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ON CERTAIN $GL(6)$ FORM AND ITS RANKIN-SELBERG CONVOLUTION

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Abstract. We consider $L_G(s)$ to be the L-function attached to a particular automorphic form G on $GL(6)$. We establish an upper bound for the mean square estimate on the critical line of Rankin-Selberg L-function $L_{G\times G}(s)$. As an application of this result, we give an asymptotic formula for the discrete sum of coefficients of $L_{G\times G}(s)$.

Keywords: Maass form; automorphic form; Rankin-Selberg convolution

MSC 2020: 11F12, 11F30, 11N75

1. Introduction

Let f be a Maass form of type v_f for $SL(2,\mathbb{Z})$ and $L_{f\times f}(s)$ denote the Rankin-Selberg L-function. Assume that f is an eigenfunction of all the Hecke operators. Due to the Gelbart-Jacquet lift (see [2]), we know that

$$L_F(s) := \frac{L_{f\times f}(s)}{\zeta(s)}$$

is the Godement-Jacquet L-function of a self-dual Maass form F of type $(\frac{1}{3}2v_f, \frac{1}{3}2v_f)$ for $SL(3,\mathbb{Z})$.

Throughout the paper, we consider

$$L_G(s) := \zeta(s)L_f(s)L_F(s)$$

being an L-function corresponding to an automorphic form G on $GL(6)$. Further, $L_G(s)$ is an automorphic L-function since it is a finite product of automorphic

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L-functions. We observe that the Rankin-Selberg L-function $L_{G\times G}(s)$ can be written as a finite product of automorphic L-functions (see Lemma 3.5 below)

$$L_{G\times G}(s) = \zeta(s)^3 L_f(s)^4 L_F(s)^4 L_{\mathrm{sym}^3 f}(s)^2 L_{\mathrm{sym}^4 f}(s).$$

Hence, $L_{G\times G}(s)$ is an automorphic L-function.

Using the Rankin-Selberg method, we would like to study the distribution of the coefficients of $L_{G\times G}(s)$. The Rankin-Selberg method was independently developed by Rankin (see [13]) and Selberg, see [14]. One of the applications of this method is to obtain bounds on the eigenvalues of a Maass form. Another major application is to obtain strong bounds for the Fourier coefficients of automorphic forms on average. In this regard, we first need to study the mean square estimate for the Rankin-Selberg convolution of G with itself, i.e., $L_{G\times G}(s)$ on the line $\Re(s) = \frac{1}{2}$.

The goal of this article is first to prove:

Theorem 1.1. *For sufficiently large T and for any $\varepsilon > 0$ we have*

$$I := \int_T^{2T} \left| L_{G\times G}\Big(\frac{1}{2} + \mathrm{i}t\Big)\right|^2 \mathrm{d}t \ll T^{1564/105+\varepsilon}.$$

In particular,

$$\left| L_{G\times G}\Big(\frac{1}{2} + \mathrm{i}t\Big)\right| \ll (|t|+1)^{782/105+\varepsilon}.$$

Remark 1.1. It is known that for a general L-function $\mathfrak{L}(s)$ of degree d, the upper bound

$$\int_T^{2T} \left| \mathfrak{L}\Big(\frac{1}{2} + \mathrm{i}t\Big)\right|^2 \mathrm{d}t \ll T^{d/2+\varepsilon}$$

holds due to the result of Perelli, see [12]. Note that $L_{G\times G}(s)$ is of degree 36, so the known upper bound for integral I is $T^{18+\varepsilon}$. Theorem 1.1 is an improvement for this upper bound as $\frac{1564}{105} \sim 14.895 < 18$.

Remark 1.2. Though we consider a particular case of an automorphic form G on $GL(6)$ such that

$$L_G(s) = \zeta(s) L_f(s) L_F(s),$$

analogous arguments can be applied to 'certain' automorphic forms of higher degrees and different representations. For example, consider the particular case corresponding to an automorphic form G' on $GL(6)$ such that

$$L_{G'}(s) := \zeta(s)^4 L_f(s).$$

It is not difficult to show that

$$L_{G'\times G'}(s) = \zeta(s)^{17} L_f(s)^8 L_F(s).$$

(Note that both $L_{G'}(s)$ and $L_{G'\times G'}(s)$ are automorphic L-functions.) Thus, we obtain a better bound in this situation as

$$\int_T^{2T} \left| L_{G'\times G'}\left(\frac{1}{2} + \mathrm{i}t\right)\right|^2 \mathrm{d}t \ll T^{2687/210+\varepsilon},$$

where $\frac{2687}{210} \sim 12.795 < 18$.

