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ON THE CHARACTERIZATION OF HARMONIC FUNCTIONS WITH INITIAL DATA IN MORREY SPACE

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Abstract. Let (X, d, μ) be a metric measure space satisfying the doubling condition and an L^2-Poincaré inequality. Consider the nonnegative operator $\mathcal{L}$ generalized by a Dirichlet form on X. We will show that a solution u to $(-\partial_t^2 + \mathcal{L})u = 0$ on $X \times \mathbb{R}_+$ satisfies an α-Carleson condition if and only if u can be represented as the Poisson integral of the operator $\mathcal{L}$ with the trace in the generalized Morrey space $L^{2,\alpha}(X)$, where α is a nonnegative function defined on a class of balls in X. This result extends the analogous characterization founded by R. Jiang, J. Xiao, D. Yang (2016) from the classical Morrey space on Euclidean space to the generalized Morrey space on the metric measure space.

Keywords: harmonic function; Dirichlet problem; Morrey space; Carleson measure; metric measure space

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1. Introduction and main result

1.1. Background. The boundary value problem of harmonic functions has a long history and has been extensively studied. It is well known that a basic tool for studying a function f on $\mathbb{R}^n$ is to transition it to a harmonic function, which is defined on the upper half-space $\mathbb{R}^{n+1}_+$ and takes f as its boundary value.

For the classical case, $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, as we all know that the Poisson integral $u(x,t) = \mathrm{e}^{-t\sqrt{-\Delta}} f(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$, is the solution to the equation

$$\begin{cases} \partial_t^2 u(x,t) + \Delta u(x,t) = 0 & \forall\, x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = f(x) & \forall\, x \in \mathbb{R}^n. \end{cases}$$

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Here $\Delta = \sum\limits_{k=1}^{n} \partial_{x_k}^2$ is the Laplace operator. In 1971, Stein-Weiss proved that a harmonic function $u(x,t)$ on $\mathbb{R}_+^{n+1}$ satisfies

$$\sup_{t>0} \int_{\mathbb{R}^n} |u(x,t)|^p \, \mathrm{d}x < \infty$$

if and only if $u(x,t) = \mathrm{e}^{-t\sqrt{-\Delta}} f(x)$ for some $f \in L^p(\mathbb{R}^n)$, see the standard textbook [49], Chapter 2 for more details.

At the end-point space $L^\infty(\mathbb{R}^n)$, it has a natural substitution, the BMO space, i.e., the space of functions of bounded mean oscillation. In the seminal work of Fefferman-Stein (see [18]), they discovered that a function $f \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if its Poisson integral $u(x,t) = \mathrm{e}^{-t\sqrt{-\Delta}} f(x)$ (which is naturally harmonic) satisfies the following Carleson condition:

$$\sup_{(x_0,r)\in\mathbb{R}^n\times\mathbb{R}_+} \left(\int_0^r \fint_{B(x_0,r)} |t\nabla u(x,t)|^2 \, \mathrm{d}x \frac{\mathrm{d}t}{t} \right)^{1/2} < \infty, \tag{1.1}$$

where $\nabla = (\nabla_x, \partial_t)$. Later on, Fabes-Johnson-Neri in [14] found that any harmonic function satisfying the Carleson condition (1.1) is the Poisson integral of a BMO function.

Since then, the study of this topic has been widely extended to more general operators such as elliptic operators (see [33], [39]) and Schrödinger operators (see [12], [25]) (instead of the Laplacian), and more general initial value spaces such as Campanato-Sobolev spaces (see [22], [27], [52]), and for domains other than $\mathbb{R}^n$ such as metric measure spaces (see [25], [28]) and Lipschitz domains, see [17], [47]. Moreover, this characterization has been further generalized to caloric functions, i.e., solutions to the heat equation, see [16], [34], [55].

Let us turn our attention to a replacement (or a generalization) of the BMO space as well as the Lebesgue space, the Morrey space, which was introduced by Morrey (see [41]) concerning regularity problems of the solutions to partial differential equations and defined as follows for any $1 \leqslant p < \infty$ and $-1/p < \lambda < 0$:

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \colon \sup_{(x_0,r)\in\mathbb{R}^n\times\mathbb{R}_+} \frac{1}{|B(x_0,r)|^{1+\lambda p}} \int_{B(x_0,r)} |f(x)|^p \, \mathrm{d}x < \infty \right\}.$$

The main fame that rests with the Morrey space is the following celebrated lemma, see [2], [41].

Morrey's Lemma. *Let the function u satisfy $|\nabla u| \in L^{p,\lambda}$ (even locally) with $\lambda > -1/n$. Then u is Hölder continuous of exponent $1 + n\lambda$.*

In the early 1960s, Campanato in [6] introduced an equivalent characterization of the Morrey space on $\mathbb{R}^n$:

$$L^{p,\lambda}(\mathbb{R}^n)$$
$$= \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \colon \sup_{(x_0,r)\in\mathbb{R}^n\times\mathbb{R}_+} \frac{1}{|B(x_0,r)|^{1+\lambda p}} \int_{B(x_0,r)} |f(x) - f_{B(x_0,r)}|^p \,\mathrm{d}x < \infty \right\},$$

see [2], [44], [46] for more details. Here and in what follows, $f_{B(x_0,r)}$ denotes the mean (or average) of f over $B(x_0,r)$, i.e.,

$$f_{B(x_0,r)} = \fint_{B(x_0,r)} f(x)\,\mathrm{d}x = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} f(x)\,\mathrm{d}x.$$

Nowadays, the Morrey space has been studied extensively in many aspects, such as its predual space (see [2], [29], [57]) and interpolation (see [56]), see also [1], [7], [10], [38], [54].

Recently, Jiang-Xiao-Yang in [27] studied the Dirichlet problem for the Laplace equation with boundary value in the Morrey space. More precisely, they proved that for any $-1/n < \lambda < 0$, a solution u to $-\partial_t^2 u - \Delta u = 0$ on $\mathbb{R}^{n+1}_+$ satisfies

$$\sup_{(x_0,r)\in\mathbb{R}^{n+1}_+} \frac{1}{|B(x_0,r)|^{\lambda}} \left(\int_0^r \fint_{B(x_0,r)} |t\nabla u(x,t)|^2 \,\mathrm{d}x \frac{\mathrm{d}t}{t} \right)^{1/2} < \infty$$

if and only if there exists a function $f \in L^{2,\lambda}(\mathbb{R}^n)$ such that u can be represented as $u = \mathrm{e}^{-t\sqrt{-\Delta}} f$. Song-Tian-Yan in [48] and Huang-Zhang in [23] extended non-trivially the study to the Schrödinger operator and parabolic Schrödinger operator, respectively, on $\mathbb{R}^n$.

The main aim of this article is to continue the line of [25], [27] to process the study of this topic on manifolds as well as general metric spaces; see our settings and the main result in the following subsection. Our result covers some general operators such as uniformly elliptic operators, and some general domains other than $\mathbb{R}^n$ such as Riemannian and sub-Riemannian manifolds; see for examples [8], Section 7.

1.2. Main results. Let us now fix our setting. We will concentrate on a complete Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfying the doubling condition (D) and admitting an L^2-Poincaré inequality (P_2). In the following, we will introduce these notation and notions, and then state the main result.

We first briefly describe the Dirichlet metric measure space, see [8], [19] for more details. Let X be a locally compact, separable, metrisable, and connected space, μ be a Borel measure that is finite on compact sets and strictly positive on nonempty open sets. We consider a strongly local, closed, and regular Dirichlet form $\mathscr{E}$ on $L^2(X,\mu)$ with dense domain $\mathscr{D} \subset L^2(X,\mu)$.

We assume that $\mathscr{E}$ admits a *carré du champ*, which means that the measure-valued nonnegative and symmetric bilinear form $\Gamma(f,g)$ is absolutely continuous with respect to μ for all $f,g \in \mathscr{D}$, see the accurate definition of Γ in [8]. In what follows, we denote by $\langle D_x f, D_x g\rangle$ the energy density $\mathrm{d}\Gamma(f,g)/\mathrm{d}\mu$, and by $|D_x f|$ the square root of $\mathrm{d}\Gamma(f,f)/\mathrm{d}\mu$.

