$ that satisfies (1.2).

Corollary 2.1. *Let $f\in\mathcal{M}_n^c$, where n and c bear their usual meaning. If $(1-p)/(1+p)<n/c$, then we get*

$$\beta_n(c)\leqslant\frac{(2c+1)^{c+1/2}(1-\varrho_p^2)^c}{(2c)^c}\Big(1-\varrho_p^2+2c\varrho_p(r_p-\varrho_p)+\frac{n\varrho_p(1-p^2)(1-r_p^2)}{(1-pr_p)^2}\Big), \tag{2.10}$$

where $r_p\in[0,1)$ is the solution of equation

$$cp^3r^3-3cp^2r^2+(3cp+n(1-p^2))r-np(1-p^2)-c=0, \tag{2.11}$$

and $\varrho_p\in(0,(2c+1)^{-1/2})$ is the solution of equation (2.2), where $R(r)$ will be replaced by $R(r_p)$. If $(1-p)/(1+p)\geqslant n/c$, then

$$\beta_n(c)\leqslant\frac{(2c+1)^{c+1/2}}{(2c)^c}(1-\varrho_p^2)^c(1-\varrho_p^2+2c\varrho_p(1-\varrho_p)), \tag{2.12}$$

where $\varrho_p\in(0,(2c+1)^{-1/2})$ is the solution of equation (2.2), where $R(r)$ will be replaced by $R(1)=1$.

Proof. The right-hand side of inequality (2.1) will be maximum for all values of $r\in[0,1)$ for which $R(r)$ defined in (2.3) is maximum. Now for the maximum value of $R(r)$ we have $R'(r)=0$, which gives (2.11). The equation (2.11) has a root in $(0,1)$ only when $(1-p)/(1+p)<n/c$. Let the root be $r_p\in(0,1)$. Then for this r_p we get a root $\varrho_p\in(0,(2c+1)^{-1/2})$ of equation (2.2). Thus, the right-hand side of inequality (2.1) will be maximum for $r=r_p$ and the maximum value is the quantity mentioned on the right-hand side of inequality (2.10). If $(1-p)/(1+p)\geqslant n/c$, then $R'(r)>0$ in $[0,1)$. Thus, $R(r)$ is monotone increasing in $[0,1)$. Since $R(r)$ is continuous in $[0,1]$, the maximum value of $R(r)$ for $r\in[0,1)$ is $R(1)$. Therefore, we deduce that $r_p=1$. Substituting $r_p=1$ into inequality (2.10), we obtain inequality (2.12). This completes the proof. □

Remark 2.1.

(i) If $c\geqslant n$ and we allow $p\to1^-$, we can derive the result in [19], Theorem 1.

(ii) Letting $p\to1^-$ and setting $c=n$ in Corollary 2.1, we get back to the result proved by Wirths in [19], Corollary 1.

The table below presents a comparison between the values of $\beta_n(c)$ and the crude estimate $2(c+1)+n$ for various combinations of p, c, and n, as promised in Section 1. By examining the table, it becomes evident that our estimated values for $\beta_n(c)$ exhibit significant improvement over the crude estimate $2(c+1)+n$.

p	c	n	r_p	ϱ_p	$\beta_n(c)$ From (2.10)	$\beta_n(c)$ From (2.12)	$2(c+1)+n$
0.10	1	2	0.529057	0.285029	3.60660	—	6
	3	2	1	0.225382	—	7.17434	10
0.25	0.5	3	0.381598	0.487307	3.74092	—	6
	5	3	1	0.204246	—	11.1520	15
0.50	1	2	0.688140	0.325569	4.01282	—	6
	6	2	1	0.195097	—	13.1435	16
0.85	24	2	0.998841	0.121393	49.0853	—	52
	25	2	1	0.119418	—	51.0831	54
0.90	1	1	0.924810	0.298806	3.73155	—	5
	20	1	1	0.130449	—	41.0913	43

Table 1. Values of $\beta_n(c)$ and $2(c+1)+n$ for different p, c, and n.

For the answer to the second question in Problem N, Wirths in [19], Theorem 2 proved the following theorem to obtain the bound for the modulus of the second Taylor's coefficient for the functions in the class $\mathcal{R}_c$ whenever the first Taylor coefficient is given.