Remark 1.3. Though there has been wide research on the classical problem of t-aspect subconvexity for lower degree L-functions, not much is known for the higher degrees. Recently in a pre-print, Nelson in [11] has announced a solution to the t-aspect subconvexity problem for standard L-functions, regardless of its degree. Theorem 1.1 indicates that we can get subconvexity bounds for higher degree L-functions whenever they have a nice decomposition into factors that have well-known subconvexity bounds.

As an application to Theorem 1.1, we study the distribution of coefficients of $L_{G\times G}(s)$ in the following theorem.

Theorem 1.2. *Let*

$$L_{G\times G}(s) := \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

which is absolutely convergent in $\Re(s) > 1$. *Then for any small* $\varepsilon > 0$ *we have*

$$\sum_{m\leqslant x} c(m) = xP_2(\log x) + O(x^{1669/1774+\varepsilon}).$$

Here $P_2(\log x)$ *denotes a polynomial in* $\log x$ *of degree two.*

Remark 1.4. From [8], for a general L-function

$$\mathfrak{L}(s) := \sum_{m=1}^{\infty} \frac{b(m)}{m^s}$$

with $b(m) \geqslant 0$, we have

$$\sum_{m\leqslant x} b(m) = xQ(\log x) + O(x^{(d-1)/(d+1)+\varepsilon}).$$

Here, Q is a polynomial of degree $\mathrm{ord}_{s=1}\mathfrak{L}(s) - 1$ and d is the degree of $\mathfrak{L}(s)$.

This result of Lau and Lü (see [8], Lemma 2.2) gives $\frac{35}{37}$ as the exponent of x in the error term. Since $L_{G\times G}(s)$ has nonnegative Dirichlet coefficients, Theorem 1.2 is an improvement.

Let us assume that

$$\mathfrak{L}(s) := \sum_{m=1}^{\infty} \frac{b(m)}{m^s}$$

is a product of two L-functions $\mathfrak{L}_1$, $\mathfrak{L}_2$ such that degree $\mathfrak{L}_1, \mathfrak{L}_2 \geqslant 2$. Let $\mathfrak{L}_1(s)$, $\mathfrak{L}_2(s)$ have Euler products and functional equations of the Riemann zeta type. Suppose that the coefficients of $\mathfrak{L}(s)$ satisfy the Ramanujan conjecture. Then for any $\varepsilon > 0$ we have (from [8])

$$\sum_{m\leqslant x} b(m) = xQ(\log x) + O(x^{1-2/d+\varepsilon}).$$

Here Q is a polynomial of degree $\operatorname{ord}_{s=1}\mathfrak{L}(s) - 1$ and d is the degree of $\mathfrak{L}(s)$.

Theorem 1.2 is also an improvement of this conditional result of Lau and Lü (see [8]) as this gives $\frac{17}{18}$ as the exponent of x in the error term.

2. Preliminaries

Let $\mathcal{L}^2(SL(2,\mathbb{Z}) \setminus \mathcal{H}^2)$ denote the vector space defined over $\mathbb{C}$, which is the completion of the subspace consisting of all smooth functions $f\colon SL(2,\mathbb{Z}) \setminus \mathcal{H}^2 \to \mathbb{C}$ satisfying the $\mathcal{L}^2$ condition

$$\iint_{SL(2,\mathbb{Z})\setminus\mathcal{H}^2} |f(z)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} < \infty.$$

Definition 2.1. Let $v_f \in \mathbb{C}$. A Maass form of type v_f for $SL(2,\mathbb{Z})$ is a nonzero function $f \in \mathcal{L}^2(SL(2,\mathbb{Z}) \setminus \mathcal{H}^2)$ which satisfies:

(1) $f(\gamma z) = f(z)$ for all $\gamma \in SL(2,\mathbb{Z})$, $z = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathcal{H}^2$;
(2) $\Delta f = v_f(1-v_f)f$;
(3) $\int_0^1 f(z)\,\mathrm{d}x = 0$.

Let f be a non-constant Maass form of type v_f for $SL(2,\mathbb{Z})$. Then for $z \in \mathcal{H}^2$, f has the Fourier-Whittaker expansion

$$f(z) = \sum_{m\neq 0} a(m)\sqrt{2\pi y}\cdot K_{v_f-1/2}(2\pi|m|y)\cdot \mathrm{e}^{2\pi \mathrm{i} mx}$$

with complex coefficients $a(m)$ and $m \in \mathbb{Z}$.

Throughout the paper, f is an eigenfunction of all the Hecke operators and is normalized so that $a(1) = 1$.