We equip the space $(X,\mu,\mathscr{E})$ with an intrinsic (pseudo-)distance d on X associated to $\mathscr{E}$, see [8]. We assume that d is a true distance and that the topology induced by d is equivalent to the original topology on X. Moreover, we assume that (X,d) is complete.

To summarize the above situation, we shall call $(X,d,\mu,\mathscr{E})$ a complete Dirichlet metric measure space endowed with a "*carré du champ*", in short a complete Dirichlet metric measure space.

We define the Sobolev space $W^{1,2}(X)$ as the domain $\mathscr{D}$ endowed with the norm $[\|f\|_{L^2}^2 + \mathscr{E}(f,f)]^{1/2}$, which is a Hilbert space. For an open set $U \subset X$, the Sobolev spaces $W^{1,p}(U)$ and $W_0^{1,p}(U)$ are defined in a usual sense, see the introduction in [8] for instance.

Corresponding to such a Dirichlet form $\mathscr{E}$, there exists an operator denoted by $\mathcal{L}$, acting on a dense domain $\mathscr{D}(\mathcal{L})$ in $L^2(X,\mu)$, $\mathscr{D}(\mathcal{L}) \subset W^{1,2}(X)$, such that for all $f \in \mathscr{D}(\mathcal{L})$ and each $g \in W^{1,2}(X)$,

$$\int_X \mathcal{L}f(x)g(x)\,\mathrm{d}\mu(x) = \mathscr{E}(f,g).$$

Throughout this paper, we denote by $\mathcal{P}_t = \mathrm{e}^{-t\sqrt{\mathcal{L}}}$ the Poisson semigroup associated with $\mathcal{L}$, and by $p_t(x,y)$ the kernel of $\mathcal{P}_t = \mathrm{e}^{-t\sqrt{\mathcal{L}}}$.

Let $B(x,r)$ denote the open ball with center x and radius r with respect to the distance d, and set $aB(x,r) = B(x,ar)$ for any $a > 0$. Moreover, unless otherwise specified for a ball B, we denote its radius and center by r_B and x_B, respectively.

We assume that μ is doubling, i.e., there exists a constant $C_d > 0$ such that for every ball $B(x,r) \subset X$,

$$\text{(D)} \qquad \mu(B(x,2r)) \leqslant C_d\mu(B(x,r)) < \infty.$$

Note that the doubling property of μ implies there exists $Q > 1$ such that for any $0 < r < R < \infty$ and $x \in X$,

$$\mu(B(x,R)) \leqslant C_d\Big(\frac{R}{r}\Big)^Q \mu(B(x,r)),$$

and reverse doubling property holds on a connected space (cf. [21], Remark 8.1.15), i.e., there exist constants $0 < \kappa \leqslant Q$ and $0 < c < 1$ such that for any $0 < r < R < \infty$

and $x \in X$,
$$c\Big(\frac{R}{r}\Big)^{\kappa}\mu(B(x,r)) \leqslant \mu(B(x,R)).$$
There also exist constants $C > 0$ and $0 \leqslant N \leqslant Q$ such that

$$\mu(B(y,r)) \leqslant C\Big(1 + \frac{d(x,y)}{r}\Big)^{N}\mu(B(x,r)) \tag{1.2}$$

uniformly for all $x, y \in X$ and $r > 0$. Indeed, property (1.2) with $N = Q$ is a direct consequence of the triangle inequality of the metric d and the doubling property (D).

An L^2-Poincaré inequality is needed. Precisely, we assume that $(X, d, \mu, \mathscr{E})$ supports an L^2-Poincaré inequality, namely, there exists a constant $C_P > 0$ such that for all balls $B = B(x_B, r_B)$ and $W^{1,2}(B)$ functions f,

$$\Big(\fint_B |f - f_B|^2 \,\mathrm{d}\mu\Big)^{1/2} \leqslant C_P r_B \Big(\fint_B |D_x f|^2 \,\mathrm{d}\mu\Big)^{1/2}. \tag{P$_2$}$$

Under the validity of a doubling measure and an L^2-Poincaré inequality on $(X, d, \mu, \mathscr{E})$, Sturm in [50] established two sides Gaussian bounds for the heat kernel. By Bochner's subordination formula, the Poisson kernel and its time derivatives admit the Poisson upper bound as follows:

$$|t^k \partial_t^k p_t(x,y)| \leqslant C_k \frac{t}{t + d(x,y)} \frac{1}{\mu(B(x, t + d(x,y)))}, \quad k = 0, 1, 2, \dots, \tag{PUB}$$

see for instance [25].

Next, let us recall the definition of harmonic functions on the upper half-space. We view $X \times \mathbb{R}$ as the product space of X and $\mathbb{R}$, and consider its product distance and product measure. The Sobolev space $W^{1,2}(X \times \mathbb{R})$ is defined as the completion of all compactly supported Lipschitz functions v on $X \times \mathbb{R}$ under the norm

$$\sqrt{\iint_{X\times\mathbb{R}} (|v|^2 + |D_x v|^2 + |\partial_t v|^2) \,\mathrm{d}\mu \,\mathrm{d}t}.$$

The Sobolev spaces $W^{1,2}_{\mathrm{loc}}(U)$, $W^{1,2}(U)$ for an open set $U \subset X \times \mathbb{R}$ are defined in the same manner as those Sobolev spaces on X. We say that $u \in W^{1,2}_{\mathrm{loc}}(X \times \mathbb{R}_+)$ is an $\mathcal{L}_+$-harmonic function in $X \times \mathbb{R}_+$ if

$$\iint_{X\times(0,\infty)} (\partial_t u \partial_t \phi + \langle D_x u, D_x \phi\rangle) \,\mathrm{d}\mu \,\mathrm{d}t = 0$$

holds for all Lipschitz functions ϕ with compact support in $X \times \mathbb{R}_+$.

In what follows, we always assume that α is a nonnegative ball function, which means its domain is a class of balls in X. The ball function α is said to be decreasing if there exists a constant $C > 0$ such that for any $x, y \in X$ and $0 < r < R < \infty$,

$$\text{(De)} \qquad \alpha(B(x,R)) \leqslant \alpha(B(x,r)).$$

Moreover, we say that the ball function α is translation invariant if there exist constants $C, c > 0$ such that for any $x, y \in X$ and $r > 0$,

$$\text{(TI)} \qquad c\alpha(B(y,r)) \leqslant \alpha(B(x,r)) \leqslant C\alpha(B(y,r)).$$

Now, let us introduce the spaces that we are focused on in this paper.

Definition 1.1. Let α be a nonnegative ball function. The harmonic mean oscillation space $\mathrm{HL}^{2,\alpha}(X \times \mathbb{R}_+)$ consists of all $\mathcal{L}_+$-harmonic functions u satisfying an α-Carleson condition

$$\|u\|_{\mathrm{HL}}^{2,\alpha} = \sup_{B \subset X} \frac{1}{\alpha(B)} \left(\frac{1}{\mu(B)} \iint_{B \times (0, r_B)} |t\nabla u(x,t)|^2 \frac{\mathrm{d}\mu(x)\,\mathrm{d}t}{t} \right)^{1/2} < \infty.$$

Here and in what follows, $\nabla = (D_x, \partial_t)$ and $|\nabla u|^2 = |D_x u|^2 + |\partial_t u|^2$.