Theorem B. *Let $g \in \mathcal{R}_c$, $c > 0$ with $g(z) = \sum\limits_{k=0}^{\infty} d_k z^k$ and $|d_0| \leqslant 1$. If*

$$|d_0| = \frac{(2c+1)^{c+1/2}}{(2c)^c}\sigma(1-\sigma^2)^c \quad \textit{for } \sigma \in [0, (2c+1)^{-1/2}],$$

then

$$|d_1| \leqslant \frac{(2c+1)^{c+1/2}}{(2c)^c}(1-(2c+1)\sigma^2)(1-\sigma^2)^c.$$

Equality holds for the function

$$g_\sigma(z) = \frac{(2c+1)^{c+1/2}}{(2c)^c(1-z^2)^c}\frac{z+\sigma}{1+\sigma z}\Big(1-\Big(\frac{z+\sigma}{1+\sigma z}\Big)^2\Big)^c, \quad z \in \mathbb{D}.$$

Given that the functions $f \in \mathcal{M}_c^n$ are holomorphic within a neighbourhood of the origin, it is reasonable to inquire about the bounds of the coefficients in the Taylor series expansion within a disc centered at the origin. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D}_p. \tag{2.13}$$

Since $f \in \mathcal{M}_c^n$, $|f(0)| \leqslant 1/p^n$, i.e., $|a_0| \leqslant 1/p^n$. We note here that the term $\sigma(1-\sigma^2)^c$ has maximum value $(2c)^c/(2c+1)^{c+1/2}$ for $\sigma \in [0,(2c+1)^{-1/2}]$. Thus, for some $\sigma \in [0,(2c+1)^{-1/2}]$, we consider

$$|a_0| = \frac{(2c+1)^{c+1/2}}{p^n(2c)^c}\sigma(1-\sigma^2)^c. \tag{2.14}$$

In the following theorem, we prove the sharp estimate for the second Taylor coefficient of the function $f \in \mathcal{M}_c^n$ when the first coefficient is prescribed as in (2.14).

Theorem 2.2. *Let $f \in \mathcal{M}_c^n$ have Taylor expansion of the form* (2.13) *with $|a_0|$ as in* (2.14) *for some $\sigma \in (0,(2c+1)^{-1/2}]$. Then*

$$|a_1| \leqslant \frac{(2c+1)^{c+1/2}}{p^n(2c)^c}(1-\sigma^2)^c\Big(1-(2c+1)\sigma^2+\frac{n\sigma(1-p^2)}{p}\Big). \tag{2.15}$$

The equality occurs in inequality (2.15) *for the function*

$$f_\sigma(z) = \frac{(2c+1)^{c+1/2}}{(2c)^c(1-z^2)^c}\frac{z+\sigma}{1+\sigma z}\Big(1-\Big(\frac{z+\sigma}{1+\sigma z}\Big)^2\Big)^c\Big(\frac{1-pz}{z-p}\Big)^n, \quad z \in \mathbb{D}. \tag{2.16}$$

P r o o f. Since $f \in \mathcal{M}_c^n$, then from equation (2.8) we get for $0 < \sigma \leqslant (2c+1)^{-1/2}$,

$$\frac{1}{2c}\frac{|f'(z)|}{|f(z)|}(1-|z|^2) \leqslant \frac{1-\sigma^2+2c\sigma(r-\sigma)}{2c\sigma} + \frac{n(1-p^2)(1-r^2)}{2c|z-p||1-pz|}.$$

Hence, setting $z=0$ we obtain

$$\frac{|f'(0)|}{|f(0)|} \leqslant \frac{1-\sigma^2-2c\sigma^2}{\sigma} + \frac{n(1-p^2)}{p}.$$

Since $f(0)=a_0$ and $f'(0)=a_1$, the previous inequality can be written as

$$\begin{aligned}|a_1| &\leqslant |a_0|\Big(\frac{1-\sigma^2-2c\sigma^2}{\sigma}+\frac{n(1-p^2)}{p}\Big)\\ &= \frac{(2c+1)^{c+1/2}}{p^n(2c)^c}\sigma(1-\sigma^2)^c\Big(\frac{1-\sigma^2-2c\sigma^2}{\sigma}+\frac{n(1-p^2)}{p}\Big)\\ &= \frac{(2c+1)^{c+1/2}}{p^n(2c)^c}(1-\sigma^2)^c\Big(1-(2c+1)\sigma^2+\frac{n\sigma(1-p^2)}{p}\Big).\end{aligned}$$