Definition 2.2. Let $s \in \mathbb{C}$. We define the L-function associated to f by

$$L_f(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s},$$

which is absolutely convergent in $\Re(s) > \frac{3}{2}$.

The generalized upper half-plane $\mathcal{H}^3$ consists of all matrices $z = x \cdot y$ with

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x_{1,2}, x_{1,3}, x_{2,3} \in \mathbb{R}$, $y_1, y_2 > 0$. Let Δ_1, Δ_2 denote the basis for the ring $\mathcal{D}^3$ of differential operators in $\partial/\partial x_{1,2}$, $\partial/\partial x_{1,3}$, $\partial/\partial x_{2,3}$, $\partial/\partial y_1$, $\partial/\partial y_2$ which commute with $GL(3, \mathbb{R})$.

Let $\mathcal{L}^2(SL(3, \mathbb{Z}) \setminus \mathcal{H}^3)$ be the vector space defined over $\mathbb{C}$ which is the completion of the subspace consisting of all smooth functions $F\colon SL(3, \mathbb{Z}) \setminus \mathcal{H}^3 \to \mathbb{C}$ satisfying the $\mathcal{L}^2$ condition

$$\int_{SL(3,\mathbb{Z})\setminus\mathcal{H}^3} |F(z)|^2 \,\mathrm{d}^* z < \infty.$$

Here $\mathrm{d}^*(z) = \mathrm{d}x_{1,2}\,\mathrm{d}x_{1,3}\,\mathrm{d}x_{2,3}\mathrm{d}y_1\,\mathrm{d}y_2/(y_1 y_2)^3$.

Definition 2.3. Let $v = (v_1, v_2) \in \mathbb{C}^2$. Let

$$a = -v_1 - 2v_2 + 1, \quad b = -v_1 + v_2, \quad c = 2v_1 + v_2 - 1$$
$$\text{and} \quad \lambda_1 = -1 - ab - bc - ca, \quad \lambda_2 = -abc.$$

A Maass form for $SL(3, \mathbb{Z})$ of type (v_1, v_2) is a smooth function $F \in \mathcal{L}^2(SL(3, \mathbb{Z})\setminus\mathcal{H}^3)$ which satisfies

(1) $F(\gamma z) = F(z)$ for all $\gamma \in SL(3, \mathbb{Z}), z \in \mathcal{H}^3$;
(2) $\Delta_1 F = \lambda_1 F$, $\Delta_2 F = \lambda_2 F$;
(3) $\int_0^1 \int_0^1 \int_0^1 F(z)\,\mathrm{d}x_{1,2}\,\mathrm{d}x_{1,3}\,\mathrm{d}x_{2,3} = 0$.

Definition 2.4. Let $F(z)$ be a Maass form of type $v = (v_1, v_2) \in \mathbb{C}^2$. Then

$$\widetilde{F}(z) := F(w \cdot (z^{-1})^\top \cdot w), \quad w = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix},$$

is a Maass form of type (v_2, v_1) for $SL(3, \mathbb{Z})$. The Maass form $\widetilde{F}$ is called the *dual Maass form.* If $A(m_1, m_2)$ is the (m_1, m_2)th Fourier coefficient of F, then $A(m_2, m_1)$ is the corresponding Fourier coefficient of $\widetilde{F}$.

Let

$$F(z) = \sum_{\gamma \in U_2(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1, m_2)}{|m_1 m_2|} \times W_J\left(\begin{pmatrix} |m_1 m_2| & & \\ & m_1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, v, \psi_{1, m_2/|m_2|}\right)$$

be a nonzero self-dual Maass form for $SL(3,\mathbb{Z})$ of type $v = (\frac{1}{3}2v_f, \frac{1}{3}2v_f)$. It is normalized so that $A(1,1) = 1$. It is a simultaneous eigenfunction of all the Hecke operators. Here, $A(m_1, m_2) \in \mathbb{C}$ and W_J is the Jacquet-Whittaker function. Further, $U_n(\mathbb{Z})$ denotes the group of $n \times n$ upper triangular matrices with 1s on the diagonal and integer entries above the diagonal and

$$\psi_{\varepsilon_1, \varepsilon_2}\left(\begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ & 1 & u_{2,3} \\ & & 1 \end{pmatrix}\right) = \mathrm{e}^{2\pi \mathrm{i}(\varepsilon_1 u_{2,3} + \varepsilon_2 u_{1,2})}$$

is a character of $U_3(\mathbb{Z})$.