The generalized Morrey space was introduced independently by Mizuhara in 1991 (see [40]) and Nakai in 1994, see [43]. An interested reader may be also referred to [3], [44]. Here, we define it as follows:

Definition 1.2. Let α be a nonnegative ball function. We say that a function $f \in L^2_{\mathrm{loc}}(X)$ belongs to the generalized Morrey space $L^{2,\alpha}(X)$ if

$$\|f\|_{L^{2,\alpha}} = \sup_{B \subset X} \frac{1}{\alpha(B)} \left(\fint_B |f(x) - f_B|^2 \,\mathrm{d}\mu(x) \right)^{1/2} < \infty.$$

Remark 1.3. If $\alpha \equiv 1$, then $L^{2,\alpha}(X) = \mathrm{BMO}(X)$. And if $\alpha(B) = [\mu(B)]^{\lambda}$ with $-\frac{1}{2} < \lambda < 0$, then $L^{2,\alpha}(X)$ is the classical Morrey space $L^{2,\lambda}(X)$ (also called Morrey-Campanato space). More generally, the ball function α can be taken as $\alpha(B) = \|\chi_B\|_{L^{p(\cdot)}}^{-1}$, where $\|\cdot\|_{L^{p(\cdot)}}$ is the norm of a variable Lebesgue function. However, the exponent 2 in $L^{2,\alpha}(X)$ cannot be extended to a variable exponent or even any other constant, which is determined by the structure of the Dirichlet form and the technology used in this paper such as the spectral theory and the Caccioppoli inequality. Therefore, the result about the variable Morrey function is not included in our paper, see [9], [51] for more about variable exponent function spaces.

The following theorem is the main result of this paper.

Theorem 1.4. *Let $(X, d, \mu, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition* (D) *and admitting an L^2-Poincaré inequality* (P_2)*. Suppose that the nonnegative ball function α is decreasing* (De)*.*

(i) *Further suppose that α is translation invariant* (TI)*. If $u \in \mathrm{HL}^{2,\alpha}(X \times \mathbb{R}_+)$, then there exists a function $f \in L^{2,\alpha}(X)$ such that $u(x,t) = \mathcal{P}_t f(x)$. Moreover, there exists a constant $C > 0$ independent of u such that*

$$\|f\|_{L^{2,\alpha}} \leqslant C\|u\|_{\mathrm{HL}^{2,\alpha}}.$$

(ii) *If $f \in L^{2,\alpha}(X)$, then $u(x,t) = \mathcal{P}_t f(x) \in \mathrm{HL}^{2,\alpha}(X \times \mathbb{R}_+)$, and there exists a constant $C > 0$ independent of f such that*

$$\|u\|_{\mathrm{HL}^{2,\alpha}} \leqslant C\|f\|_{L^{2,\alpha}}.$$

Some remarks are in order.

Remark 1.5. Due to some technical reasons, in Theorem 1.4 (i), we assume an additional condition that α satisfies the translation invariant (TI). This is due to the following reasons. Firstly, in Lemma 2.2, we obtain the pointwise upper bound of the time regularity of the $\mathrm{HL}^{2,\alpha}$-function as follows:

$$|t\partial_t u(x,t)| \leqslant C\alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}}. \tag{1.3}$$

With the help of this regularity, in Proposition 2.6, we show that a harmonic function is bounded from above and below by a bound similar to (1.3). If we want to take advantage of the Liouville theorem, we need the translation invariant property to ensure that the upper bound in (1.3) is at least independent of the center of the ball. This is the first time the assumption (TI) appears and the substantial reason why it appears. However, in the cases of Euclidean space, Lie group of polynomial growth, and metric space with a uniformly distributed measure, if $\alpha(B) = [\mu(B)]^\lambda$ with $-\frac{1}{2} < \lambda < 0$, the assumption (TI) naturally holds.

Remark 1.6. Regarding the proof of Theorem 1.4, the main difficulty lies in that the pointwise bound for the gradient of the Poisson kernel does not hold in a general metric space, unless assuming a strong nonnegative curvature condition or group structure, see [36], [37]. Indeed, it fails already for uniformly elliptic operators, see for instance [8], [26]. To overcome this difficulty, based on the structure of the elliptic equation, we adopt a space-time conversion technology in Proposition 5.2 of [25] to control the spatial derivation part in a tent domain $B(x_B, r_B) \times (0, r_B)$; see Proposition 3.2 for more details.

Remark 1.7. As we already said in Remark 1.3, if $\alpha \equiv 1$, then $L^{2,\alpha}(X) = \mathrm{BMO}(X)$. And it is worth noting that Jin-Li-Shen in [28] studied the Dirichlet problem for the Laplace equation with boundary value in the BMO space. However, in view of the proof from the solution to the initial value, the essential ingredients in this paper are different from that of [28]. The argument of Jin-Li-Shen (see [28]) depends heavily on three nontrivial results:

(i) the Hardy space $H^1(X)$ is complete;

(ii) the Hardy-BMO dual theorem of Fefferman-Stein;

(iii) the time derivative of the Poisson kernel is in the Hardy space.

However, the theory about the predual of the generalized Morrey space is still somewhat incomplete, see [2], [45]. To overcome the lack of these properties, in this paper, we adopt a method from [27], which is based on the basic L^2-theory via the feature of Morrey functions and BMO functions; see the proof of Theorem 3.1 for more details. As a consequence, our results work with both the Morrey space and the BMO space.

Remark 1.8. In the case $X = \mathbb{R}^n$ and $\alpha(B) = |B|^\lambda$ with $-\frac{1}{2} < \lambda < 0$, we write $L^{2,\alpha}(\mathbb{R}^n) = L^{2,\lambda}(\mathbb{R}^n)$ and $\mathrm{HL}^{2,\alpha}(\mathbb{R}^n) = \mathrm{HL}^{2,\lambda}(\mathbb{R}^n)$ for simplicity. Combining some known results, we can observe the following facts.

(i) For a function f defined on $\mathbb{R}^n$, the following statements are equivalent.

(a) $\|f\|_{L^{2,\lambda}} < \infty$,

(b) $\sup\limits_{B\subset\mathbb{R}^n} |B|^{-1/2-\lambda}(\iint_{B\times(0,r_B)} |t\nabla\mathcal{P}_t f(x)|^2(\mathrm{d}x\,\mathrm{d}t)/t)^{1/2} < \infty$,

(c) $\sup\limits_{B\subset\mathbb{R}^n} |B|^{-1/2-\lambda}(\iint_{B\times(0,r_B)} |t\nabla\mathscr{P}_t f(x)|^2(\mathrm{d}x\,\mathrm{d}t)/t)^{1/2} < \infty$.

The equivalence of (a) and (b) is included in Theorem 1.4; the equivalence of (a) and (c) is discussed in [48]. Here $\mathscr{P}_t = \mathrm{e}^{-t\sqrt{\mathscr{L}}}$ is the Poisson semigroup associated with the Schrödinger operator $\mathscr{L} = \mathcal{L} + V$ with V belonging to the reverse Hölder class $RH_q(\mathbb{R}^n)$ for some $q \geqslant n$. These equivalences are the counterpart of the famous words that Stein-Weiss wrote at the beginning of Chapter 2 of [49]. Namely, we find that a function f can be verified to belong to the Morrey space $L^{2,\lambda}(\mathbb{R}^n)$, by checking that its Poisson integral $\mathcal{P}_t f$ (or $\mathscr{P}_t f$) satisfies the Carleson condition. Although the different natures of Schrödinger operator and Laplace operator, a bit surprisingly, they induce the same Carlson condition.

(ii) For a function u defined on $\mathbb{R}^{n+1}_+$, if u is harmonic, then there exist constants $C, c > 0$ such that

$$c\|u\|_{L^\infty(L^{2,\lambda})} \leqslant \|u\|_{\mathrm{HL}^{2,\lambda}} \leqslant C\|u\|_{L^\infty(L^{2,\lambda})}, \tag{1.4}$$

where

$$\|u\|_{L^\infty(L^{2,\lambda})} = \sup_{t>0}\sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^\lambda}\left(\fint_B |u(x,t)|^2\,\mathrm{d}\mu(x)\right)^{1/2}.$$

This result is obtained by using the Morrey space $L^{2,\lambda}$ as a bridge, and combining Theorem 1.4 with Therorem 2.17 of [35]. Let us distinguish between the two norms involved in (1.4). The finite norm $\|u\|_{\mathrm{HL}^{2,\lambda}} < \infty$ shows that u induces a Carleson type measure $t|\nabla u(x,t)|^2 \,\mathrm{d}\mu(x)\,\mathrm{d}t$. However, the finite norm $\|u\|_{L^\infty(L^{2,\lambda})} < \infty$ means that for any $t > 0$, $u(\cdot,t)$ belongs to the Morrey space $L^{2,\lambda}(\mathbb{R}^n)$. Note that the Carleson condition $\|u\|_{\mathrm{HL}^{2,\lambda}}$ tells us the information about the gradient of u on a tent domain, whereas the essentially-bounded-Morrey condition $\|u\|_{L^\infty(L^{2,\lambda})}$ indicates the integrability of u on a ball. This fact surprised us since we can utilize the non-derivative term of u to describe its derivative term ∇u. A similar but different phenomenon have appeared in [20]. Without the classical derivative structure, the author of [20] takes advantage of a novel "derivative", nowadays named the "Hajłasz gradient" after him, to define the Sobolev space on the metric measure space, see [31], [32], [53] for more details.