Hence, assertion (2.15) is proved for all $\sigma \in (0,(2c+1)^{-1/2}]$. Now we discuss the equality case of inequality (2.15). The function f_σ in (2.16) can be written as

$$f_\sigma(z) = \frac{(2c+1)^{c+1/2}}{(2c)^c}\frac{z+\sigma}{1+\sigma z}\frac{(1-\sigma^2)^c}{(1+\sigma z)^{2c}}\Big(\frac{1-pz}{z-p}\Big)^n, \quad z \in \mathbb{D}.$$

A short calculation yields that for all $z \in \mathbb{D}$, $\sigma \in (0, (2c+1)^{-1/2}]$,

$$(1-|z|^2)^c|\eta_{f_\sigma}^n(z)| = \frac{(2c+1)^{c+1/2}}{(2c)^c}\Big|\frac{z+\sigma}{1+\sigma z}\Big|\Big(1-\Big|\frac{z+\sigma}{1+\sigma z}\Big|^2\Big)^c \leqslant 1,$$

and the equality occurs when $|(z+\sigma)/(1+\sigma z)| = (2c+1)^{-1/2}$. Therefore, the function f_σ given in (2.16) belongs to $\mathcal{M}_c^n$ for $\sigma \in (0, (2c+1)^{-1/2}]$. Also, for $z \in \mathbb{D}_p$

$$\begin{aligned} f'_\sigma(z) = \frac{(2c+1)^{c+1/2}}{(2c)^c}\Big(\Big(\frac{(1-\sigma^2)^{c+1}}{(1+\sigma z)^{2c+2}} - \frac{2c\sigma(z+\sigma)(1-\sigma^2)^c}{(1+\sigma z)^{2c+2}}\Big)\Big(\frac{1-pz}{z-p}\Big)^n \\ + \frac{z+\sigma}{1+\sigma z}\Big(1-\Big(\frac{z+\sigma}{1+\sigma z}\Big)^2\Big)^c\Big(\frac{1-pz}{z-p}\Big)^{n-1}\frac{n(p^2-1)}{(z-p)^2(1-z^2)^c}\Big). \end{aligned}$$

Therefore,

$$|f_\sigma(0)| = \frac{(2c+1)^{c+1/2}}{p^n(2c)^c}\sigma(1-\sigma^2)^c = |a_0|$$

and

$$|f'_\sigma(0)| = \frac{(2c+1)^{c+1/2}}{p^n(2c)^c}(1-\sigma^2)^c\Big(1-(2c+1)\sigma^2 + \frac{n\sigma(1-p^2)}{p}\Big) = |a_1|.$$

This completes the proof. □

Remark 2.2. If we consider $c = n$ and let $p \to 1^-$ in Theorem 2.2, we get Theorem B.

3. Meromorphic Bloch-type functions and coefficient estimates

In the following theorem, we prove that the Bloch constant of the class $\mathcal{A}(p)$ is equal to that of its subclass $\Omega(p)$.

Theorem 3.1. *The Bloch constant for the class $\Omega(p)$ is $B(p)$.*

Proof. Let $B_{\Omega(p)}$ be the Bloch constant for the class $\Omega(p)$, i.e., let

$$B_{\Omega(p)} = \inf\{B_f(p)\colon\ f \in \Omega(p)\}. \tag{3.1}$$

Since $\Omega(p) \subsetneq \mathcal{A}(p)$, then it follows from the definition of the Bloch constant that