Definition 2.5. Let $s \in \mathbb{C}$ with $\Re(s) > 2$. We define the Godement-Jacquet L-function $L_F(s)$ attached to F by the absolutely convergent series

$$L_F(s) = \sum_{m=1}^{\infty} \frac{A(m,1)}{m^s},$$

which is of degree three.

2.1. Euler products. Riemann zeta function has the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

further $L_f(s)$ has the Euler product

$$L_f(s) = \prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\beta_p}{p^s}\right)^{-1},$$

where for each prime p, α_p and β_p are complex numbers satisfying $\alpha_p \cdot \beta_p = 1$, $\alpha_p + \beta_p = a(p)$.

The Rankin-Selberg L-function $L_{f \times f}(s)$ has the Euler product

$$L_{f \times f}(s) = \prod_p \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{\beta_p^2}{p^s}\right)^{-1}.$$

From the Gelbart-Jacquet lift, we see that $L_F(s)$ has the Euler product

$$L_F(s) = \prod_p \Big(1 - \frac{\alpha_p^2}{p^s}\Big)^{-1} \Big(1 - \frac{1}{p^s}\Big)^{-1} \Big(1 - \frac{\beta_p^2}{p^s}\Big)^{-1}.$$

The jth symmetric power L-function of f has the Euler product

$$L_{\mathrm{sym}^j f}(s) = \prod_p \prod_{l=0}^{j} \Big(1 - \frac{\alpha_p^{j-l}\beta_p^l}{p^s}\Big)^{-1}.$$

2.2. Functional equations. It is well known that $\zeta(s)$ has meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = 1$ and satisfies the functional equation

$$\Lambda(s) := \pi^{-s/2}\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \Lambda(1-s).$$

The L-function $L_f(s)$ has a holomorphic continuation to all $s \in \mathbb{C}$ and satisfies the functional equation

$$\Lambda_f(s) := \pi^{-s}\Gamma\Big(\frac{s+\delta-1/2+v_f}{2}\Big)\Gamma\Big(\frac{s+\delta+1/2-v_f}{2}\Big)L_f(s) = (-1)^{\delta}\cdot \Lambda_f(1-s),$$

where $\delta = 0$ if f is even and $\delta = 1$ if f is odd.

The L-functions $L_F(s)$ and $L_{\widetilde{F}}(s)$ have holomorphic continuations to all $s \in \mathbb{C}$ and satisfy the functional equation

$$H_v(s)L_F(s) = \widetilde{H}_v(1-s)L_{\widetilde{F}}(1-s),$$

where

$$H_v(s) = \pi^{-3s/2}\Gamma\Big(\frac{s+1-2v_1-v_2}{2}\Big)\Gamma\Big(\frac{s+v_1-v_2}{2}\Big)\Gamma\Big(\frac{s-1+v_1+2v_2}{2}\Big),$$
$$\widetilde{H}_v(s) = \pi^{-3s/2}\Gamma\Big(\frac{s+1-v_1-2v_2}{2}\Big)\Gamma\Big(\frac{s-v_1+v_2}{2}\Big)\Gamma\Big(\frac{s-1+2v_1+v_2}{2}\Big).$$

Since in our case, $L_F(s)$ is a self-dual Maass form of type $(\frac{1}{3}2v_f, \frac{1}{3}2v_f)$, we have

$$H_v(s) = \widetilde{H}_v(s) = \pi^{-3s/2}\Gamma\Big(\frac{s+1-2v_f}{2}\Big)\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s-1+2v_f}{2}\Big),$$

and the functional equation takes the form

$$H_v(s)L_F(s) = H_v(1-s)L_F(1-s).$$

Let g be a Maass form for $SL(n,\mathbb{Z})$ and

$$L_g(s) = \prod_p \prod_{i=1}^{n} \Big(1 - \frac{\alpha_{p,i}}{p^s}\Big)^{-1}$$

be the L-function associated to g. Then the jth symmetric power L-function is given by

$$L_{\mathrm{sym}^j g}(s) = \prod_p \prod_{1\leqslant i_1\leqslant i_2\leqslant\ldots\leqslant i_j\leqslant n} \Big(1 - \frac{\alpha_{p,i_1}\alpha_{p,i_2}\ldots\alpha_{p,i_j}}{p^s}\Big)^{-1}.$$

Langlands in [7] conjectured that $L_{\mathrm{sym}^j g}(s)$ is the L-function related to a Maass form on $SL(M,\mathbb{Z})$, where

$$M = \sum_{1\leqslant i_1\leqslant i_2\leqslant\ldots\leqslant i_j\leqslant n} 1.$$

This is known for $n = 2$, $j = 2$ due to the Gelbart-Jacquet lift (see [2]) from $SL(2,\mathbb{Z})$ to $SL(3,\mathbb{Z})$. Kim and Shahidi in [6] proved this conjecture for $n = 2$, $j = 3$, which is the symmetric cube lift from $SL(2,\mathbb{Z})$ to $SL(4,\mathbb{Z})$. Kim in [5] obtained the symmetric fourth power lift from $SL(2,\mathbb{Z})$ to $SL(5,\mathbb{Z})$, which is the case when $n = 2$, $j = 4$.