To the best of our knowledge, Theorem 1.4 is new even for the second order degenerate elliptic operator $L = -\operatorname{div} A\nabla$, where $A = A(x)$ is an $n \times n$ matrix of real, symmetric, bounded measurable coefficients, defined on $\mathbb{R}^n$, and satisfies certain degenerate conditions such as A_2-weight, i.e., there exist constants $0 < \lambda \leqslant \Lambda < \infty$ such that

$$\lambda w(x)|\xi|^2 \leqslant \langle A\xi, \xi\rangle \leqslant \Lambda w(x)|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^n,$$

where $w \geqslant 0$ is an A_2-weight, see [42]. The doubling condition is a consequence of the A_2-weight, and it was known from [15] that an L^2-Poincaré inequality holds.

Corollary 1.9. *Let $L = -\operatorname{div} A\nabla$ be a second order degenerate elliptic operator on $\mathbb{R}^n$. Then an $\mathcal{L}_+$-harmonic function $u \in \mathrm{HL}^{2,\alpha}(\mathbb{R}^{n+1}_+)$ if and only if there exists a function $f \in L^{2,\alpha}(\mathbb{R}^n)$ such that $u(x,t) = \mathcal{P}_t f(x)$. Moreover, there exists a constant $C > 1$ such that*

$$C^{-1}\|f\|_{L^{2,\alpha}(\mathbb{R}^n)} \leqslant \|u\|_{\mathrm{HL}^{2,\alpha}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{2,\alpha}(\mathbb{R}^n)}.$$

The paper is organized as follows. In Section 2, we obtain some properties for harmonic functions satisfying the α-Carleson condition. Sections 3 will be devoted to the proof of the main result (i.e., the proof of Theorem 1.4).

The letter C (or c) will denote a positive constant that may vary from line to line but will remain independent of the main variables.

2. Some properties of harmonic functions satisfying α-Carleson

This section is devoted to studying some properties of the $\mathrm{HL}^{2,\alpha}$-function u, including the pointwise upper bound of $|t\partial_t u(x,t)|$, the Calderón reproducing formula, and the uniform bounedness of $u(\cdot,\varepsilon)$ in $L^{2,\alpha}(X)$ for any $\varepsilon > 0$. These properties will play a significant role in the proof of Theorem 1.4 (i).

Lemma 2.1. *Assume that the Dirichlet metric measure space $(X,d,\mu,\mathscr{E})$ satisfies* (D) *and admits* (P_2). *There exist constants $0<\theta<1$ and $C>0$ such that if $g\in W^{1,2}(B(x,2r)\times(s,s+2r))$ is an $\mathcal{L}_+$-harmonic function, then it holds that*

$$|g(y,t)-g(z,t')| \leqslant C\Big(\frac{d(y,z)+|t-t'|}{2r}\Big)^{\theta}\frac{1}{r\mu(B(x,2r))}\iint_{B(x,2r)\times(s,s+2r)}|g(z,s')|\,\mathrm{d}\mu(z)\,\mathrm{d}s' \tag{2.1}$$

for any ball $B(x,r)$, $s>0$ and $(y,t),(z,t')\in B(x,r)\times(s+\frac12 r, s+\frac32 r)$.

Proof. For the proof, we refer to [50], Proposition 3.2, Corollary 3.3. See also [24], Lemma 2.8 and [4], Theorem 1.1, Corollary 1.2. □

Lemma 2.2. *Assume that the Dirichlet metric measure space $(X,d,\mu,\mathscr{E})$ satisfies* (D) *and admits* (P_2). *Suppose that the nonnegative ball function α satisfies* (De). *If $u\in\mathrm{HL}^{2,\alpha}(X\times\mathbb{R}_+)$, then $\partial_t u(x,t)$ is also an $\mathcal{L}_+$-harmonic function on $X\times\mathbb{R}_+$. Moreover, there exists a constant $C>0$ such that for any $x\in X$ and $t>0$,*

$$|t\partial_t u(x,t)| \leqslant C\alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}}.$$

Proof. By [13], Remark 2.3 and Proposition 4.1, we know that there is a weak L^2-Poincaré inequality on $X\times\mathbb{R}$, i.e., there exists $\sigma\geqslant 1$ such that for each $g\in W^{1,2}(B(y,\sigma r)\times(s-\sigma r,s+\sigma r))$ with $(y,s)\in X\times\mathbb{R}$ and $r>0$, it holds that

$$\iint_{B(y,r)\times(s-r,s+r)}|g-g_{B(y,r)\times(s-r,s+r)}|^2\,\mathrm{d}\mu\,\mathrm{d}t \leqslant Cr^2\iint_{B(y,\sigma r)\times(s-\sigma r,s+\sigma r)}|\nabla g|^2\,\mathrm{d}\mu\,\mathrm{d}t, \tag{2.2}$$

where $\nabla=(D_x,\partial_t)$. By the self-improvement of the Poincaré inequality from [30], we see that an $L^{2-\delta}$-Poincaré inequality holds on $X\times\mathbb{R}$ for some $0<\delta<1$, which means that (2.2) can be improved to

$$\frac{1}{r\mu(B(y,r))}\iint_{B(y,r)\times(s-r,s+r)}|g-g_{B(y,r)\times(s-r,s+r)}|\,\mathrm{d}\mu\,\mathrm{d}t \leqslant Cr\Big(\frac{1}{r\mu(B(y,r))}\iint_{B(y,\sigma r)\times(s-\sigma r,s+\sigma r)}|\nabla g|^{2-\delta}\,\mathrm{d}\mu\,\mathrm{d}t\Big)^{1/(2-\delta)}. \tag{2.3}$$

Let $\varepsilon > 0$. For any $-\frac{1}{8}\varepsilon < h < \frac{1}{8}\varepsilon$, we define the following difference quotient function:

$$u(x, t; h) = \frac{u(x, t+h) - u(x, t)}{h}.$$

It follows that for any $-\frac{1}{8}\varepsilon < h < \frac{1}{8}\varepsilon$, $u(\cdot, \cdot; h)$ is an $\mathcal{L}_+$-harmonic function on $X \times (\varepsilon, \infty)$; see for example [25], Lemma 4.1. Therefore, we can conclude by the mean value property (see [4], Theorem 5.4) that for any $t > 3\varepsilon$,

(2.4)

$$|u(x, t; h)| \leqslant C\bigg(\frac{1}{t\mu(B(x,t))} \iint_{B(x,t/(2\sigma))\times((1-1/(2\sigma))t,(1+1/(2\sigma))t)} |u(y, s; h)|^2 \,\mathrm{d}\mu(y)\,\mathrm{d}s\bigg)^{1/2}.$$

Set $E_{x,t} = B(x, \frac{3}{4}t) \times (\frac{1}{4}t, \frac{7}{4}t)$. Applying the Poincaré inequality (2.3) and the argument in Theorem 8.1.7 of [21], we see that for all $(y, s) \in B(x, t/(2\sigma)) \times ((1 - 1/(2\sigma))t, (1 + 1/(2\sigma))t)$ and $|h| \ll t$, it holds that

$$\begin{aligned} (2.5)\qquad &|u(y, s+h) - u(y, s)| \\ &\qquad \leqslant Ch[\mathcal{M}_{2-\delta}(|\nabla u|\chi_{E_{x,t}})(y, s+h) + \mathcal{M}_{2-\delta}(|\nabla u|\chi_{E_{x,t}})(y, s)], \end{aligned}$$

where $\mathcal{M}_{2-\delta}$ is the $L^{(2-\delta)}$-Hardy-Littlewood maximal operator on $X \times \mathbb{R}_+$ defined by

$$\mathcal{M}_{2-\delta}(v)(x, s) = \sup_{\widetilde{B}\ni(x,s)} \bigg(⨍_{\widetilde{B}} |v|^{2-\delta} \,\mathrm{d}\mu\,\mathrm{d}t\bigg)^{1/(2-\delta)}$$

for any function $v \in L^1_{\mathrm{loc}}(X \times \mathbb{R}_+)$. Here $\widetilde{B}$ is the open ball in $X \times \mathbb{R}_+$.

