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ON A SUM INVOLVING THE INTEGRAL PART FUNCTION

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Abstract. Let $[t]$ be the integral part of a real number t, and let f be the arithmetic function satisfying some simple condition. We establish a new asymptotical formula for the sum $S_f(x) = \sum_{n\leqslant x} f([x/n])$, which improves the recent result of J. Stucky (2022).

Keywords: asymptotical formula; exponential sum; exponential pair; integral part

MSC 2020: 11L07, 11N37

1. Introduction

Let $[t]$ denote the integral part of a real number t. The problem of evaluating the sum

$$S_f(x) = \sum_{n\leqslant x} f\Big(\Big[\frac{x}{n}\Big]\Big),$$

where f is an arithmetic function, has received much attention. For example, when $f(n) = n$, the problem becomes the Dirichlet divisor problem. If f is any other arithmetic function, the problem was first studied by Bordellès et al., see [3]. Subsequently, Wu in [10] and Zhai in [11] independently showed that

$$S_f(x) = C_f x + O(x^{(1+\alpha)/2}(\log x)^{\theta}),$$

where

$$C_f = \sum_{n=1}^{\infty} \frac{f(n)}{n(n+1)} \quad \text{and} \quad f(n) \ll n^{\alpha}(\log n)^{\theta}$$

for some $\alpha \in [0,1)$ and $\theta \geqslant 0$. Most recently, if f satisfies $f(n) = \sum_{d\mid n} g(d)$ with

$$\sum_{d\leqslant x} |g(d)| \ll x^{\alpha}(\log x)^{\theta}$$

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for some $\alpha \in [0,1)$ and $\theta \geqslant 0$, Stucky in [9] showed that

$$S_f(x) = C_f x + O(x^{(1+\alpha)/(3-\alpha)}(\log x)^\theta),$$

where θ should be replaced by $\max\{1, \theta\}$ if $\alpha = 0$. For some specific f, one can also refer to [5], [7], [6].

In this paper, motived by the work of Stucky (see [9]), we prove Theorem 1.1, which is indeed an improvement of the result of Stucky (see [9]) in the exponent of x, but a possible loss in the exponent of $\log x$.

Theorem 1.1. *Suppose* $f(n) = \sum_{d \mid n} g(d)$ *and, further,*

$$\sum_{d \leqslant x} |g(d)| \ll x^\alpha (\log x)^\theta \quad (x \to \infty) \tag{1.1}$$

for some $\alpha \in [0,1)$ *and* $\theta \geqslant 0$.

(1) *If* $0 \leqslant \alpha < \frac{1}{3}$, *then*

$$S_f(x) = C_f x + O(x^{1/3} \log x).$$

(2) *If* $\frac{1}{3} \leqslant \alpha < 1$, *then*

$$S_f(x) = C_f x + O(x^\alpha (\log x)^{1+\theta}).$$

Remark 1.1. Clearly, $\frac{1}{3}$ is smaller than $(1+\alpha)/(3-\alpha)$ when $0 < \alpha < \frac{1}{3}$. On the other hand, when $\frac{1}{3} \leqslant \alpha < 1$, we have $\alpha < (1+\alpha)/(3-\alpha) \Leftrightarrow 3\alpha - \alpha^2 < 1 + \alpha \Leftrightarrow 0 < (\alpha - 1)^2$. Thus, the exponent of x in Theorem 1.1 is better than that in [9].

Corollary 1.1. *Let* f *be the divisor function*

$$f(n) := \sigma_{-1/2}(n) = \sum_{d \mid n} d^{-1/2}.$$

Then, we have

$$S_f(x) = C_f x + O(x^{1/2} \log x).$$

Proof. Clearly $g(d) = d^{-1/2}$ at this time, thus $\sum_{d \leqslant x} |g(d)| \ll x^{1/2}$. Taking $\alpha = \frac{1}{2}$ in Theorem 1.1, we are done. $\square$

For $n \geqslant 1$, the Euler totient function $\varphi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n.