The L-functions related to the symmetric cube and symmetric fourth power of f have holomorphic continuations to all $s\in\mathbb{C}$ and satisfy the functional equations

$$\Lambda_{\mathrm{sym}^3 f}(s) := \pi^{-2s}\prod_{i=1}^{4}\Gamma\Big(\frac{s-\lambda_i(v_{\mathrm{sym}^3 f})}{2}\Big)L_{\mathrm{sym}^3 f}(s) = \Lambda_{\mathrm{sym}^3 f}(1-s),$$
$$\Lambda_{\mathrm{sym}^4 f}(s) := \pi^{-5s/2}\prod_{i=1}^{5}\Gamma\Big(\frac{s-\lambda_i(v_{\mathrm{sym}^4 f})}{2}\Big)L_{\mathrm{sym}^4 f}(s) = \Lambda_{\mathrm{sym}^4 f}(1-s).$$

Note that f is self-dual since a Maass form for $SL(2,\mathbb{Z})$ is a self-dual Maass form.

2.3. Good factor. We call an L-function to be a good L-function if it has a subconvexity bound better than what any known moment bound would give. For example, the Riemann zeta function $\zeta(s)$ is a good L-function. We have the twelfth power moment of $\zeta(s)$ due to Heath-Brown (see [4])

$$\int_T^{2T}\Big|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\Big|^{12}\,\mathrm{d}t \ll T^{2+\varepsilon},$$

which yields that

$$\Big|\zeta\Big(\frac{1}{2}+\mathrm{i}T\Big)\Big| \ll T^{1/6+\varepsilon}.$$

But we know a better bound due to the work of Bourgain in [1], specifically,

$$\Big|\zeta\Big(\frac{1}{2}+\mathrm{i}T\Big)\Big| \ll T^{13/84+\varepsilon}.$$

We call a factor of good L-functions to be a good factor.

3. Lemmas

Lemma 3.1 ([1]). *For $\varepsilon > 0$ we have*

$$\Big|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\Big| \ll (|t|+1)^{13/84+\varepsilon}.$$

Lemma 3.2 ([10]). *Let $\varepsilon > 0$ and f be a Hecke-Maass form for $SL(2,\mathbb{Z})$. Then we have the weyl bound*

$$\Big|L_f\Big(\frac{1}{2}+\mathrm{i}t\Big)\Big| \ll (|t|+1)^{1/3+\varepsilon}.$$

Lemma 3.3 ([9]). *Let F be a self-dual Hecke-Maass cusp form for $SL(3,\mathbb{Z})$. Then for $\varepsilon > 0$ we have*

$$\Big|L_F\Big(\frac{1}{2}+\mathrm{i}t\Big)\Big| \ll (|t|+1)^{3/5+\varepsilon}.$$

Lemma 3.4 ([12]). *For a general L-function $\mathfrak{L}(s)$ of degree d and any $\varepsilon > 0$ we have*

$$\int_T^{2T} |\mathfrak{L}(\sigma+\mathrm{i}t)|^2\,\mathrm{d}t \ll T^{d(1-\sigma)+\varepsilon}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$ and $T \geqslant 1$. Moreover,

$$|\mathfrak{L}(\sigma+\mathrm{i}t)| \ll (|t|+1)^{\max\{d(1-\sigma)/2,0\}+\varepsilon}$$

uniformly for $-\varepsilon \leqslant \sigma \leqslant 1+\varepsilon$.

Lemma 3.5. *If $L_G(s) = \zeta(s)L_f(s)L_F(s)$, then the Rankin-Selberg convolution of G with itself has the decomposition*

$$L_{G\times G}(s) = \zeta(s)^3 L_f(s)^4 L_F(s)^4 L_{\mathrm{sym}^3 f}(s)^2 L_{\mathrm{sym}^4 f}(s). \tag{3.1}$$

Proof. If

$$L_G(s) = \zeta(s)L_f(s)L_F(s),$$

then $L_G(s)$ has the Euler product

$$L_G(s) = \prod_p \Big(1-\frac{\alpha_p^2}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_p}{p^s}\Big)^{-1}\Big(1-\frac{1}{p^s}\Big)^{-2}\Big(1-\frac{\beta_p}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p^2}{p^s}\Big)^{-1}.$$

