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LIPSCHITZ CONSTANTS FOR A HYPERBOLIC TYPE METRIC UNDER MÖBIUS TRANSFORMATIONS

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Abstract. Let D be a nonempty open set in a metric space (X,d) with $\partial D \neq \emptyset$. Define

$$h_{D,c}(x,y) = \log\left(1 + c\frac{d(x,y)}{\sqrt{d_D(x)d_D(y)}}\right),$$

where $d_D(x) = d(x,\partial D)$ is the distance from x to the boundary of D. For every $c \geqslant 2$, $h_{D,c}$ is a metric. We study the sharp Lipschitz constants for the metric $h_{D,c}$ under Möbius transformations of the unit ball, the upper half space, and the punctured unit ball.

Keywords: Lipschitz constant; hyperbolic type metric; Möbius transformation

MSC 2020: 51M10, 30C65

1. Introduction

In geometric function theory, numerous hyperbolic type metrics including the quasihyperbolic metric, distance ratio metric, and Apollonian metric, as extensions of the classical hyperbolic metric have been used and become standard tools, see [4], [7], [8], [9], [10], [11], [24]. Very recently, Mocanu in [19] studied metric-preserving functions with respect to intrinsic metrics of hyperbolic type which yield new metrics. Conformal invariants and conformally invariant metrics are the key notions of geometric function theory and of quasiconformal mapping theory, see [1], [2], [4], [24]. However, for hyperbolic type metrics we cannot anymore expect the same invariance properties as the classical hyperbolic metric under conformal mappings. Fortunately, the hyperbolic type metrics still have some type of "near-invariance" or "quasi-invariance" as desirable feature under (quasi)conformal mappings. It is natural to ask what the Lipschitz constants are for these hyperbolic type metrics under

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(quasi)conformal mappings. For example, Gehring, Osgood, and Palka proved that these metrics are not changed by more than a factor 2 under Möbius transformations, see [9], [10], and the similar results for the other hyperbolic type metrics can be found in [5], [11], [12], [13], [15], [16], [17], [18], [20], [25], [26].

Let D be a nonempty open set in a metric space (X, d) with $\partial D \neq \emptyset$. Define

$$h_{D,c}(x,y) = \log\biggl(1 + c\frac{d(x,y)}{\sqrt{d_D(x)d_D(y)}}\biggr),$$

where $d_D(x) = d(x, \partial D)$ is the distance from x to the boundary of D. Dovgoshey, Hariri, and Vuorinen in [6] proved that $h_{D,c}$ is a metric for every $c \geqslant 2$ and the constant $c = 2$ is the best possible in the sense that for each $c \in (0,2)$, there exists an open set D such that the triangle inequality for $h_{D,c}$ fails. Before the work [6], Hästö in [14] has studied $h_{D,c}$ in the case when $X = \mathbb{R}^n$ and $D = \mathbb{R}^n\backslash\{0\}$. The metric $h_{D,c}$, as other hyperbolic type metrics, has applications in the study of geometric function theory, such as estimates of the Kobayashi and quasihyperbolic distances (see [21]), regularity of quasiconformal maps, see [6]. In particular, the distortion of the metric $h_{D,c}$ under quasiconformal mappings (see [6], Theorem 4.9) is as follows: for all $c > 0$, there exists a constant $C > 1$ such that

$$h_{f(D),c}(f(x), f(y)) \leqslant C \max\{h_{D,c}(x,y), h_{D,c}(x,y)^{\alpha}\},$$

where $D \subsetneqq \mathbb{R}^n$ is a uniform domain, $f\colon\ D \to f(D)$ is a K-quasiconformal mapping, and $\alpha = K^{1/(1-n)}$. Quasiconformal mappings are widely studied by use of various hyperbolic type metrics these days. The interested readers are referred to, e.g., [5], [7], [9], [10], [11], [12], [13].

In this paper we investigate the Lipschitz constants for the metric $h_{D,c}$ under Möbius transformations of the unit ball, the upper half-space, and the punctured unit ball, respectively. The main results are as follows.

Theorem 1.1. *Let $a \in \mathbb{B}^n$ and $f\colon\ \mathbb{B}^n \to \mathbb{B}^n = f(\mathbb{B}^n)$ be a Möbius transformation with $f(a) = 0$. Then for all $x, y \in \mathbb{B}^n$,*

$$\frac{1}{1+|a|}h_{\mathbb{B}^n,c}(x,y) \leqslant h_{\mathbb{B}^n,c}(f(x), f(y)) \leqslant (1+|a|)h_{\mathbb{B}^n,c}(x,y),$$

where the constant $1 + |a|$ is the best possible.

Theorem 1.2. *Let $f\colon\ \mathbb{B}^n \to \mathbb{H}^n = f(\mathbb{B}^n)$ be a Möbius transformation. Then for all $x, y \in \mathbb{B}^n$,*

$$h_{\mathbb{B}^n,c}(x,y) \leqslant h_{\mathbb{H}^n,c}(f(x), f(y)) \leqslant 2h_{\mathbb{B}^n,c}(x,y),$$

where the constants 1 and 2 are the best possible.