Corollary 1.2. *Let f be the function*

$$f(n) := \frac{n}{\varphi(n)}.$$

Then, we have

$$S_f(x) = C_f x + O(x^{1/3}\log x).$$

P r o o f. Since

$$\frac{n}{\varphi(n)} = \sum_{d\mid n} \frac{\mu^2(d)}{\varphi(d)}$$

(see Exercise 2.3 in [1]) and

$$\sum_{d\leqslant x} \frac{1}{\varphi(d)} = O(\log x)$$

(see Exercise 3.10 in [1]), we have

$$\sum_{d\leqslant x} |g(d)| = \sum_{d\leqslant x} \frac{1}{\varphi(d)} \ll \log x$$

at this time. Taking $\alpha = 0$ in Theorem 1.1, we are done. □

2. Notation

Throughout this paper, $U \ll V$ and $U = O(V)$ both mean $|U| \leqslant CV$ with some constant $C > 0$. We also write $n \asymp N$ to denote that $N < n \leqslant 2N$. We let, as usual, $\psi(x) = x - [x] - \frac{1}{2}$ and $e(t) = \mathrm{e}^{2\pi \mathrm{i} t}$.

3. Preliminary lemmas

To prove Theorem 1.1, we need some lemmas.

Lemma 3.1. *Let $H \geqslant 1$. There are functions $a(h)$ and $b(h)$, such that for $|h| \leqslant H$ we have*

$$a(h) \ll \frac{1}{|h|}, \quad b(h) \ll \frac{1}{H}, \quad \text{and} \quad \Bigl|\psi(t) - \sum_{1\leqslant |h|\leqslant H} a(h)e(ht)\Bigr| \leqslant \sum_{0\leqslant |h|\leqslant H} b(h)e(ht).$$

P r o o f. See Theorem A.6 in [4]. □

The next two lemmas play a key role in the proof of Theorem 1.1. To prove them we make use of exponent pairs, the definition and basic properties of which can be found in Section 3.3 in [4] or Section 6.6.3 in [2].

Lemma 3.2.

$$\sum_{n\asymp N}\psi\Big(\frac{x/d}{n+\delta}\Big)\ll\Big(\frac{Hx}{dN}\Big)^{1/2}+\frac{dN^2}{x}+\frac{N}{H}\quad(x\to\infty),\tag{3.1}$$

where $1\leqslant H\leqslant N\leqslant x/d$ *and* $0\leqslant\delta\leqslant 1$.

Moreover, we get that

$$\sum_{n\leqslant\sqrt{x/d}}\psi\Big(\frac{x/d}{n+\delta}\Big)\ll\Big(\frac{x}{d}\Big)^{1/3}\log x\quad(x\to\infty).\tag{3.2}$$

P r o o f. From Lemma 3.1, we find that

$$\sum_{n\asymp N}\psi\Big(\frac{x/d}{n+\delta}\Big)\ll T_1+T_2+T_3,\tag{3.3}$$

where

$$T_1=\sum_{n\asymp N}\sum_{1\leqslant|h|\leqslant H}a(h)e\Big(\frac{hx/d}{n+\delta}\Big),\quad T_2=\sum_{n\asymp N}\sum_{1\leqslant|h|\leqslant H}b(h)e\Big(\frac{hx/d}{n+\delta}\Big),$$

and

$$T_3=\sum_{n\asymp N}b(0).$$

Applying the exponent pair (k,λ) to the sum over n, we obtain

$$\sum_{n\asymp N}e\Big(\frac{hx/d}{n+\delta}\Big)\ll\Big(\frac{hx}{dN^2}\Big)^k(N)^\lambda+\frac{dN^2}{hx}.\tag{3.4}$$

Taking $(k,\lambda)=(\frac12,\frac12)$ in (3.4), we derive that

$$T_1\ll\sum_{1\leqslant|h|\leqslant H}a(h)\Big(\Big(\frac{hx}{dN}\Big)^{1/2}+\frac{dN^2}{hx}\Big)\ll\Big(\frac{Hx}{dN}\Big)^{1/2}+\frac{dN^2}{x}.\tag{3.5}$$

Clearly

$$T_2\ll T_1,\tag{3.6}$$

and

$$T_3\ll\frac{N}{H}.\tag{3.7}$$

From (3.3), (3.5), (3.6) and (3.7), we obtain (3.1). Now we write

$$\sum_{n\leqslant\sqrt{x/d}}\psi\Big(\frac{x/d}{n+\delta}\Big)=S_1+S_2,$$

where

$$S_1 = \sum_{n \leqslant (x/d)^{1/3}} \psi\Big(\frac{x/d}{n+\delta}\Big), \quad \text{and} \quad S_2 = \sum_{(x/d)^{1/3} < n \leqslant \sqrt{x/d}} \psi\Big(\frac{x/d}{n+\delta}\Big).$$

Clearly $S_1 \ll (x/d)^{1/3}$. Let us proceed to estimate S_2.