From this, we can write the Euler product of the convolution L-function as

$$L_{G\times G}(s) = \prod_p \Big(1-\frac{\alpha_p^4}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_p^3}{p^s}\Big)^{-2}\Big(1-\frac{\alpha_p^2}{p^s}\Big)^{-5}\Big(1-\frac{\alpha_p}{p^s}\Big)^{-6}\Big(1-\frac{1}{p^s}\Big)^{-8}$$
$$\times \Big(1-\frac{\beta_p}{p^s}\Big)^{-6}\Big(1-\frac{\beta_p^2}{p^s}\Big)^{-5}\Big(1-\frac{\beta_p^3}{p^s}\Big)^{-2}\Big(1-\frac{\beta_p^4}{p^s}\Big)^{-1}.$$

Each term on the right-hand side of identity (3.1) has the following Euler products:

$$\zeta(s) = \prod_p \Big(1-\frac{1}{p^s}\Big)^{-1},$$
$$L_f(s) = \prod_p \Big(1-\frac{\alpha_p}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p}{p^s}\Big)^{-1},$$
$$L_F(s) = \prod_p \Big(1-\frac{\alpha_p^2}{p^s}\Big)^{-1}\Big(1-\frac{1}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p^2}{p^s}\Big)^{-1},$$
$$L_{\mathrm{sym}^3 f}(s) = \prod_p \Big(1-\frac{\alpha_p^3}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_p}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p^3}{p^s}\Big)^{-1},$$
$$L_{\mathrm{sym}^4 f}(s) = \prod_p \Big(1-\frac{\alpha_p^4}{p^s}\Big)^{-1}\Big(1-\frac{\alpha_p^2}{p^s}\Big)^{-1}\Big(1-\frac{1}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p^2}{p^s}\Big)^{-1}\Big(1-\frac{\beta_p^4}{p^s}\Big)^{-1}.$$

Comparing Euler products on both sides, we get the identity. □

Remark 3.1. It should be noted that there is a typo in a similar identity on page 257 of [3].

4. Proof of Theorem 1.1

From Section 2.2, we can see that each factor on the right-hand side of equation (3.1) has a functional equation of the Riemann zeta type so that we have $0 \leqslant \Re(s) \leqslant 1$ as the critical strip for $L_{G\times G}(s)$. It should be understood that the coefficients of the L-functions $L_f(s)$, $L_F(s)$, $L_{\mathrm{sym}^3 f}(s)$, $L_{\mathrm{sym}^4 f}(s)$ are normalized in such a way that the critical strip for these L-functions is $0 \leqslant \Re(s) \leqslant 1$.

Further, by Lemmas 3.1, 3.2 and 3.3, it is easy to see that $\zeta(s)$, $L_f(s)$ and $L_F(s)$ are good L-functions. Hence, in the decomposition of $L_{G\times G}(s)$ given in Lemma 3.5, the factor $\zeta(s)^3 L_f(s)^4 L_F(s)^4$ is a good factor. For this good factor, we use the subconvexity bounds given in Lemmas 3.1, 3.2 and 3.3 and use the general mean square result given in Lemma 3.4 for the remaining factor. Thus, we obtain

$$
\begin{aligned}
&\int_T^{2T} \left|L_{G\times G}\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \mathrm{d}t \\
&\quad = \int_T^{2T} \left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)^3 L_f\Big(\frac{1}{2}+\mathrm{i}t\Big)^4 L_F\Big(\frac{1}{2}+\mathrm{i}t\Big)^4 L_{\mathrm{sym}^3 f}\Big(\frac{1}{2}+\mathrm{i}t\Big)^2 L_{\mathrm{sym}^4 f}\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \mathrm{d}t \\
&\quad \ll \max_{T\leqslant t\leqslant 2T}\left(\left|\zeta\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^6 \left|L_f\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^8 \left|L_F\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^8\right) \\
&\qquad \times \int_T^{2T} \left|L_{\mathrm{sym}^3 f}\Big(\frac{1}{2}+\mathrm{i}t\Big)^2 L_{\mathrm{sym}^4 f}\Big(\frac{1}{2}+\mathrm{i}t\Big)\right|^2 \mathrm{d}t \\
&\quad \ll T^{(6\times 13/84)+(8\times 1/3)+(8\times 3/5)+13/2+\varepsilon} \ll T^{1564/105+\varepsilon}.
\end{aligned}
$$

This proves the theorem. □

5. Proof of Theorem 1.2

From Section 2.2, we can see that each L-function on the right-hand side of identity 3.1 of Lemma 3.5 except the Riemann zeta function is entire. Also, $L_{G\times G}(s)$ has nonnegative Dirichlet coefficients. By Landau's lemma, a Dirichlet series with nonnegative coefficients must be absolutely convergent up to its first pole. This implies that $L_{G\times G}(s)$ converges absolutely for $\Re(s) > 1$.