Theorem 1.3. *Let $f\colon \mathbb{H}^n \to \mathbb{H}^n = f(\mathbb{H}^n)$ be a Möbius transformation. Then for all $x, y \in \mathbb{H}^n$,*

$$h_{\mathbb{H}^n,c}(f(x), f(y)) = h_{\mathbb{H}^n,c}(x, y).$$

Theorem 1.4. *Let $a \in \mathbb{B}^n$ and $f\colon \mathbb{B}^n \to \mathbb{B}^n = f(\mathbb{B}^n)$ be a Möbius transformation with $f(0) = a$. Then for all $x, y \in \mathbb{B}^n \setminus \{0\}$ we have*

$$h_{\mathbb{B}^n\setminus\{a\},c}(f(x), f(y)) \leqslant (1 + |a|)h_{\mathbb{B}^n\setminus\{0\},c}(x, y).$$

2. Möbius transformations and some inequalities

We use the standard notations as in [2]. The standard basis in the Euclidean space $\mathbb{R}^n$, $n > 2$, is denoted by $\{e_1, e_2, \ldots, e_n\}$. A point $x \in \mathbb{R}^n$ is represented as $(x_1, x_2, \ldots, x_n)$. The ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ is $B^n(x, r) = \{y \in \mathbb{R}^n\colon |x-y| < r\}$ and the sphere with the same center and radius is $S^{n-1}(x, r) = \{y \in \mathbb{R}^n\colon |x - y| = r\}$. We use the abbreviation $\mathbb{B}^n = B^n(0, 1)$. The upper half space is denoted by $\mathbb{H}^n = \{x \in \mathbb{R}^n\colon x_n > 0\}$ and $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \{\infty\}$. For basic facts about Möbius transformations the reader is referred to [3], [22], [24]. We denote $x^* = x/|x|^2$ for $x \in \mathbb{R}^n \setminus \{0\}$, and $0^* = \infty$, $\infty^* = 0$. For $a \in \mathbb{B}^n \setminus \{0\}$, let

$$\sigma_a(x) = a^* + r^2(x - a^*)^*, \quad r^2 = |a|^{-2} - 1$$

be the inversion in the sphere $S^{n-1}(a^*, r)$. Then $\sigma_a(a)=0$, $\sigma_a(a^*)=\infty$, see [3], (3.1.5)

$$|\sigma_a(x) - \sigma_a(y)| = \frac{r^2|x - y|}{|x - a^*||y - a^*|}. \tag{2.1}$$

The following lemmas show some fundamental results on Möbius transformations.

Lemma 2.1 ([3], Theorem 3.5.1)**.** *Let $a \in \mathbb{B}^n \setminus \{0\}$, and let f be a Möbius transformation of the unit ball with $f(a) = 0$. Then*

$$f(x) = (A \circ \sigma_a)(x),$$

where A is an orthogonal transformation.

Lemma 2.2 ([22], Theorem 4.4.8)**.** *Let $f\colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be a Möbius transformation with $f(\mathbb{B}^n) = \mathbb{B}^n$. Then f is an orthogonal transformation from $\mathbb{R}^n$ onto itself if and only if $f(0) = 0$.*

Lemma 2.3 ([22], Theorem 4.3.2). *Let $f\colon \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ be a Möbius transformation. Then f is a similarity from $\mathbb{R}^n$ onto itself if and only if $f(\infty)=\infty$.*

The following inequalities are useful in the proofs of main results.

Lemma 2.4 ([23], Lemma 3.2). *Let $a, b \in \mathbb{B}^n$. Then*

(1) $$|a|^2|b-a^*|^2 - |b-a|^2 = (1-|a|^2)(1-|b|^2);$$

(2) $$\frac{||b|-|a||}{1-|a||b|} \leqslant \frac{|b-a|}{|a||b-a^*|} \leqslant \frac{|b|+|a|}{1+|a||b|}.$$

Lemma 2.5. *For $a, z \in \mathbb{B}^n$, there holds*

$$|a||z-a^*| \geqslant 1-|a||z|.$$

Proof. By the triangle inequality, we have

$$|a||z-a^*| \geqslant |a|||z|-|a^*|| = 1-|a||z|.$$

□

Lemma 2.6. *For $a, z \in \mathbb{B}^n$ with $\frac{1}{2} \leqslant |z| \leqslant 1$, there holds*

$$|z|(1-|a||z|) \geqslant \Big(1-\frac{|a|}{2}\Big)(1-|z|).$$

Proof. It is easy to see that

$$\begin{aligned}
|z|(1-|a||z|) - \Big(1-\frac{|a|}{2}\Big)(1-|z|) &= 2|z| - 1 + \frac{|a|}{2} - |a||z|^2 - \frac{|a||z|}{2} \\
&= 2|z| - 1 + \frac{|a|}{2} - |a||z| - |a||z|^2 + \frac{|a||z|}{2} \\
&= 2|z| - 1 - (2|z|-1)\frac{|a|}{2} - (2|z|-1)\frac{|a||z|}{2} \\
&= (2|z|-1)\Big(1-\frac{|a|}{2}-\frac{|a||z|}{2}\Big) \geqslant 0.
\end{aligned}$$

□

For the reference, we record Bernoulli's inequality here:

(2.2) $$r\log(1+t) \geqslant \log(1+rt), \quad r \geqslant 1,\ t>0.$$

3. Proofs of the main results

Proof of Theorem 1.1. For $a=0$, Lemma 2.2 implies that f is an orthogonal transformation. Since $h_{\mathbb{B}^n,c}$ is unchanged under orthogonal transformations for all $x,y\in\mathbb{B}^n$ we have

$$h_{\mathbb{B}^n,c}(f(x),f(y)) = h_{\mathbb{B}^n,c}(x,y)$$

as desired.