$$B(p) \leqslant B_{\Omega(p)}. \tag{3.2}$$

Now we prove that $B_{\Omega(p)} \leqslant B(p)$. Here $\mathcal{A}(p) = \Omega(p) \cup \Omega^c(p)$, where $\Omega^c(p)$ is the complement of $\Omega(p)$, i.e., the class of functions $f \in \mathcal{A}(p)$ such that $|f'|$ is unbounded on $\partial\mathbb{D}$ and f has no logarithmic singularity on $\partial\mathbb{D}$. From (3.1) we get $B_{\Omega(p)} \leqslant B_f(p)$ for all $f \in \Omega(p)$. Now for $f \in \Omega^c(p)$ we have to prove that $B_{\Omega(p)} \leqslant B_f(p)$. To this end, let $f \in \Omega^c(p)$ and define $g(z) = f(rz)/r$, $z \in \mathbb{D}$ and $p < r < 1$. It is clear that g has a simple pole at $z = p/r$ and $g(\mathbb{D}) = f(\mathbb{D}_r)/r$. Thus, $g \in \Omega(p/r)$. Therefore, from (3.1) we get

$$B_{\Omega(p/r)} \leqslant B_g\Big(\frac{p}{r}\Big).$$

If we allow $r \to 1^-$, then $B_{\Omega(p/r)} \to B_{\Omega(p)}$ and $B_g(p/r) \to B_f(p)$. Hence, $B_{\Omega(p)} \leqslant B_f(p)$ for $f \in \Omega^c(p)$. Thus, $f \in \mathcal{A}(p)$ implies either $f \in \Omega(p)$ or $f \in \Omega^c(p)$, and in both cases we get $B_{\Omega(p)} \leqslant B_f(p)$. Therefore,

$$B_{\Omega(p)} \leqslant \inf\{B_f(p)\colon\ f \in \mathcal{A}(p)\} = B(p). \tag{3.3}$$

From (3.2) and (3.3) we conclude that $B(p) = B_{\Omega(p)}$. This completes the proof. □

Remark 3.1. In article [3], it was proved that $B(p) > 4pB|f'(0)|/(1+p)^2$, where B is the Bloch constant for the class $\mathcal{O}(\mathbb{D})$. Consequently, based on the above theorem, we can assert that the lower bound of the Bloch constant for the class $\Omega(p)$ is $4pB|f'(0)|/(1+p)^2$. Additionally, in [3] it was conjectured that $B(p) = 4pB|f'(0)|/(1+p)^2$. If one proves this conjecture for the class $\Omega(p)$, then the conjecture will be validated for the whole class $\mathcal{A}(p)$ as inferred from the aforementioned theorem.

The meromorphic Bloch-type functions are invariant under the left composition with the following type of disc automorphisms:

$$\varphi(z) = \frac{w((z-p)/(1-pz)) + p}{1 + pw((z-p)/(1-pz))}, \quad z \in \mathbb{D},\ w \in \partial\mathbb{D}, \tag{3.4}$$

i.e., if $f \in \Omega(p)$, then $f \circ \varphi \in \Omega(p)$. With the help of this invariance property, we prove the following result. This theorem is analogous to Landau's reduction for the class of holomorphic Bloch functions $\mathcal{B}$, see [10], Theorem 4.2.2. More concretely, let γ be the circle $|z - p/(1+p^2)| = p/(1+p^2)$ inside $\mathbb{D}$. Let us consider the following two subclasses of $\Omega(p)$:

$$\Omega_\gamma(p) := \{f \in \Omega(p)\colon\ \psi_f(z_0) = N_f,\ z_0 \in \gamma,\ \text{and}\ f'(0) = p^{-2}\},$$

and

$$\Omega^1_\gamma(p) := \{f \in \Omega_\gamma(p)\colon\ N_f = 1\}.$$

We are now ready to prove the following theorem.

Theorem 3.2. *The Bloch constant $B_\gamma(p)$ for the class $\Omega_\gamma(p)$ is the same as the Bloch constant for the class $\Omega^1_\gamma(p)$.*