Taking $N/H = (x/d)^{1/3}$ in (3.1), we get that

$$\sum_{n \asymp N} \psi\Big(\frac{x/d}{n+\delta}\Big) \ll \Big(\frac{x}{d}\Big)^{1/3} + \frac{dN^2}{x}.$$

Hence,

$$S_2 \ll \Big(\frac{x}{d}\Big)^{1/3} \log x.$$

Therefore, we arrive at (3.2). □

Lemma 3.3 is Euler's summation formula, which is Theorem 3.1 in [1].

Lemma 3.3. *If f has a continuous derivative f' on the interval $[y,x]$, where $0 < y < x$, then*

$$\sum_{y < n \leqslant x} f(n) = \int_y^x f(t)\,\mathrm{d}t + \int_y^x (t - [t]) f'(t)\,\mathrm{d}t + f(x)([x] - x) - f(y)([y] - y).$$

Lemma 3.4. *Let $\sigma_u(x)$ be defined by*

$$\sigma_u(x) = \sum_{n \leqslant u} \Big(\psi\Big(\frac{x/d}{n}\Big) - \psi\Big(\frac{x/d}{n+1/d}\Big)\Big),$$

where $\sqrt{x/d} < u \leqslant x/d$. Then we have

$$\sigma_u(x) \ll \Big(\frac{x}{d}\Big)^{1/3} \log x \quad (x \to \infty)$$

uniformly in $d \leqslant x$.

Proof. Following the calculation of $\sigma_u(x)$ on page 219 in [8], we write

$$\sigma_u(x) = H_0(x,u) - H_{1/d}(x,u), \quad H_\delta(x,u) := \sum_{n \leqslant u} \Big\{\frac{x/d}{n+\delta}\Big\}.$$

Let us consider the planar domain $D_\delta\colon\ \sqrt{x/d} < a \leqslant u$, $0 < b \leqslant x/(d(a+\delta))$; denote by $L(D_\delta)$ the number of lattice points in D_δ and by $V(D_\delta)$ the area. Then we have on the one hand (with $F(a) := x/(d(a+\delta))$)

$$L(D_\delta) = \sum_{\sqrt{x/d} < a \leqslant u} [F(a)] = \sum_{\sqrt{x/d} < a \leqslant u} F(a) - H_\delta(x,u) + \sum_{a \leqslant \sqrt{x/d}} \{F(a)\}.$$

By Lemma 3.2, we get that

$$\sum_{a\leqslant\sqrt{x/d}}\psi(F(a))=O\Big(\Big(\frac{x}{d}\Big)^{1/3}\log x\Big). \tag{3.8}$$

Hence, applying Lemma 3.3 to $\sum F(a)$ and using the estimate

$$\int_{\sqrt{x/d}}^{u}\psi(a)F'(a)\,\mathrm{d}a\ll\max_{\sqrt{x/d}\leqslant a\leqslant u}\|F'(a)\|\ll 1$$

(based on the second mean-value theorem), we obtain

$$\begin{aligned}L(D_\delta)=V(D_\delta)-\psi(u)F(u)+\psi\Big(\sqrt{\frac{x}{d}}\Big)F\Big(\sqrt{\frac{x}{d}}\Big)-H_\delta(x,u)\\+\frac{1}{2}\sqrt{\frac{x}{d}}+O\Big(\Big(\frac{x}{d}\Big)^{1/3}\log x\Big).\end{aligned} \tag{3.9}$$

On the other hand (counting the lattice points "in the other direction") we get (with $G=F^{-1}$, i.e, $G(b)=x/(db)-\delta$)

$$L(D_\delta)=\Big([u]-\Big[\sqrt{\frac{x}{d}}\Big]\Big)[F(u)]+\sum_{F(u)<b\leqslant F(\sqrt{x/d})}[G(b)]-\Big(\Big[F\Big(\sqrt{\frac{x}{d}}\Big)\Big]-[F(u)]\Big)\Big[\sqrt{\frac{x}{d}}\Big].$$

Similar to the estimate of (3.8), we infer that

$$\sum_{F(u)<b\leqslant F(\sqrt{x/d})}\psi(G(b))=O\Big(\Big(\frac{x}{d}\Big)^{1/3}\log x\Big).$$

Now we apply Lemma 3.3 to $\sum G(b)$, simplify and collect the terms which represent $V(D_\delta)$. This yields

$$\begin{aligned}L(D_\delta)=V(D_\delta)-\frac{u}{2}+\frac{\sqrt{x/d}}{2}-\psi(u)F(u)+\psi\Big(\sqrt{\frac{x}{d}}\Big)F\Big(\sqrt{\frac{x}{d}}\Big)\\+\int_{F(u)}^{F(\sqrt{x/d})}\psi(b)G'(b)\,\mathrm{d}b+O\Big(\Big(\frac{x}{d}\Big)^{1/3}\log x\Big).\end{aligned} \tag{3.10}$$