Now we assume $a\neq 0$. Since $f=A\circ\sigma_a$ by Lemma 2.1, we have for all $x,y\in\mathbb{B}^n$

$$h_{\mathbb{B}^n,c}(f(x),f(y)) = h_{\mathbb{B}^n,c}(\sigma_a(x),\sigma_a(y)).$$

Equation (2.1) implies that

$$|\sigma_a(x)-\sigma_a(y)| = \frac{(|a|^{-2}-1)|x-y|}{|x-a^*||y-a^*|} \quad \text{and} \quad 1-|\sigma_a(x)| = \frac{|a||x-a^*|-|x-a|}{|a||x-a^*|}.$$

We first prove the right-hand side of the inequalities. By use of Lemma 2.4 and Bernoulli's inequality (2.2), we have

$$\begin{aligned}
h_{\mathbb{B}^n,c}(\sigma_a(x),\sigma_a(y)) &= \log\left(1+c\frac{|\sigma_a(x)-\sigma_a(y)|}{\sqrt{(1-|\sigma_a(x)|)(1-|\sigma_a(y)|)}}\right)\\
&= \log\left(1+c\frac{|x-y|}{|a|\sqrt{|x-a^*||y-a^*|}}\sqrt{\frac{1}{(|a||x-a^*|-|x-a|)(|a||y-a^*|-|y-a|)}}\right)\\
&= \log\left(1+c\frac{|x-y|}{|a|\sqrt{|x-a^*||y-a^*|}}\sqrt{\frac{(|a||x-a^*|+|x-a|)(|a||y-a^*|+|y-a|)}{(1-|x|^2)(1-|y|^2)}}\right)\\
&= \log\left(1+c\frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}}\sqrt{\left(1+\frac{|x-a|}{|a||x-a^*|}\right)\left(1+\frac{|y-a|}{|a||y-a^*|}\right)}\right)\\
&\leqslant \log\left(1+c\frac{|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}}\sqrt{\left(1+\frac{|x|+|a|}{1+|x||a|}\right)\left(1+\frac{|y|+|a|}{1+|y||a|}\right)}\right)\\
&= \log\left(1+c\frac{|x-y|}{\sqrt{(1-|x|)(1-|y|)}}\sqrt{\frac{(1+|a|)^2}{(1+|x||a|)(1+|y||a|)}}\right)\\
&\leqslant \log\left(1+c\frac{(1+|a|)|x-y|}{\sqrt{(1-|x|)(1-|y|)}}\right) \leqslant (1+|a|)\log\left(1+c\frac{|x-y|}{\sqrt{(1-|x|)(1-|y|)}}\right)\\
&= (1+|a|)h_{\mathbb{B}^n,c}(x,y).
\end{aligned}$$

Therefore,

$$h_{\mathbb{B}^n,c}(f(x),f(y)) \leqslant (1+|a|)h_{\mathbb{B}^n,c}(x,y).$$

Next, we prove the left-hand side of the inequalities. Since $f(x) = (A \circ \sigma_a)(x)$ by Lemma 2.1, $f^{-1}(x) = (A \circ \sigma_a)^{-1}(x) = (\sigma_a \circ A^T)(x)$. Hence, for all $u, v \in \mathbb{B}^n$, we have

$$\begin{aligned} h_{\mathbb{B}^n,c}(f^{-1}(u), f^{-1}(v)) = h_{\mathbb{B}^n,c}(\sigma_a(A^T(u)), \sigma_a(A^T(v))) &\leqslant (1+|a|)h_{\mathbb{B}^n,c}(A^T(u), A^T(v)) \\ &= (1+|a|)h_{\mathbb{B}^n,c}(u,v). \end{aligned}$$

Now let $u = f(x)$ and $v = f(y)$. We have

$$h_{\mathbb{B}^n,c}(f(x), f(y)) \geqslant \frac{1}{1+|a|} h_{\mathbb{B}^n,c}(x,y).$$

For the sharpness of the constants we set $x = ta/|a| = -y$ with $t \in (0, |a|)$. Then

$$|\sigma_a(x) - \sigma_a(y)| = \frac{2t(1-|a|^2)}{1-|a|^2t^2}$$

and

$$1 - |\sigma_a(x)| = \frac{(1-|a|)(1+t)}{1-t|a|}, \quad 1 - |\sigma_a(y)| = \frac{(1-|a|)(1-t)}{1+t|a|}.$$

Hence,

$$\begin{aligned} (3.1) \qquad \lim_{t\to 0^+} \frac{h_{\mathbb{B}^n,c}(f(x), f(y))}{h_{\mathbb{B}^n,c}(x,y)} &= \lim_{t\to 0^+} \frac{h_{\mathbb{B}^n,c}(\sigma_a(x), \sigma_a(y))}{h_{\mathbb{B}^n,c}(x,y)} \\ &= \lim_{t\to 0^+} \frac{\log(1+2ct(1+|a|)/\sqrt{(1-t^2)(1-|a|^2t^2)})}{\log(1+2ct/(1-t))} \\ &= \lim_{t\to 0^+} \frac{(1+|a|)(1-t)}{\sqrt{(1-|a|^2t^2)(1-t^2)}} = 1+|a|. \end{aligned}$$

It follows from (3.1) and $\sigma_a(\sigma_a(x)) = x$ that

$$\begin{aligned} \lim_{t\to 0^+} \frac{h_{\mathbb{B}^n,c}(f(\sigma_a(x)), f(\sigma_a(y)))}{h_{\mathbb{B}^n,c}(\sigma_a(x), \sigma_a(y))} &= \lim_{t\to 0^+} \frac{h_{\mathbb{B}^n,c}(\sigma_a(\sigma_a(x)), \sigma_a(\sigma_a(y)))}{h_{\mathbb{B}^n,c}(\sigma_a(x), \sigma_a(y))} \\ &= \lim_{t\to 0^+} \frac{h_{\mathbb{B}^n,c}(x,y)}{h_{\mathbb{B}^n,c}(\sigma_a(x), \sigma_a(y))} = \frac{1}{1+|a|}. \end{aligned}$$