P r o o f. Since $\Omega^1_\gamma(p) \subset \Omega_\gamma(p)$, it is obvious that $B_\gamma(p) \leqslant \inf\{B_f(p)\colon\ f \in \Omega^1_\gamma(p)\}$. Now the theorem will be proved if we show that $\inf\{B_f(p)\colon\ f \in \Omega^1_\gamma(p)\} \leqslant B_\gamma(p)$. Let $f \in \Omega_\gamma(p)$, then ψ_f attains maximum value $N_f > 1$ at a point $z_0 \in \gamma$. Also let

$$h(z) = \frac{\mathrm{e}^{-\mathrm{i}\theta}(f \circ \varphi)(z)}{N_f}, \quad z \in \mathbb{D}, \tag{3.5}$$

where $\theta = \arg((f \circ \varphi)'(0))$, and φ is a disc automorphism of the form (3.4). Here $\varphi(p) = p$ and $\varphi(\mathbb{D}) = \mathbb{D}$. Thus, h has simple pole at p. Now

$$h'(0) = \frac{\mathrm{e}^{-\mathrm{i}\theta} f'(\varphi(0))\varphi'(0)}{N_f} = \frac{|f'(\varphi(0))||\varphi'(0)|}{N_f} = \frac{|f'(z_0)|(1-p^2)^2}{N_f|1-p^2w|^2},$$

where $z_0 = \varphi(0) = p(1-w)/(1-p^2w)$ lies on the circle γ for some $w \in \partial\mathbb{D}$. Since $N_f = \psi_f(z_0)$, then $h'(0) = p^{-2}$. Also,

$$\begin{aligned}
&\Big|\frac{z-p}{1-pz}\Big|^2(1-|z|^2)|(f\circ\varphi)'(z)| \\
&\quad = \Big|\frac{z-p}{1-pz}\Big|^2(1-|z|^2)|f'(\varphi(z))||\varphi'(z)| \\
&\quad = \Big|\frac{\varphi(z)-p}{1-p\varphi(z)}\Big|^2 \frac{(1-p^2)^2(1-|z|^2)|f'(\varphi(z))|}{|1-pz|^2|1+pw(z-p)/(1-pz)|^2} \\
&\quad = \Big|\frac{\varphi(z)-p}{1-p\varphi(z)}\Big|^2 \frac{(1-p^2)(1-|z|^2)|f'(\varphi(z))|(1-|\varphi(z)|^2)}{|1-pz|^2(1-|(z-p)/(1-pz)|^2)} \\
&\quad = \Big|\frac{\varphi(z)-p}{1-p\varphi(z)}\Big|^2(1-|\varphi(z)|^2)|f'(\varphi(z))|.
\end{aligned}$$

Therefore for $z \in \mathbb{D}$,

$$\begin{aligned}
\psi_h(z) &= \frac{1}{N_f}\Big|\frac{z-p}{1-pz}\Big|^2(1-|z|^2)|(f\circ\varphi)'(z)| \\
&= \frac{1}{N_f}\Big|\frac{\varphi(z)-p}{1-p\varphi(z)}\Big|^2(1-|\varphi(z)|^2)|f'(\varphi(z))| \leqslant 1.
\end{aligned}$$

The above inequality and $\psi_h(0) = 1$ imply that $h \in \Omega^1_\gamma(p)$. Thus, for each $f \in \Omega_\gamma(p)$ we get a function $h \in \Omega^1_\gamma(p)$ of the form (3.5) with $h(\mathbb{D}) = f(\mathbb{D})/N_f$. Therefore, $B_h(p) \leqslant B_f(p)$, which implies that for each $f \in \Omega_\gamma(p)$,

$$\inf\{B_h(p)\colon\ h \in \Omega^1_\gamma(p)\} \leqslant B_f(p).$$

The above inequality implies that

$$\inf\{B_h(p)\colon\ h \in \Omega^1_\gamma(p)\} \leqslant \inf\{B_f(p)\colon\ f \in \Omega_\gamma(p)\} = B_\gamma(p).$$

This completes the proof. $\square$

Remark 3.2. The Landau reduction does not hold for the whole class $\Omega(p)$ because this class is not invariant under all disc automorphisms of $\mathbb{D}$.