Combining (2.5) with (2.4) and the fact that $\mathcal{M}_{2-\delta}$ is bounded on $L^2(X \times \mathbb{R}_+)$, we further deduce that

$$\begin{aligned} &|u(x, t; h)| \\ &\quad \leqslant C\bigg(\frac{1}{t\mu(B(x,t))} \iint_{B(x,t/(2\sigma))\times((1-1/((2\sigma)))t,(1+1/(2\sigma))t)} |u(y, s; h)|^2 \,\mathrm{d}\mu(y)\,\mathrm{d}s\bigg)^{1/2} \\ &\quad \leqslant C\bigg(\frac{1}{t\mu(B(x,t))} \iint_{B(x,t/(2\sigma))\times((1-1/(2\sigma))t,(1+1/(2\sigma))t)} [\mathcal{M}_{2-\delta}(|\nabla u|\chi_{E_{x,t}})(y, s+h) \\ &\qquad + \mathcal{M}_{2-\delta}(|\nabla u|\chi_{E_{x,t}})(y, s)]^2 \,\mathrm{d}\mu(y)\,\mathrm{d}s\bigg)^{1/2} \\ &\quad \leqslant C\bigg(\frac{1}{t\mu(B(x,t))} \iint_{B(x,3t/4)\times(t/4,7t/4)} |\nabla u|^2 \,\mathrm{d}\mu(y)\,\mathrm{d}s\bigg)^{1/2} \leqslant \frac{C}{t}\alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}}. \end{aligned}$$

Letting $h \to 0$, we conclude that for any $t > 3\varepsilon$,

$$|t\partial_t u(x, t)| \leqslant C\alpha(B(x, t))\|u\|_{\mathrm{HL}^{2,\alpha}}.$$

Since ε is arbitrary and the constant $C > 0$ is harmless, we can obtain the required estimate, i.e.,

$$|t\partial_t u(x,t)| \leqslant C\alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}}$$

for any $x \in X$ and $t > 0$.

It remains to show that $\partial_t u(x,t)$ is also $\mathcal{L}_+$-harmonic. Since $u(x,t;h)$ is an $\mathcal{L}_+$-harmonic function on $X \times (\varepsilon,\infty)$ for all $-\frac{1}{8}\varepsilon < h < \frac{1}{8}\varepsilon$, and

$$|u(x,t;h)| \leqslant \frac{C}{t}\alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}}$$

for all $t > 3\varepsilon$, (2.1) shows that $u(x,t;h)$ is uniformly continuous on any compact set of $X \times (3\varepsilon,\infty)$ for all $h \in (-\frac{1}{8}\varepsilon, \frac{1}{8}\varepsilon)$. This implies that $\partial_t u(x,t)$ is locally uniformly continuous on $X \times (3\varepsilon,\infty)$, and the convergence $u(x,t;h) \to \partial_t u(x,t)$ is locally uniform on $X \times (3\varepsilon,\infty)$. Hence, we can apply Theorem 7.25 of [5] with the locally uniform convergences $\pm u(x,t;h) \to \pm\partial_t u(x,t)$ to deduce that $\pm\partial_t u(x,t)$ are both super-harmonic functions on $X \times (3\varepsilon,\infty)$. Since ε is arbitrary, it follows that $\partial_t u(x,t)$ is $\mathcal{L}_+$-harmonic on $X \times \mathbb{R}_+$, which completes the proof. □

Lemma 2.3. *Assume that the Dirichlet metric measure space* $(X, d, \mu, \mathscr{E})$ *satisfies* (D) *and admits* (P$_2$). *Suppose that the nonnegative ball function* α *satisfies* (De). *If* $u \in \mathrm{HL}^{2,\alpha}(X \times \mathbb{R}_+)$, *then there exists a constant* $C > 0$ *such that*

$$|u(x,t)-u(y,t)| \leqslant \begin{cases} C\Big(\dfrac{d(x,y)}{t}\Big)^{\theta}\alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}}, & d(x,y) \leqslant t/4\sigma, \\ C\Big(\dfrac{d(x,y)}{t}\Big)^{\theta}[\alpha(B(x,t)) + \alpha(B(y,t))]\|u\|_{\mathrm{HL}^{2,\alpha}}, & d(x,y) > t/4\sigma, \end{cases}$$

where $\sigma \geqslant 1$ *as in* (2.2).

P r o o f. For the large time case $t \geqslant 4\sigma d(x,y)$, it follows from (2.1) that

$$\begin{aligned} |u(x,t) - u(y,t)| &= |(u(x,t) - M) - (u(y,t) - M)| \\ &\leqslant C\Big(\frac{d(x,y)}{t}\Big)^{\theta}\frac{1}{t\mu(B(x,t))} \\ &\quad\times \iint_{B(x,t/(2\sigma))\times((1-1/(2\sigma))t,(1+1/(2\sigma))t)} |u(z,s) - M|\,\mathrm{d}\mu(z)\,\mathrm{d}s \end{aligned}$$

for any constant $M \in \mathbb{R}$. Taking $M = u_{B(x,t/(2\sigma))\times((1-1/(2\sigma))t,(1+1/(2\sigma))t)}$ and using (2.2) leads to

$$\text{(2.6)}\quad |u(x,t) - u(y,t)| \leqslant Ct\Big(\frac{d(x,y)}{t}\Big)^{\theta}\Big(\frac{1}{t\mu(B(x,t))}\iint_{B(x,t/2)\times(t/2,3t/2)} |\nabla u(z,s)|^2\,\mathrm{d}\mu(z)\,\mathrm{d}s\Big)^{1/2}$$

$$
\begin{aligned}
&\leqslant C\Big(\frac{d(x,y)}{t}\Big)^{\theta}\Big(\frac{1}{\mu(B(x,t))}\iint_{B(x,2t)\times(0,2t)}|s\nabla u(z,s)|^2\frac{\mathrm{d}\mu(z)\,\mathrm{d}s}{s}\Big)^{1/2}\\
&\leqslant C\Big(\frac{d(x,y)}{t}\Big)^{\theta}\alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}}.
\end{aligned}
$$

For the small time case $t < 4\sigma d(x,y)$, we let $t' = 4\sigma d(x,y)$. By Lemma 2.2, it holds that

$$
\begin{aligned}
|u(x,t)-u(x,t')| &\leqslant \int_t^{t'}|s\partial_s u(x,s)|\frac{\mathrm{d}s}{s}\leqslant C\int_t^{t'}\alpha(B(x,s))\frac{\mathrm{d}s}{s}\|u\|_{\mathrm{HL}^{2,\alpha}}\\
&\leqslant C\log\Big(\frac{4\sigma d(x,y)}{t}\Big)\alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}},
\end{aligned}
$$

and similarly,

$$
|u(y,t)-u(y,t')| \leqslant C\log\Big(\frac{4\sigma d(x,y)}{t}\Big)\alpha(B(y,t))\|u\|_{\mathrm{HL}^{2,\alpha}}.
$$