This completes the proof. □

P r o o f of Theorem 1.2. The right-hand side of the inequalities was proved in [6], Lemma 2.11. Now we prove the left-hand side of the inequalities. Let $\varphi_s\colon \mathbb{H}^n \to \mathbb{H}^n$ be a similarity with $\varphi_s(e_n) = a = f(0)$, and let $\sigma\colon \mathbb{B}^n \to \mathbb{H}^n$ be the inversion in the sphere $S^{n-1}(-e_n, \sqrt{2})$ with $\sigma(0) = e_n$. Then $f^{-1} \circ \varphi_s \circ \sigma\colon \mathbb{B}^n \to \mathbb{B}^n$ is a Möbius transformation with $f^{-1}(\varphi_s(\sigma(0))) = f^{-1}(\varphi_s(e_n)) = f^{-1}(a) = 0$. By Lemma 2.2, we see that $\psi_o \equiv f^{-1} \circ \varphi_s \circ \sigma$ is an orthogonal transformation and $f = \varphi_s \circ \sigma \circ {\psi_o}^{-1}$.

Since σ is an inversion in $S^{n-1}(-e_n, \sqrt{2})$, we have

$$\sigma(x) = -e_n + \frac{2(x+e_n)}{|x+e_n|^2}.$$

Let $\sigma(x) = (\sigma_1(x), \sigma_2(x), \ldots, \sigma_n(x))$, then

$$\sigma_n(x) = \frac{1-|x|^2}{|x+e_n|^2}.$$

It follows from (2.1) that

$$|\sigma(x) - \sigma(y)| = \frac{2|x-y|}{|x+e_n||y+e_n|}.$$

By the invariance of $h_{\mathbb{H}^n,c}$ under similarities, we have for all $x, y \in \mathbb{B}^n$,

$$\begin{aligned}
&h_{\mathbb{H}^n,c}(f(x), f(y)) = h_{\mathbb{H}^n,c}(\sigma(\psi_o^{-1}(x)), \sigma(\psi_o^{-1}(y)))\\
&= \log\left(1 + c\frac{|\sigma(\psi_o^{-1}(x)) - \sigma(\psi_o^{-1}(y))|}{\sqrt{\sigma_n(\psi_o^{-1}(x))\sigma_n(\psi_o^{-1}(y))}}\right)\\
&= \log\left(1 + c\frac{2|\psi_o^{-1}(x) - \psi_o^{-1}(y)|}{\sqrt{(1+|\psi_o^{-1}(x)|)(1+|\psi_o^{-1}(y)|)}\sqrt{(1-|\psi_o^{-1}(x)|)(1-|\psi_o^{-1}(y)|)}}\right)\\
&= \log\left(1 + c\frac{2|x-y|}{\sqrt{(1+|x|)(1+|y|)}\sqrt{(1-|x|)(1-|y|)}}\right)\\
&\geqslant \log\left(1 + c\frac{|x-y|}{\sqrt{(1-|x|)(1-|y|)}}\right) = h_{\mathbb{B}^n,c}(x,y).
\end{aligned}$$

For the sharpness of the constants, set $x = -y = te_1$ with $t \in (0,1)$. Then we have

$$h_{\mathbb{H}^n,c}(\sigma(x), \sigma(y)) = \log\left(1 + c\frac{4t}{1-t^2}\right) \quad \text{and} \quad h_{\mathbb{B}^n,c}(x,y) = \log\left(1 + c\frac{2t}{1-t}\right).$$

Hence,

$$\lim_{t\to 0^+} \frac{h_{\mathbb{H}^n,c}(f(x), f(y))}{h_{\mathbb{B}^n,c}(x,y)} = \lim_{t\to 0^+} \frac{h_{\mathbb{H}^n,c}(\sigma(x), \sigma(y))}{h_{\mathbb{B}^n,c}(x,y)} = \lim_{t\to 0^+} \frac{2}{1+t} = 2$$

and

$$\begin{aligned}
\lim_{t\to 1^-} \frac{h_{\mathbb{H}^n,c}(f(x), f(y))}{h_{\mathbb{B}^n,c}(x,y)} &= \lim_{t\to 1^-} \frac{h_{\mathbb{H}^n,c}(\sigma(x), \sigma(y))}{h_{\mathbb{B}^n,c}(x,y)}\\
&= \lim_{t\to 1^-} \frac{\log(1-t^2)}{\log(1-t)} = \lim_{t\to 1^-} \frac{2t}{1+t} = 1.
\end{aligned}$$

This completes the proof. $\square$

P r o o f of Theorem 1.3. We assume that $f(a)=\infty$ for some $a\in\partial\mathbb{H}^n$. If $a=\infty$, Lemma 2.3 implies that f is a similarity. By the invariance of $h_{\mathbb{H}^n,c}$ under similarities, we have for all $x,y\in\mathbb{H}^n$,

$$h_{\mathbb{H}^n,c}(f(x),f(y))=h_{\mathbb{H}^n,c}(x,y).$$

Now we assume $a\neq\infty$. Let $\varphi_t\colon\ \overline{\mathbb{H}^n}\to\overline{\mathbb{H}^n}$ be a translation with $\varphi_t(x)=x-a$, then $\varphi_t(a)=0$. Let $\sigma\colon\ \mathbb{H}^n\to\mathbb{H}^n$ be the inversion in the sphere $S^{n-1}(0,1)$, then $\sigma(0)=\infty$ and $\sigma(\infty)=0$. We see that $f\circ{\varphi_t}^{-1}\circ\sigma^{-1}\colon\ \mathbb{H}^n\to\mathbb{H}^n$ is a Möbius transformation with $f({\varphi_t}^{-1}(\sigma^{-1}(\infty)))=f({\varphi_t}^{-1}(0))=f(a)=\infty$. It follows from Lemma 2.3 that $\varphi_s\equiv f\circ{\varphi_t}^{-1}\circ\sigma^{-1}$ is a similarity, and hence $f=\varphi_s\circ\sigma\circ\varphi_t$.