Using Perron's formula, we get

$$
\sum_{m\leqslant x} c(m) = \frac{1}{2\pi i}\int_{1+\varepsilon-\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T} \frac{L_{G\times G}(s)x^s}{s}\,\mathrm{d}s + O\Big(\frac{x^{1+\varepsilon}}{T}\Big).
$$

We move the line of integration to $\Re(s) = \frac{1}{2}$. From equation (3.1), we can see that $s = 1$ is a pole of $L_{G\times G}(s)$ of order three, which contributes a residue $xP_2(\log x)$. Here $P_2(\log x)$ is a degree two polynomial in $\log x$.

By Cauchy's residue formula,

$$
\begin{aligned}
\sum_{m\leqslant x} c(m) = xP_2(\log x) + \frac{1}{2\pi i}\left[\int_{1+\varepsilon-\mathrm{i}T}^{1/2-\mathrm{i}T} + \int_{1/2-\mathrm{i}T}^{1/2+\mathrm{i}T} + \int_{1/2+\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T}\right] \frac{L_{G\times G}(s)x^s}{s}\,\mathrm{d}s \\
+ O\Big(\frac{x^{1+\varepsilon}}{T}\Big).
\end{aligned}
$$

Horizontal lines contribute in absolute value:

$$
\begin{aligned}
&\left|\frac{1}{2\pi \mathrm{i}}\int_{1/2+\mathrm{i}T}^{1+\varepsilon+\mathrm{i}T} \frac{L_{G\times G}(s)x^s}{s}\,\mathrm{d}s\right| \\
&\qquad = \left|\frac{1}{2\pi \mathrm{i}}\int_{1/2}^{1+\varepsilon} \frac{L_{G\times G}(\sigma+\mathrm{i}T)x^{\sigma+\mathrm{i}T}}{\sigma+\mathrm{i}T}\,\mathrm{d}\sigma\right| \ll \int_{1/2}^{1+\varepsilon} \frac{|L_{G\times G}(\sigma+\mathrm{i}T)|x^{\sigma}}{|\sigma+\mathrm{i}T|}\,\mathrm{d}\sigma
\end{aligned}
$$

$$\ll \frac{1}{T} \max_{1/2 \leqslant \sigma \leqslant 1+\varepsilon} x^{\sigma} T^{1564(1-\sigma)/105+\varepsilon} \ll T^{1459/105} \max_{1/2 \leqslant \sigma \leqslant 1+\varepsilon} \Big(\frac{x}{T^{1564/105}}\Big)^{\sigma}$$
$$\ll x^{1/2} T^{677/105} + \frac{x^{1+\varepsilon}}{T}.$$

Here we have used the fact that $(x/T^{1564/105})^{\sigma}$ is a monotonic function of σ and hence attains its maximum at one of the endpoints of the interval $[\frac{1}{2}, 1+\varepsilon]$.

The left vertical line contributes in absolute value:

$$\Big|\frac{1}{2\pi \mathrm{i}} \int_{1/2-\mathrm{i}T}^{1/2+\mathrm{i}T} \frac{L_{G\times G}(s) x^s}{s}\, \mathrm{d}s\Big| \leqslant \Big|\frac{1}{2\pi \mathrm{i}} \int_{\substack{|t|\leqslant t_0 \\ \sigma=1/2}} \frac{L_{G\times G}(s) x^s}{s}\, \mathrm{d}s\Big| + \Big|\frac{1}{2\pi \mathrm{i}} \int_{\substack{t_0<|t|\leqslant T \\ \sigma=1/2}} \frac{L_{G\times G}(s) x^s}{s}\, \mathrm{d}s\Big|.$$