Combining these two estimates above and (2.6), we deduce from the triangle inequality that

$$
\begin{aligned}
|u(x,t)-u(y,t)| &\leqslant |u(x,t)-u(x,t')|+|u(x,t')-u(y,t')|+|u(y,t')-u(y,t)|\\
&\leqslant C\Big[\log\Big(\frac{4\sigma d(x,y)}{t}\Big)\alpha(B(x,t))+\Big(\frac{d(x,y)}{4\sigma d(x,y)}\Big)^{\theta}\alpha(B(x,4\sigma d(x,y)))\\
&\quad+\log\Big(\frac{4\sigma d(x,y)}{t}\Big)\alpha(B(y,t))\Big]\|u\|_{\mathrm{HL}^{2,\alpha}}\\
&\leqslant C\Big(\frac{d(x,y)}{t}\Big)^{\theta}[\alpha(B(x,t))+\alpha(B(y,t))]\|u\|_{\mathrm{HL}^{2,\alpha}},
\end{aligned}
$$

where we used the trivial inequality $\log r \leqslant Cr^{\theta}$ for all $r > 1$. This completes the proof. □

Remark 2.4. If the ball function α satisfies (TI), the bound for the small time case in Lemma 2.3 would coincide with the large time case, and hence it holds for any $x, y \in X$ and $t > 0$, that

$$
|u(x,t)-u(y,t)| \leqslant C\Big(\frac{d(x,y)}{t}\Big)^{\theta}\alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}}.
$$

In the sequel, for the convenience of writing, we set

$$
u_s(x) = u(x,s) \quad \forall\, x\in X,\ s>0.
$$

Lemma 2.5. *Assume that the Dirichlet metric measure space $(X, d, \mu, \mathscr{E})$ satisfies* (D) *and admits* (P_2). *Suppose that the nonnegative ball function α satisfies* (De) *and* (TI). *If $u \in \mathrm{HL}^{2,\alpha}(X \times \mathbb{R}_+)$, then the Poisson integral $\mathcal{P}_t(u_s)$ of u_s is well-defined. Moreover, there exists a constant $C > 0$ such that*

$$|t\partial_t \mathcal{P}_t(u_s)(x)| \leqslant C\Big(\frac{t}{s}\Big)^{\theta} \alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}}.$$

Proof. It follows from $\mathcal{P}_t 1 \equiv 1$ that for any $x \in X$,

$$\mathcal{P}_t(u_s)(x) = \mathcal{P}_t(u_s - u_s(x))(x) + u_s(x). \tag{2.7}$$

Using (PUB), Remark 2.4 and the classical annuli argument, one has

$$\begin{aligned} &|\mathcal{P}_t(u_s - u_s(x))(x)| \\ &\quad\leqslant C\int_X \frac{t}{t+d(x,y)} \frac{|u(y,s)-u(x,s)|}{\mu(B(x,t+d(x,y)))}\,\mathrm{d}\mu(y) \\ &\quad\leqslant C\int_X \frac{t}{t+d(x,y)} \Big(\frac{d(x,y)}{s}\Big)^{\theta} \frac{\alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}}}{\mu(B(x,t+d(x,y)))}\,\mathrm{d}\mu(y) \\ &\quad\leqslant C\Big(\frac{t}{s}\Big)^{\theta}\alpha(B(x,s))\|u\|_{\mathrm{HL}^{2,\alpha}} \int_X \Big(\frac{t}{t+d(x,y)}\Big)^{1-\theta} \frac{\mathrm{d}\mu(y)}{\mu(B(x,t+d(x,y)))} \\ &\quad\leqslant C\Big(\frac{t}{s}\Big)^{\theta}\alpha(B(x,s))\|u\|_{\mathrm{HL}^{2,\alpha}}. \end{aligned} \tag{2.8}$$

This together with (2.7) gives us that

$$|\mathcal{P}_t(u_s)(x)| \leqslant C\Big(\frac{t}{s}\Big)^{\theta}\alpha(B(x,s))\|u\|_{\mathrm{HL}^{2,\alpha}} + |u_s(x)|, \tag{2.9}$$

which means that $\mathcal{P}_t(u_s)(x)$ is well-defined.

Now, it remains to estimate $t\partial_t\mathcal{P}_t(u_s)(x)$. We can use the conservation property $\mathcal{P}_t 1 \equiv 1$, (PUB) and a similar argument as (2.8) to deduce that

$$\begin{aligned} |t\partial_t\mathcal{P}_t(u_s)(x)| &= \Big|\int_X t\partial_t p_t(x,y)(u_s(y)-u_s(x))\,\mathrm{d}\mu(y)\Big| \\ &\leqslant \int_X \frac{t}{t+d(x,y)} \frac{|u_s(y)-u_s(x)|}{\mu(B(x,t+d(x,y)))}\,\mathrm{d}\mu(y) \\ &\leqslant C\Big(\frac{t}{s}\Big)^{\theta}\alpha(B(x,s))\|u\|_{\mathrm{HL}^{2,\alpha}}, \end{aligned}$$

which completes the proof. ∎

Proposition 2.6. *Assume that the Dirichlet metric measure space* $(X,d,\mu,\mathscr{E})$ *satisfies* (D) *and admits* (P_2). *Suppose that the nonnegative ball function* α *satisfies* (De) *and* (TI). *If* $u\in \mathrm{HL}^{2,\alpha}(X\times\mathbb{R}_+)$, *then for any* $x\in X$ *and* $s,t>0$ *it holds that*

$$u(x,t+s)=\mathcal{P}_t(u(\cdot,s))(x).$$

Proof. For any $x\in X$ and $t>0$, set

$$v(x,t)=\partial_t u(x,t+s)-\mathcal{P}_t(\partial_s u(\cdot,s))(x).$$

By Lemma 2.2, $\partial_t u(x,t+s)$ is $\mathcal{L}_+$-harmonic and therefore Hölder continuous on $X\times(-s,\infty)$. It follows that

$$v(x,0)=\lim_{t\to 0^+}v(x,t)=\lim_{t\to 0^+}\{\partial_t u(x,t+s)-\mathcal{P}_t(\partial_s u(\cdot,s))(x)\}=0.$$

We can extend $v(x,t)$ to a function $w(x,t)$ on $X\times\mathbb{R}$ as follows:

$$w(x,t)=\begin{cases} v(x,t), & t\geqslant 0;\\ -v(x,-t), & t<0.\end{cases}$$

Obviously, w is also harmonic on $X\times\mathbb{R}$. We claim that $w\equiv 0$ on $X\times\mathbb{R}$. On the one hand, there holds by Lemma 2.2 that for any $x\in X$ and $t>0$,

$$|\partial_t u(x,t+s)|\leqslant \frac{C}{t+s}\alpha(B(x,t+s))\|u\|_{\mathrm{HL}^{2,\alpha}}\leqslant \frac{C}{s}\alpha(B(x,s))\|u\|_{\mathrm{HL}^{2,\alpha}}.$$

On the other hand, it follows from Lemma 2.2, (TI) and the conservation property $\mathcal{P}_t 1\equiv 1$ that for any $x\in X$ and $t>0$,

$$\begin{aligned}|\mathcal{P}_t(\partial_s u(\cdot,s))(x)|&\leqslant \int_X p_t(x,y)|\partial_s u(y,s)|\,\mathrm{d}\mu(y)\\ &\leqslant \frac{C}{s}\|u\|_{\mathrm{HL}^{2,\alpha}}\int_X p_t(x,y)\alpha(B(y,s))\,\mathrm{d}\mu(y)\leqslant \frac{C}{s}\alpha(B(x,s))\|u\|_{\mathrm{HL}^{2,\alpha}}.\end{aligned}$$

Based on the above argument, we see that for any $x\in X$ and $t\in\mathbb{R}$,

$$|w(x,t)|\leqslant \frac{C}{s}\alpha(B(x,s))\|u\|_{\mathrm{HL}^{2,\alpha}}.$$

This together with (TI) implies that w is a bounded harmonic function on $X\times\mathbb{R}$. However, since $X\times\mathbb{R}$ satisfies a doubling condition and supports an L^2-Poincaré inequality, the Liouville property implies that there is no bounded harmonic function on $X\times\mathbb{R}$ other than constant, see [4], [50]. Noticing $w(x,0)=0$, we know that $w\equiv 0$ on $X\times\mathbb{R}$.