Since σ is the inversion in the sphere $S^{n-1}(0,1)$, we have

$$\sigma(x)=\frac{x}{|x|^2}\quad\text{and}\quad\sigma_n(x)=\frac{x_n}{|x|^2}.$$

By (2.1), we have

$$|\sigma(x)-\sigma(y)|=\frac{|x-y|}{|x||y|}.$$

Since $h_{\mathbb{H}^n,c}$ is unchanged under similarities, we have for all $x,y\in\mathbb{H}^n$,

$$\begin{aligned}h_{\mathbb{H}^n,c}(f(x),f(y))&=h_{\mathbb{H}^n,c}(\sigma(\varphi_t(x)),\sigma(\varphi_t(y)))=\log\biggl(1+c\frac{|\sigma(\varphi_t(x))-\sigma(\varphi_t(y))|}{\sqrt{\sigma_n(\varphi_t(x))\sigma_n(\varphi_t(y))}}\biggr)\\&=\log\biggl(1+c\frac{|\varphi_t(x)-\varphi_t(y)|}{\sqrt{(\varphi_t)_n(x)(\varphi_t)_n(y)}}\biggr)=\log\biggl(1+c\frac{|x-y|}{\sqrt{x_ny_n}}\biggr)=h_{\mathbb{H}^n,c}(x,y).\end{aligned}$$

This completes the proof. □

P r o o f of Theorem 1.4. Since $h_{D,c}$-metric is preserved under orthogonal transformations, it follows from Lemma 2.1 that for $x,y,a\in\mathbb{B}^n$, it suffices to prove the theorem for $f=\sigma_a$.

Without loss of generality, we assume that $|y|\leqslant|x|$. Let $D=\mathbb{B}^n\setminus\{0\}$ and $D'=\mathbb{B}^n\setminus\{a\}$. Then we have the following cases for $h_{D,c}(x,y)$: If $\frac12\leqslant|y|\leqslant|x|$, then

$$h_{D,c}(x,y)=\log\biggl(1+\frac{c|x-y|}{\sqrt{1-|x|}\sqrt{1-|y|}}\biggr);$$

If $|y|\leqslant\frac12\leqslant|x|$, then

$$h_{D,c}(x,y)=\log\biggl(1+\frac{c|x-y|}{\sqrt{1-|x|}\sqrt{|y|}}\biggr);$$

If $|y|\leqslant|x|\leqslant\frac12$, then

$$h_{D,c}(x,y)=\log\biggl(1+\frac{c|x-y|}{\sqrt{|x|}\sqrt{|y|}}\biggr).$$

We also have
$$h_{D',c}(\sigma_a(x),\sigma_a(y)) = \log\Big(1+\frac{c|\sigma_a(x)-\sigma_a(y)|}{T}\Big),$$
where
$$T=\sqrt{\min\{|\sigma_a(x)-a|,1-|\sigma_a(x)|\}}\sqrt{\min\{|\sigma_a(y)-a|,1-|\sigma_a(y)|\}}.$$

Now we prove the inequality according to the following cases.
Case 1: $T=\sqrt{|\sigma_a(x)-a||\sigma_a(y)-a|}$. Since
$$|\sigma_a(x)-a| = |\sigma_a(x)-\sigma_a(0)| = \frac{s^2|x|}{|x-a^*||a^*|}, \quad s^2=|a|^{-2}-1,$$
it follows that
$$|\sigma_a(y)-a| = \frac{s^2|y|}{|y-a^*||a^*|} \quad \text{and} \quad T=\frac{s^2}{|a^*|}\frac{\sqrt{|x||y|}}{\sqrt{|x-a^*||y-a^*|}}.$$
Hence,
$$h_{D',c}(\sigma_a(x),\sigma_a(y)) = \log\left(1+\frac{c|x-y|}{|a|\sqrt{|x||y||x-a^*||y-a^*|}}\right).$$
By Lemma 2.5, we have
$$h_{D',c}(\sigma_a(x),\sigma_a(y)) \leqslant \log\Big(1+\frac{c|x-y|}{\sqrt{|x||y|(1-|a||x|)(1-|a||y|)}}\Big).$$

Subcase 1.1: $\frac{1}{2}\leqslant |y|\leqslant |x|$. By Lemma 2.6 and Bernoulli's inequality (2.2), we have
$$\begin{aligned}
h_{D',c}(\sigma_a(x),\sigma_a(y)) &\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{|x||y|(1-|a||x|)(1-|a||y|)}}\right)\\
&\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{(1-|a|/2)^2(1-|x|)(1-|y|)}}\right)\\
&\leqslant \frac{1}{1-|a|/2}\log\left(1+\frac{c|x-y|}{\sqrt{(1-|x|)(1-|y|)}}\right) = \frac{2}{2-|a|}h_{D,c}(x,y).
\end{aligned}$$

Subcase 1.2: $|y|\leqslant \frac{1}{2}\leqslant |x|$. In this case, we have
$$\begin{aligned}
h_{D',c}(\sigma_a(x),\sigma_a(y)) &\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{|x||y|(1-|a||x|)(1-|a||y|)}}\right)\\
&\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{(1-|a|/2)(1-|x|)|y|(1-|a|/2)}}\right)\\
&\leqslant \frac{1}{1-|a|/2}\log\left(1+\frac{c|x-y|}{\sqrt{(1-|x|)|y|}}\right) = \frac{2}{2-|a|}h_{D,c}(x,y).
\end{aligned}$$

Subcase 1.3: $|y| \leqslant |x| \leqslant \frac{1}{2}$. We see that

$$\begin{aligned} h_{D',c}(\sigma_a(x),\sigma_a(y)) &\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{|x||y|(1-|a||x|)(1-|a||y|)}}\right) \\ &\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{|x||y|(1-|a|/2)(1-|a|/2)}}\right) \\ &\leqslant \frac{1}{1-|a|/2}\log\left(1+\frac{c|x-y|}{\sqrt{|x||y|}}\right) = \frac{2}{2-|a|}h_{D,c}(x,y). \end{aligned}$$

Hence, for Case 1 we obtain

$$h_{D',c}(\sigma_a(x),\sigma_a(y)) \leqslant \frac{2}{2-|a|}h_{D,c}(x,y).$$

Case 2: $T = \sqrt{|\sigma_a(x)-a|(1-|\sigma_a(y)|)}$.