The estimation of the moduli of the Taylor coefficients is a common challenge encountered in geometric function theory. Furthermore, when considering functions $g \in \Omega(p)$ with $N_g = 1$, it implies that $g' \in \mathcal{M}_1^2$. Consequently, we will try to estimate certain Taylor coefficients for functions $g \in \Omega(p)$, which is equivalent to finding coefficient estimates for functions in $\mathcal{M}_1^2$. Let $g \in \Omega(p)$ have Taylor series expansion of the form

$$g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}_p. \tag{3.6}$$

It is easy to show that $g \in \Omega(p)$ implies $g'/N_g \in \mathcal{M}_1^2$ and $|b_1| \leqslant N_g/p^2$. Thus, for some $\sigma \in [0, 1/\sqrt{3}]$ we consider

$$|b_1| = \frac{3\sqrt{3}}{2p^2}\sigma(1-\sigma^2)N_g. \tag{3.7}$$

In the following theorem, we present a sharp estimate for the Taylor coefficient b_2 of functions $g \in \Omega(p)$ when $|b_1|$ is given by (3.7).

Theorem 3.3. *Let $g \in \Omega(p)$ have Taylor expansion of the form* (3.6). *If for some $\sigma \in (0, 1/\sqrt{3}]$, $|b_1|$ is given by* (3.7), *then*

$$|b_2| \leqslant \frac{3\sqrt{3}}{4p^2}(1-\sigma^2)\Big(1 - 3\sigma^2 + \frac{2\sigma(1-p^2)}{p}\Big)N_g. \tag{3.8}$$

If $p \in (0, 1/\sqrt{3})$, equality occurs in the above inequality for the function g_σ, where g'_σ is given by

$$g'_\sigma(z) = \frac{3\sqrt{3}}{2(1-z^2)}\frac{z+\sigma}{1+\sigma z}\Big(1 - \Big(\frac{z+\sigma}{1+\sigma z}\Big)^2\Big)\Big(\frac{1-pz}{z-p}\Big)^2, \quad z \in \mathbb{D}, \tag{3.9}$$

with $\sigma = (\sqrt{3}(1-p^2) - 2p)/(3-p^2) \in (-1, 1)$.

P r o o f. Since $g \in \Omega(p)$, we have $g'/N_g \in \mathcal{M}_1^2$. If we consider $f(z) = g'(z)/N_g$ and for some $\sigma \in (0, 1/\sqrt{3}]$,

$$|g'(0)| = |b_1| = \frac{3\sqrt{3}}{2p^2}\sigma(1-\sigma^2)N_g,$$

then from Theorem 2.2 we get

$$|g''(0)| \leqslant \frac{3\sqrt{3}}{2p^2}(1-\sigma^2)\Big(1 - 3\sigma^2 + \frac{2\sigma(1-p^2)}{p}\Big)N_g.$$

This implies

$$|b_2| \leqslant \frac{3\sqrt{3}}{4p^2}(1-\sigma^2)\Big(1-3\sigma^2+\frac{2\sigma(1-p^2)}{p}\Big)N_g.$$

Now we show the equality case of inequality (3.8). Since $\psi_{g_\sigma}(z) \leqslant 1$, $z \in \mathbb{D}$, for the function g_σ, where g'_σ is defined by (3.9), we have $g_\sigma \in \Omega(p)$ with $N_{g_\sigma} = 1$. Also

$$\sigma = \frac{\sqrt{3}(1-p^2)-2p}{3-p^2} \in \Big(0, \frac{1}{\sqrt{3}}\Big) \quad \text{for } 0 < p < \frac{1}{\sqrt{3}}.$$

Therefore,

$$g'_\sigma(0) = \frac{3\sqrt{3}\sigma(1-\sigma^2)}{2p^2} = b_1$$

and

$$\begin{aligned} g''_\sigma(z) = \frac{3\sqrt{3}}{2}\bigg(&\Big(\frac{(1-\sigma^2)^2}{(1+\sigma z)^4} - \frac{2\sigma(z+\sigma)(1-\sigma^2)}{(1+\sigma z)^4}\Big)\Big(\frac{1-pz}{z-p}\Big)^2 \\ &+ \frac{z+\sigma}{1+\sigma z}\Big(1-\Big(\frac{z+\sigma}{1+\sigma z}\Big)^2\Big)\frac{1-pz}{z-p}\,\frac{2(p^2-1)}{(z-p)^2(1-z^2)}\bigg). \end{aligned}$$

Thus,

$$b_2 = \frac{g''_\sigma(0)}{2} = \frac{3\sqrt{3}}{4p^2}(1-\sigma^2)\Big(1-3\sigma^2+\frac{2\sigma(1-p^2)}{p}\Big).$$

This proves the equality case of inequality (3.8). □

In the following theorem, we will estimate the weighted sum of moduli of the Taylor coefficients of $g \in \Omega(p)$ by using Parseval's formula.