Hence, for any $s,t>0$, it holds that

$$\partial_s u(x,t+s)=\partial_t u(x,t+s)=\mathcal{P}_t(\partial_s u(\cdot,s))(x)=\partial_s\mathcal{P}_t(u(\cdot,s))(x),$$

and further,

$$\partial_s\partial_t u(x,t+s)=\partial_s\partial_t\mathcal{P}_t(u(\cdot,s))(x).$$

This implies that for any $s,s'>0$,

$$\partial_t u(x,t+s)-\partial_t\mathcal{P}_t(u(\cdot,s))(x)=\partial_t u(x,t+s')-\partial_t\mathcal{P}_t(u(\cdot,s'))(x).$$

Notice that Lemma 2.5 yields

$$|\partial_t\mathcal{P}_t(u(\cdot,s'))(x)|\leqslant\frac{C}{t}\Big(\frac{t}{s'}\Big)^{\theta}\alpha(B(x,s'))\|u\|_{\mathrm{HL}^{2,\alpha}},$$

and Lemma 2.2 yields

$$|\partial_t u(x,t+s')|\leqslant\frac{C}{t+s'}\alpha(B(x,t+s'))\|u\|_{\mathrm{HL}^{2,\alpha}}\leqslant\frac{C}{s'}\alpha(B(x,t))\|u\|_{\mathrm{HL}^{2,\alpha}},$$

which both converge to 0 as $s'\to\infty$. It holds that

$$\partial_t u(x,t+s)-\partial_t\mathcal{P}_t(u(\cdot,s))(x)\equiv 0,$$

and hence there exits a function $f(x,s)$ such that for any $t>0$,

$$u(x,t+s)-\mathcal{P}_t(u(\cdot,s))(x)=f(x,s).$$

Note that for every $x\in X$ and $s>0$,

$$f(x,s)=\lim_{t\to 0^+}f(x,s)=u(x,s)-\lim_{t\to 0^+}\mathcal{P}_t(u(\cdot,s))(x)=0.$$

One has

$$u(x,t+s)=\mathcal{P}_t(u(\cdot,s))(x).$$

The proof is complete. □

Lemma 2.7. *Assume that the Dirichlet metric measure space* $(X,d,\mu,\mathscr{E})$ *satisfies* (D) *and admits* (P_2). *Suppose that the nonnegative ball function* α *satisfies* (De) *and* (TI). *If* $u\in\mathrm{HL}^{2,\alpha}(X\times\mathbb{R}_+)$, *then there exists a constant* $C>0$ *such that for any* $\varepsilon>0$,

$$\sup_{B\subset X}\frac{1}{\alpha(B)}\bigg(\int_0^{r_B}\fint_B|t\partial_t\mathcal{P}_t u_\varepsilon(x)|^2\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}\bigg)^{1/2}\leqslant C\|u\|_{\mathrm{HL}^{2,\alpha}}.$$

P r o o f. The proof of this lemma is similar to that of [14], Theorem 1.4, see also [12], Lemma 3.4 or [16], (1.5). We left the details to the interested reader. $\square$

Lemma 2.8. *Assume that the Dirichlet metric measure space* $(X, d, \mu, \mathscr{E})$ *satisfies* (D) *and admits* (P_2). *Suppose that the nonnegative ball function* α *satisfies* (De) *and* (TI). *For any* $\varepsilon > 0$ *we set*

$$\begin{cases} F_\varepsilon(x,t) = t\sqrt{\mathcal{L}}\mathcal{P}_t u_\varepsilon(x), \\ G(x,t) = t\sqrt{\mathcal{L}}\mathcal{P}_t(I - \mathcal{P}_{r_B})g(x), \end{cases}$$

where $u \in \mathrm{HL}^{2,\alpha}(X \times \mathbb{R}_+)$ *and* $g \in L^2(B)$. *Then the following statements are valid.*

(i) *There exists a constant* $C > 0$, *independent of* ε, *such that*

$$\int_0^\infty \int_X |F_\varepsilon(x,t)G(x,t)| \,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} \leqslant C\alpha(B)[\mu(B)]^{1/2}\|u\|_{\mathrm{HL}^{2,\alpha}}\|g\|_{L^2(B)}.$$

(ii) *It holds that*

$$\int_X u_\varepsilon(I - \mathcal{P}_{r_B})g \,\mathrm{d}\mu = 4\int_0^\infty \int_X F_\varepsilon(x,t)G(x,t) \,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}.$$

P r o o f. (i) One writes

$$\begin{aligned} \int_0^\infty \int_X |F_\varepsilon(x,t)G(x,t)| \,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} &= \biggl\{\int_{T(2B)} + \sum_{k=1}^\infty \int_{T(2^{k+1}B)\setminus T(2^kB)}\biggr\} \ldots \,\mathrm{d}\mu\frac{\mathrm{d}t}{t} \\ &= \mathrm{I}_0 + \sum_{k=1}^\infty \mathrm{I}_k, \end{aligned}$$

where $T(B)$ is the tent over B, i.e., $T(B) = B \times (0, r_B)$.

For the term I_0, since $(I - \mathcal{P}_{r_B})g \in L^2(X)$, we conclude by the spectral theory that

$$\int_X \int_0^\infty |t\sqrt{\mathcal{L}}\mathcal{P}_t(I - \mathcal{P}_{r_B})g|^2 \frac{\mathrm{d}t}{t} \,\mathrm{d}\mu = \frac{1}{4}\int_X |(I - \mathcal{P}_{r_B})g|^2 \,\mathrm{d}\mu.$$

This together with Lemma 2.7, Hölder's inequality and (D), implies that

$$\begin{aligned} \mathrm{I}_0 &\leqslant \biggl(\int_{T(2B)} |F_\varepsilon(x,t)|^2 \,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}\biggr)^{1/2}\biggl(\int_X \int_0^\infty |G(x,t)|^2\frac{\mathrm{d}t}{t}\,\mathrm{d}\mu(x)\biggr)^{1/2} \\ &\leqslant C[\mu(2B)]^{1/2}\biggl(\int_0^{2r_B}\int_{2B} |t\sqrt{\mathcal{L}}\mathcal{P}_t u_\varepsilon(x)|^2 \,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}\biggr)^{1/2} \\ &\quad\times \biggl(\int_X |(I - \mathcal{P}_{r_B})g(x)|^2 \,\mathrm{d}\mu(x)\biggr)^{1/2} \\ &\leqslant C\alpha(B)[\mu(B)]^{1/2}\|u\|_{\mathrm{HL}^{2,\alpha}}\|g\|_{L^2(B)}. \end{aligned}$$

For the term I_k, by (PUB), it holds that for any $(x,t)\in T(2^{k+1}B)\setminus T(2^kB)$,

$$\begin{aligned}
|G(x,t)| &= \left| t\sqrt{\mathcal{L}}\mathcal{P}_t \int_0^{r_B} s\sqrt{\mathcal{L}}\mathcal{P}_s g(x)\frac{\mathrm{d}s}{s}\right| \\
&= \left|\int_0^{r_B} \frac{ts}{(t+s)^2}\frac{\mathrm{d}s}{s}\int_X r^2\frac{\partial^2 p_r(x,y)}{\partial r^2}\bigg|_{r=t+s} g(y)\,\mathrm{d}\mu(y)\right| \\
&\leqslant C\int_0^{r_B}\frac{ts}{(t+s)^2}\frac{\mathrm{d}s}{s}\int_B \frac{t+s}{t+s+d(x,y)}\frac{|g(y)|}{\mu(B(x,t+s+d(x,y)))}\,\mathrm{d}\mu(y).
\end{aligned}$$