Since

$$|\sigma_a(y)| = |\sigma_a(y)-\sigma_a(a)| = \frac{s^2|y-a|}{|y-a^*||a-a^*|},$$

we get

$$\begin{aligned} &h_{D',c}(\sigma_a(x),\sigma_a(y)) = \log\left(1+\frac{c|\sigma_a(x)-\sigma_a(y)|}{\sqrt{|\sigma_a(x)-a|(1-|\sigma_a(y)|)}}\right) \\ &= \log\left(1+\frac{cs^2|x-y|}{\sqrt{|x-a^*||y-a^*|}\sqrt{|x|(|y-a^*||a-a^*|-s^2|y-a|)}}\right) \\ &= \log\left(1+\frac{cs^2|x-y|}{\sqrt{|x-a^*||y-a^*|}\sqrt{|x||a-a^*|}\sqrt{|y-a^*|-|a^*||y-a|}}\right) \\ &= \log\left(1+\frac{c(1-|a|^2)|x-y|}{|a|\sqrt{|x-a^*||y-a^*|}\sqrt{|x|(1-|a|^2)}\sqrt{|a||y-a^*|-|y-a|}}\right) \\ &= \log\left(1+\frac{c(1-|a|^2)|x-y|\sqrt{|a||y-a^*|+|y-a|}}{|a|\sqrt{|x-a^*||y-a^*|}\sqrt{|x|(1-|a|^2)}\sqrt{(1-|a|^2)(1-|y|^2)}}\right) \\ &= \log\left(1+\frac{c|x-y|}{\sqrt{1-|y|^2}}\frac{\sqrt{|a||y-a^*|+|y-a|}}{\sqrt{|a||y-a^*|}}\frac{1}{\sqrt{|x||a||x-a^*|}}\right) \\ &\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{1-|y|^2}}\frac{\sqrt{|a||y-a^*|+|y-a|}}{\sqrt{|a||y-a^*|}}\frac{1}{\sqrt{(1-|a||x|)|x|}}\right) \\ &= \log\left(1+\frac{c|x-y|}{\sqrt{1-|y|^2}}\sqrt{1+\frac{|y-a|}{|a||y-a^*|}}\frac{1}{\sqrt{(1-|a||x|)|x|}}\right) \\ &\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{1-|y|^2}}\sqrt{1+\frac{|a|+|y|}{1+|a||y|}}\frac{1}{\sqrt{(1-|a||x|)|x|}}\right) \\ &= \log\left(1+\frac{c|x-y|}{\sqrt{1-|y|}}\sqrt{\frac{1+|a|}{1+|a||y|}}\frac{1}{\sqrt{(1-|a||x|)|x|}}\right). \end{aligned}$$

Subcase 2.1: $\frac{1}{2} \leqslant |y| \leqslant |x|$. In this case, we have

$$\begin{aligned}
h_{D',c}(\sigma_a(x),\sigma_a(y)) &\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{1-|y|}}\sqrt{\frac{1+|a|}{1+|a||y|}}\frac{1}{\sqrt{(1-|a||x|)|x|}}\right)\\
&\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{1-|y|}}\sqrt{\frac{1+|a|}{1+|a||y|}}\frac{1}{\sqrt{(1-|a|/2)(1-|x|)}}\right)\\
&\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{(1-|y|)(1-|x|)}}\sqrt{\frac{2(1+|a|)}{2+|a|}}\sqrt{\frac{2}{2-|a|}}\right)\\
&= \log\left(1+\frac{c|x-y|}{\sqrt{(1-|x|)(1-|y|)}}2\sqrt{\frac{1+|a|}{4-|a|^2}}\right)\leqslant 2\sqrt{\frac{1+|a|}{4-|a|^2}}h_{D,c}(x,y).
\end{aligned}$$

Subcase 2.2: $|y| \leqslant \frac{1}{2} \leqslant |x|$. Similarly,

$$\begin{aligned}
h_{D',c}(\sigma_a(x),\sigma_a(y)) &\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{1-|y|}}\sqrt{\frac{1+|a|}{1+|a||y|}}\frac{1}{\sqrt{(1-|a||x|)|x|}}\right)\\
&\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{|y|}}\sqrt{\frac{1+|a|}{1+|a||y|}}\frac{1}{\sqrt{(1-|a|/2)(1-|x|)}}\right)\\
&= \log\left(1+\frac{c|x-y|}{\sqrt{(1-|x|)|y|}}\sqrt{\frac{1+|a|}{1+|a||y|}}\sqrt{\frac{2}{2-|a|}}\right)\\
&\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{(1-|x|)|y|}}\sqrt{1+|a|}\sqrt{\frac{2}{2-|a|}}\right)\leqslant \sqrt{\frac{2(1+|a|)}{2-|a|}}h_{D,c}(x,y).
\end{aligned}$$