Theorem 3.4. *If $g \in \Omega(p)$ and g has the expansion of the form* (3.6), *then for* $|z| = r < p \in (0,1)$,

$$\sum_{n=1}^{\infty} n^2|b_n|^2 r^{2n-2} \leqslant \frac{((1-2r^2+p^2r^2)^2(p^2-r^2)+2r^2(1-p^2)^2(1-r^2)^2)N_g}{(p^2-r^2)^3(1-r^2)^2}. \tag{3.10}$$

If $p \in (1/\sqrt{3}, 1)$, equality occurs in the above inequality for the function g_σ such that g'_σ is given by (3.9) *at the points of intersection of the circles C_1 and C_2, where*

$$C_1 := \{z \in \mathbb{C}\colon\ |z| = r < p\} \quad \text{and} \quad C_2 := \Big\{z \in \mathbb{C}\colon\ \Big|\frac{z+\sigma}{1+\sigma z}\Big| = \frac{1}{\sqrt{3}}\Big\}.$$

P r o o f. Since $g \in \Omega(p)$, then $\psi_g(z) \leqslant N_g$ for $z \in \mathbb{D}$. If $z \in \mathbb{D}_p$ with $|z| = r$, then from Parseval's formula we get

$$\begin{aligned}\sum_{n=1}^{\infty} n^2 |b_n|^2 r^{2n-2} &= \frac{1}{2\pi} \int_0^{2\pi} |g'(z)|^2 \, \mathrm{d}\theta \leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{N_g}{(1-r^2)^2} \left| \frac{1 - pr\mathrm{e}^{\mathrm{i}\theta}}{r\mathrm{e}^{\mathrm{i}\theta} - p} \right|^4 \mathrm{d}\theta \\ &= \frac{1}{2\pi} \frac{N_g}{(1-r^2)^2} \int_0^{2\pi} \left(\frac{1 + p^2 r^2 - 2pr\cos\theta}{r^2 + p^2 - 2pr\cos\theta} \right)^2 \mathrm{d}\theta = \frac{\lambda(r) N_g}{(1-r^2)^2},\end{aligned}$$

where

$$\begin{aligned}\lambda(r) &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1 + p^2 r^2 - 2pr\cos\theta}{r^2 + p^2 - 2pr\cos\theta} \right)^2 \mathrm{d}\theta \\ &= \frac{(1 - 2r^2 + p^2 r^2)^2 (p^2 - r^2) + 2r^2 (1-p^2)^2 (1-r^2)^2}{(p^2 - r^2)^3}.\end{aligned}$$

Thus, inequality (3.10) holds for $|z| = r$, $z \in \mathbb{D}_p$. Now for the equality case, we consider the function g_σ, where g'_σ is given in equation (3.9). Since $\psi_{g_\sigma}(z) \leqslant 1$ for $z \in \mathbb{D}$ and $\psi_{g_\sigma}(z) = 1$ for $|(z+\sigma)/(1+\sigma z)| = 1/\sqrt{3}$, $g_\sigma \in \Omega(p)$ with $N_{g_\sigma} = 1$. Thus, the equality occurs in (3.10) for the function g_σ, $\sigma = (\sqrt{3}(1-p^2) - 2p)/(3-p^2) \in (-1, 1)$ at the intersections of two circles C_1 and C_2. To ensure $C_1 \cap C_2 \neq \emptyset$, we argue as follows. The circle $|z| = p$ always touches the circle C_2 at the point $(p, 0)$. If $1/\sqrt{3} < p < 1$, then C_2 lies inside the circle, $|z| = p$ and in this case, the circles C_1 and C_2 have intersections only when $\max\{0, -(\sqrt{3}(1-\sigma^2) + 2\sigma)/(3-\sigma^2)\} \leqslant r < p$ (here $(-(\sqrt{3}(1-\sigma^2) + 2\sigma)/(3-\sigma^2), 0)$ is the left end point of the diameter of the circle C_2 on the real axis). This completes the proof. $\square$

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