By Theorem 1.1, we have

$$\begin{aligned}\Big|\frac{1}{2\pi \mathrm{i}} \int_{\substack{|t|\leqslant t_0 \\ \sigma=1/2}} \frac{L_{G\times G}(s) x^s}{s}\, \mathrm{d}s\Big| &= \Big|\frac{1}{2\pi \mathrm{i}} \int_{|t|\leqslant t_0} \frac{L_{G\times G}(1/2+\mathrm{i}t) x^{1/2+\mathrm{i}t}}{1/2+\mathrm{i}t} \mathrm{i}\, \mathrm{d}t\Big| \\ &\ll \int_{|t|\leqslant t_0} \Big|L_{G\times G}\Big(\frac{1}{2}+\mathrm{i}t\Big)\Big| x^{1/2}\, \mathrm{d}t \\ &\ll x^{1/2} \int_{|t|\leqslant t_0} (|t|+1)^{782/105+\varepsilon}\, \mathrm{d}t \ll x^{1/2}\end{aligned}$$

and

$$\begin{aligned}\Big|\frac{1}{2\pi \mathrm{i}} \int_{\substack{t_0<|t|\leqslant T \\ \sigma=1/2}} \frac{L_{G\times G}(s) x^s}{s}\, \mathrm{d}s\Big| &= \Big|\frac{1}{2\pi \mathrm{i}} \int_{t_0<|t|\leqslant T} \frac{L_{G\times G}(1/2+\mathrm{i}t) x^{1/2+\mathrm{i}t}}{1/2+\mathrm{i}t} \mathrm{i}\, \mathrm{d}t\Big| \\ &\ll \int_{t_0<|t|\leqslant T} \frac{t^{782/105+\varepsilon} x^{1/2}}{t}\, \mathrm{d}t \ll x^{1/2} T^{782/105+\varepsilon}.\end{aligned}$$

Combining all these estimates, we get

$$\begin{aligned}\sum_{m\leqslant x} c(m) &= xP_2(\log x) + O(x^{1/2}T^{677/105}) + O(x^{1/2}) \\ &\quad + O(x^{1/2}T^{782/105+\varepsilon}) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big) \\ &= xP_2(\log x) + O(x^{1/2}T^{782/105+\varepsilon}) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big).\end{aligned}$$

We choose T such that

$$x^{1/2}T^{782/105+\varepsilon} \sim \frac{x^{1+\varepsilon}}{T} \quad \text{i.e., } T \sim x^{105/1774}.$$

Substituting this value of T, we get the desired result.

References

[1] *J. Bourgain*: Decoupling, exponential sums and the Riemann zeta function. J. Am. Math. Soc. *30* (2017), 205–224. zbl MR doi

[2] *S. Gelbart, H. Jacquet*: A relation between automorphic representations of GL(2) and GL(3). Ann. Sci. Éc. Norm. Supér. (4) *11* (1978), 471–542. zbl MR doi

[3] *D. Goldfeld*: Automorphic Forms and L-Functions for the Group $\mathrm{GL}(n,\mathbb{R})$. Cambridge Studies in Advanced Mathematics 99. Cambridge University Press, Cambridge, 2006. zbl MR doi

[4] *D. R. Heath-Brown*: The twelfth power moment of the Riemann-function. Q. J. Math. *29* (1978), 443–462. zbl MR doi

[5] *H. H. Kim*: Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2. J. Am. Math. Soc. *16* (2003), 139–183. zbl MR doi

[6] *H. H. Kim, F. Shahidi*: Functorial products for $\mathrm{GL}_2 \times \mathrm{GL}_3$ and the symmetric cube for GL_2. Ann. Math. (2) *155* (2002), 837–893. zbl MR doi

[7] *R. P. Langlands*: Problems in the theory of automorphic forms. Lectures in Modern Analysis and Applications III. Lecture Notes in Mathematics 170. Springer, Berlin, 1970, pp. 18–61. zbl MR doi

[8] *Y.-K. Lau, G. Lü*: Sums of Fourier coefficients of cusp forms. Q. J. Math. *62* (2011), 687–716. zbl MR doi

[9] *Y. Lin, R. Nunes, Z. Qi*: Strong subconvexity for self-dual GL(3) L-functions. Int. Math. Res. Not. *153* (2022), 11453–11470. zbl MR doi

[10] *T. Meurman*: On the order of the Maass L-function on the critical line. Number Theory. Volume 1. Colloquia Mathematica Societatis János Bolyai 51. North-Holland, Amsterdam, 1990, pp. 325–354. zbl MR

[11] *P. D. Nelson*: Bounds for standard L-functions. Available at `https://arxiv.org/abs/2109.15230` (2021), 237 pages. doi

[12] *A. Perelli*: General L-functions. Ann. Mat. Pura Appl., IV. Ser. *130* (1982), 287–306. zbl MR doi

[13] *R. A. Rankin*: Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions I. Proc. Camb. Philos. Soc. *35* (1939), 351–356. zbl MR doi

[14] *A. Selberg*: Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist. Arch. Math. Naturvid. B *43* (1940), 47–50. (In German.) zbl MR

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