Subcase 2.3: $|y| \leqslant |x| \leqslant \frac{1}{2}$. We see that

$$\begin{aligned}
h_{D',c}(\sigma_a(x),\sigma_a(y)) &\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{1-|y|}}\sqrt{\frac{1+|a|}{1+|a||y|}}\frac{1}{\sqrt{(1-|a||x|)|x|}}\right)\\
&\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{|y|}}\sqrt{\frac{1+|a|}{1+|a||y|}}\frac{1}{\sqrt{(1-|a|/2)|x|}}\right)\\
&= \log\left(1+\frac{c|x-y|}{\sqrt{|x||y|}}\sqrt{\frac{1+|a|}{1+|a||y|}}\sqrt{\frac{2}{2-|a|}}\right)\\
&\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{|x||y|}}\sqrt{1+|a|}\sqrt{\frac{2}{2-|a|}}\right)\leqslant \sqrt{\frac{2(1+|a|)}{2-|a|}}h_{D,c}(x,y).
\end{aligned}$$

Hence, for Case 2, we obtain

$$h_{D',c}(\sigma_a(x),\sigma_a(y)) \leqslant \sqrt{\frac{2(1+|a|)}{2-|a|}}h_{D,c}(x,y).$$

Case 3: $T=\sqrt{(1-|\sigma_a(x)|)|\sigma_a(y)-a|}$. Since

$$|\sigma_a(x)|=|\sigma_a(x)-\sigma_a(a)|=\frac{s^2|x-a|}{|x-a^*||a-a^*|},$$

similarly as in Case 2, we have

$$h_{D',c}(\sigma_a(x),\sigma_a(y))\leqslant\log\left(1+\frac{c|x-y|}{\sqrt{1-|x|}}\sqrt{\frac{1+|a|}{1+|a||x|}}\frac{1}{\sqrt{(1-|a||y|)|y|}}\right).$$

Subcase 3.1: $\frac{1}{2}\leqslant|y|\leqslant|x|$. Then

$$\begin{aligned}h_{D',c}(\sigma_a(x),\sigma_a(y))&\leqslant\log\left(1+\frac{c|x-y|}{\sqrt{1-|x|}}\sqrt{\frac{1+|a|}{1+|a||x|}}\frac{1}{\sqrt{(1-|a||y|)|y|}}\right)\\&\leqslant\log\left(1+\frac{c|x-y|}{\sqrt{1-|x|}}\sqrt{\frac{1+|a|}{1+|a||x|}}\frac{1}{\sqrt{(1-|a|/2)(1-|y|)}}\right)\\&=\log\left(1+\frac{c|x-y|}{\sqrt{(1-|y|)(1-|x|)}}\sqrt{\frac{1+|a|}{1+|a||x|}}\sqrt{\frac{2}{2-|a|}}\right)\\&\leqslant\log\left(1+\frac{c|x-y|}{\sqrt{(1-|y|)(1-|x|)}}\sqrt{\frac{2(1+|a|)}{2+|a|}}\sqrt{\frac{2}{2-|a|}}\right)\\&\leqslant2\sqrt{\frac{1+|a|}{4-|a|^2}}\log\left(1+\frac{c|x-y|}{\sqrt{(1-|x|)(1-|y|)}}\right)=2\sqrt{\frac{1+|a|}{4-|a|^2}}h_{D,c}(x,y).\end{aligned}$$

Subcase 3.2: $|y|\leqslant\frac{1}{2}\leqslant|x|$. Similarly,

$$\begin{aligned}h_{D',c}(\sigma_a(x),\sigma_a(y))&\leqslant\log\left(1+\frac{c|x-y|}{\sqrt{1-|x|}}\sqrt{\frac{1+|a|}{1+|a||x|}}\frac{1}{\sqrt{(1-|a||y|)|y|}}\right)\\&\leqslant\log\left(1+\frac{c|x-y|}{\sqrt{1-|x|}}\sqrt{\frac{1+|a|}{1+|a||x|}}\frac{1}{\sqrt{(1-|a|/2)|y|}}\right)\\&=\log\left(1+\frac{c|x-y|}{\sqrt{|y|(1-|x|)}}\sqrt{\frac{1+|a|}{1+|a||x|}}\sqrt{\frac{2}{2-|a|}}\right)\\&\leqslant\log\left(1+\frac{c|x-y|}{\sqrt{|y|(1-|x|)}}\sqrt{\frac{2(1+|a|)}{2+|a|}}\sqrt{\frac{2}{2-|a|}}\right)\\&\leqslant2\sqrt{\frac{1+|a|}{4-|a|^2}}\log\left(1+\frac{c|x-y|}{\sqrt{|y|(1-|x|)}}\right)=2\sqrt{\frac{1+|a|}{4-|a|^2}}h_{D,c}(x,y).\end{aligned}$$

Subcase 3.3: $|y| \leqslant |x| \leqslant \frac{1}{2}$. We see that

$$\begin{aligned}
h_{D',c}(\sigma_a(x),\sigma_a(y)) &\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{1-|x|}}\sqrt{\frac{1+|a|}{1+|a||x|}}\frac{1}{\sqrt{(1-|a||y|)|y|}}\right)\\
&\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{|x|}}\sqrt{\frac{1+|a|}{1+|a||x|}}\frac{1}{\sqrt{(1-|a|/2)|y|}}\right)\\
&\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{|x||y|}}\sqrt{\frac{1+|a|}{1+|a||x|}}\sqrt{\frac{2}{2-|a|}}\right)\\
&\leqslant \log\left(1+\frac{c|x-y|}{\sqrt{|x||y|}}\sqrt{1+|a|}\sqrt{\frac{2}{2-|a|}}\right)\\
&\leqslant \sqrt{\frac{2(1+|a|)}{2-|a|}}\log\left(1+\frac{c|x-y|}{\sqrt{|x||y|}}\right)=\sqrt{\frac{2(1+|a|)}{2-|a|}}h_{D,c}(x,y).
\end{aligned}$$