Let us put

$$J_\delta(x,u):=\int_{1/2}^{F(u)}\psi(b)G'(b)\,\mathrm{d}b,\quad C:=\int_{1/2}^{\infty}\psi(b)b^{-2}\,\mathrm{d}b$$

and observe that

$$\int_{F(\sqrt{x/d})}^{\infty}\psi(b)G'(b)\,\mathrm{d}b=O(1),$$

then

$$\int_{F(u)}^{F(\sqrt{x/d})} \psi(b)G'(b)\,\mathrm{d}b = -\frac{Cx}{d} - J_\delta(x,u) + O(1).$$

Substituting this into (3.10) and comparing with (3.9) we obtain

$$H_\delta(x,u) = \frac{u}{2} + \frac{Cx}{d} + J_\delta(x,u) + O\Big(\Big(\frac{x}{d}\Big)^{1/3}\log x\Big).$$

Now

$$J_0(x,u) - J_{1/d}(x,u) = -\frac{x}{d}\int_{x/(du+1)}^{x/(du)} \psi(b)b^{-2}\,\mathrm{d}b \ll \frac{x}{d}\int_{x/(du+1)}^{x/(du)} b^{-2}\,\mathrm{d}b \ll 1,$$

hence,

$$\sigma_u(x) = H_0(x,u) - H_{1/d}(x,u) = O\Big(\Big(\frac{x}{d}\Big)^{1/3}\log x\Big)$$

for $\sqrt{x/d} \leqslant u \leqslant x/d$, uniformly in $d \leqslant x$. □

4. Proof of Theorem 1.1

Taking $m = [x/n]$, we then have $m \leqslant x/n < m+1$ and

$$\begin{aligned}
\sum_{n\leqslant x} f\Big(\Big[\frac{x}{n}\Big]\Big) &= \sum_{m\leqslant x} f(m) \sum_{x/(m+1)<n\leqslant x/m} 1 \\
&= \sum_{m\leqslant x} f(m)\Big(\frac{x}{m} - \psi\Big(\frac{x}{m}\Big) - \frac{x}{m+1} + \psi\Big(\frac{x}{m+1}\Big)\Big) \\
&= W_1 + W_2,
\end{aligned} \tag{4.1}$$

where

$$W_1 = x\sum_{m\leqslant x} \frac{f(m)}{m(m+1)} \quad \text{and} \quad W_2 = \sum_{m\leqslant x} f(m)\Big(\psi\Big(\frac{x}{m+1}\Big) - \psi\Big(\frac{x}{m}\Big)\Big).$$

Recalling the definition of f, we get that

$$W_1 = C_f x + O\bigg(x\sum_{m>x} \frac{m^\alpha(\log m)^\theta}{m(m+1)}\bigg) = C_f x + O(x^\alpha(\log x)^\theta). \tag{4.2}$$

We proceed to estimate W_2. By Lemma 3.4 and partial summation, we have

$$\begin{aligned}
W_2 &= \sum_{m\leqslant x} f(m)\Big(\psi\Big(\frac{x}{m+1}\Big) - \psi\Big(\frac{x}{m}\Big)\Big) = \sum_{d\leqslant x} g(d)(-\sigma_{x/d}(x)) \\
&\ll \sum_{d\leqslant x} g(d)\Big(\frac{x}{d}\Big)^{1/3}\log x \ll (1 + x^{\alpha-1/3}(\log x)^\theta)x^{1/3}\log x.
\end{aligned} \tag{4.3}$$

Case 1: $0 \leqslant \alpha < \frac{1}{3}$. In this case,

$$W_2 \ll x^{1/3} \log x.$$

Substituting this and (4.2) into (4.1), we obtain

$$\sum_{n \leqslant x} f\Big(\Big[\frac{x}{n}\Big]\Big) = C_f x + O(x^{1/3} \log x).$$

Case 2: $\frac{1}{3} \leqslant \alpha < 1$. In this case,

$$W_2 \ll x^{\alpha} (\log x)^{1+\theta}.$$

Substituting this and (4.2) into (4.1), we obtain

$$\sum_{n \leqslant x} f\Big(\Big[\frac{x}{n}\Big]\Big) = C_f x + O(x^{\alpha} (\log x)^{1+\theta}).$$

The theorem is proved. □

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