Therefore, for Case 3 we have

$$h_{D',c}(\sigma_a(x),\sigma_a(y)) \leqslant \sqrt{\frac{2(1+|a|)}{2-|a|}}h_{D,c}(x,y).$$

Case 4: $T=\sqrt{1-|\sigma_a(x)|}\sqrt{|1-|\sigma_a(y)|}$. Then

$$\begin{aligned}
&h_{D',c}(\sigma_a(x),\sigma_a(y)) = \log\left(1+\frac{c|\sigma_a(x)-\sigma_a(y)|}{\sqrt{1-|\sigma_a(x)|}\sqrt{|1-|\sigma_a(y)|}}\right)\\
&=\log\left(1+\frac{c(1-|a|^2)|x-y|}{|a|^2\sqrt{|x-a^*||y-a^*|}}\frac{1}{\sqrt{|x-a^*|-|a^*||x-a|}\sqrt{|y-a^*|-|a^*||y-a|}}\right)\\
&=\log\left(1+\frac{c(1-|a|^2)|x-y|}{|a|\sqrt{|x-a^*||y-a^*|}}\frac{1}{\sqrt{|a||x-a^*|-|x-a|}\sqrt{|a||y-a^*|-|y-a|}}\right)\\
&=\log\left(1+\frac{c(1-|a|^2)|x-y|}{|a|\sqrt{|x-a^*||y-a^*|}}\frac{\sqrt{|a||x-a^*|+|x-a|}\sqrt{|a||y-a^*|+|y-a|}}{\sqrt{(1-|a|^2)(1-|x|^2)}\sqrt{(1-|a|^2)(1-|y|^2)}}\right)\\
&=\log\left(1+\frac{c|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}}\sqrt{1+\frac{|x-a|}{|a||x-a^*|}}\sqrt{1+\frac{|y-a|}{|a||y-a^*|}}\right)\\
&\leqslant\log\left(1+\frac{c|x-y|}{\sqrt{(1-|x|^2)(1-|y|^2)}}\sqrt{1+\frac{|x|+|a|}{1+|a||x|}}\sqrt{1+\frac{|y|+|a|}{1+|a||y|}}\right)\\
&=\log\left(1+\frac{c|x-y|}{\sqrt{(1-|x|)(1-|y|)}}\frac{1+|a|}{\sqrt{(1+|a||x|)(1+|a||y|)}}\right).
\end{aligned}$$

Subcase 4.1: $\frac{1}{2} \leqslant |y| \leqslant |x|$. We see that

$$\begin{aligned} h_{D',c}(\sigma_a(x), \sigma_a(y)) &\leqslant \log\left(1 + \frac{c|x-y|}{\sqrt{(1-|x|)(1-|y|)}} \frac{1+|a|}{\sqrt{(1+|a||x|)(1+|a||y|)}}\right) \\ &\leqslant \log\left(1 + \frac{c|x-y|}{\sqrt{(1-|x|)(1-|y|)}} \frac{2(1+|a|)}{2+|a|}\right) \\ &\leqslant \frac{2(1+|a|)}{2+|a|} h_{D,c}(x,y). \end{aligned}$$

Subcase 4.2: $|y| \leqslant \frac{1}{2} \leqslant |x|$. Similarly,

$$\begin{aligned} h_{D',c}(\sigma_a(x), \sigma_a(y)) &\leqslant \log\left(1 + \frac{c|x-y|}{\sqrt{(1-|x|)(1-|y|)}} \frac{1+|a|}{\sqrt{(1+|a||x|)(1+|a||y|)}}\right) \\ &\leqslant \log\left(1 + \frac{c|x-y|}{\sqrt{(1-|x|)|y|}} \frac{1+|a|}{\sqrt{1+|a|/2}}\right) \\ &\leqslant \frac{\sqrt{2}(1+|a|)}{\sqrt{2+|a|}} h_{D,c}(x,y). \end{aligned}$$

Subcase 4.3: $|y| \leqslant |x| \leqslant \frac{1}{2}$. We have

$$\begin{aligned} h_{D',c}(\sigma_a(x), \sigma_a(y)) &\leqslant \log\left(1 + \frac{c|x-y|}{\sqrt{(1-|x|)(1-|y|)}} \frac{1+|a|}{\sqrt{(1+|a||x|)(1+|a||y|)}}\right) \\ &\leqslant \log\left(1 + \frac{c|x-y|}{\sqrt{|x||y|}}(1+|a|)\right) \\ &\leqslant (1+|a|) h_{D,c}(x,y). \end{aligned}$$

Therefore, for Case 4, we have

$$h_{D',c}(\sigma_a(x), \sigma_a(y)) \leqslant (1+|a|) h_{D,c}(x,y).$$

Since $\max\{2/(2-|a|), \sqrt{2(1+|a|)/(2-|a|)}, 1+|a|\} = 1+|a|$, we finally obtain

$$h_{D',c}(\sigma_a(x), \sigma_a(y)) \leqslant (1+|a|) h_{D,c}(x,y).$$

By Lemma 2.1, $f^{-1}(x) = (A \circ \sigma_a)(x)$ and $f(x) = (\sigma_a \circ A^T)(x)$. Then

$$\begin{aligned} h_{D',c}(f(x), f(y)) &= h_{D',c}((\sigma_a \circ A^T)(x), (\sigma_a \circ A^T)(y)) \\ &\leqslant (1+|a|) h_{D,c}(A^T(x), A^T(y)) \\ &= (1+|a|) h_{D,c}(x,y). \end{aligned}$$

This completes the proof. □

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