Notice that for any $(x,t)\in T(2^{k+1}B)\setminus T(2^kB)$ and $y\in B$, it holds that

$$t+s+d(x,y)\geqslant t+d(x,y)\geqslant 2^{k-1}r_B.$$

Therefore, one has

$$\begin{aligned}
|G(x,t)| &\leqslant C\int_0^{r_B}\frac{ts}{(t+s)^2}\frac{\mathrm{d}s}{s}\int_B\frac{t+s}{2^kr_B}\frac{|g(y)|}{\mu(B(x,2^{k+1}r_B))}\,\mathrm{d}\mu(y) \\
&\leqslant \frac{C}{2^kr_B}\frac{\|g\|_{L^1(B)}}{\mu(2^{k+1}B)}\int_0^{r_B}\frac{ts}{t+s}\frac{\mathrm{d}s}{s} \\
&\leqslant \frac{C}{2^kr_B}\frac{\|g\|_{L^1(B)}}{\mu(2^{k+1}B)}\int_0^{r_B}(ts)^{1/2}\frac{\mathrm{d}s}{s}\leqslant \frac{C}{2^k}\frac{\|g\|_{L^1(B)}}{\mu(2^{k+1}B)}\Big(\frac{t}{r_B}\Big)^{1/2}.
\end{aligned}$$

By substituting this estimate into I_k, using Hölder's inequality, and invoking Lemma 2.7, we deduce that

$$\begin{aligned}
\mathrm{I}_k &\leqslant \int_{T(2^{k+1}B)\setminus T(2^kB)}|F_\varepsilon(x,t)G(x,t)|\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} \\
&\leqslant \bigg(\int_{T(2^{k+1}B)\setminus T(2^kB)}|F_\varepsilon(x,t)|^2\,\mathrm{d}\mu\frac{\mathrm{d}t}{t}\bigg)^{1/2} \\
&\quad\times\bigg(\int_{T(2^{k+1}B)\setminus T(2^kB)}|G(x,t)|^2\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}\bigg)^{1/2} \\
&\leqslant C\frac{\|g\|_{L^1(B)}}{2^k}\bigg(\int_0^{2^{k+1}r_B}\fint_{2^{k+1}B}|t\sqrt{\mathcal{L}}\mathcal{P}_tu_\varepsilon(x)|^2\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}\bigg)^{1/2} \\
&\quad\times\bigg(\int_0^{2^{k+1}r_B}\frac{t}{r_B}\frac{\mathrm{d}t}{t}\bigg)^{1/2} \\
&\leqslant C2^{-k/2}\alpha(2^{k+1}B)\|u\|_{\mathrm{HL}^{2,\alpha}}\|g\|_{L^1(B)} \\
&\leqslant C2^{-k/2}\alpha(B)[\mu(B)]^{1/2}\|u\|_{\mathrm{HL}^{2,\alpha}}\|g\|_{L^2(B)}.
\end{aligned}$$

Summing over k leads to

$$\int_0^\infty\int_X|F_\varepsilon(x,t)G(x,t)|\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}=\mathrm{I}_0+\sum_{k=1}^\infty\mathrm{I}_k\leqslant C\alpha(B)[\mu(B)]^{1/2}\|u\|_{\mathrm{HL}^{2,\alpha}}\|g\|_{L^2(B)}.$$

(ii) Let us verify the identity

$$\int_X u_\varepsilon (I-\mathcal{P}_{r_B})g \,\mathrm{d}\mu = 4\int_0^\infty \int_X F_\varepsilon(x,t)G(x,t)\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}.$$

From (i), the RHS of this identity is bounded. Consequently, we can invoke Lebesgue's dominated convergence theorem to obtain

$$\int_0^\infty \int_X F_\varepsilon(x,t)G(x,t)\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t} = \lim_{\delta\to 0^+}\int_\delta^{1/\delta} \int_X F_\varepsilon(x,t)G(x,t)\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t},$$

which together with Fubini's theorem implies that

$$\begin{aligned}
&\int_0^\infty \int_X F_\varepsilon(x,t)G(x,t)\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}\\
&\qquad = \lim_{\delta\to 0^+}\int_\delta^{1/\delta}\int_X u_\varepsilon(x)(t\sqrt{\mathcal{L}}\mathcal{P}_t)^2(I-\mathcal{P}_{r_B})g(x)\,\mathrm{d}\mu(x)\frac{\mathrm{d}t}{t}\\
&\qquad = \lim_{\delta\to 0^+}\int_X u_\varepsilon(x)\bigg(\int_\delta^{1/\delta}(t\sqrt{\mathcal{L}}\mathcal{P}_t)^2(I-\mathcal{P}_{r_B})g(x)\frac{\mathrm{d}t}{t}\bigg)\,\mathrm{d}\mu(x)\\
&\qquad = \lim_{\delta\to 0^+}\int_{4B}\dots\,\mathrm{d}\mu + \lim_{\delta\to 0^+}\int_{X\setminus 4B}\dots\,\mathrm{d}\mu.
\end{aligned}$$

For the local part, since $(I-\mathcal{P}_{r_B})g\in L^2(X)$, the spectral theory tells us that

$$(I-\mathcal{P}_{r_B})g(x) = 4\lim_{\delta\to 0^+}\int_\delta^{1/\delta}(t\sqrt{\mathcal{L}}\mathcal{P}_t)^2(I-\mathcal{P}_{r_B})g(x)\frac{\mathrm{d}t}{t},$$

where the integral converges strongly in $L^2(X)$, see [11], page 958. Therefore, one has

$$\begin{aligned}
\lim_{\delta\to 0^+}\int_{4B} u_\varepsilon(x)\bigg(\int_\delta^{1/\delta}(t\sqrt{\mathcal{L}}\mathcal{P}_t)^2(I-\mathcal{P}_{r_B})g(x)\frac{\mathrm{d}t}{t}\bigg)\,\mathrm{d}\mu(x)&\\
= \frac{1}{4}\int_{4B} u_\varepsilon(x)(I-\mathcal{P}_{r_B})g(x)\,\mathrm{d}\mu(x).&
\end{aligned}$$

For the global part, it follows from (PUB) and (D) that for any $x\in X\setminus 4B$,

$$\begin{aligned}
(2.10)\quad &\sup_{\delta>0}\bigg|\int_\delta^{1/\delta}(t\sqrt{\mathcal{L}}\mathcal{P}_t)^2(I-\mathcal{P}_{r_B})g(x)\frac{\mathrm{d}t}{t}\bigg|\\
&\quad = \sup_{\delta>0}\bigg|\int_\delta^{1/\delta}(t\sqrt{\mathcal{L}}\mathcal{P}_t)^2\int_0^{r_B} s\sqrt{\mathcal{L}}\mathcal{P}_s g(x)\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\bigg|\\
&\quad \leqslant \sup_{\delta>0}\int_\delta^{1/\delta}\int_0^{r_B}\frac{t^2 s}{(2t+s)^3}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\int_X\bigg|r^3\frac{\partial^3 p_r(x,y)}{\partial r^3}\bigg|_{r=2t+s}\bigg||g(y)|\,\mathrm{d}\mu(y)\\
&\quad \leqslant C\int_0^\infty\int_0^{r_B}\frac{t^2 s}{(2t+s)^3}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_B \Big(\frac{2t+s}{2t+s+d(x,y)}\Big)^{\gamma} \frac{|g(y)|}{\mu(B(x,2t+s+d(x,y)))}\,\mathrm{d}\mu(y)\\
&\leqslant C\int_0^{\infty}\int_0^{r_B} \frac{t^2 s}{(t+s)^{3-\gamma}}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t}\\
&\quad \times \int_B \Big(\frac{1}{r_B+d(x,x_B)}\Big)^{\gamma} \frac{|g(y)|}{\mu(B(x_B,r_B+d(x,x_B)))}\,\mathrm{d}\mu(y)\\
&\leqslant C\Big(\frac{r_B}{r_B+d(x,x_B)}\Big)^{\gamma}\frac{\|g\|_{L^1(B)}}{\mu(B(x_B,r_B+d(x,x_B)))},
\end{aligned}
$$

where γ is a positive constant satisfying $\theta<\gamma<1$. Above, in the last inequality, we used the fact $\gamma<1$ to deduce that

$$
\begin{aligned}
\int_0^{\infty}\int_0^{r_B} \frac{t^2 s}{(t+s)^{3-\gamma}}\frac{\mathrm{d}s}{s}\frac{\mathrm{d}t}{t} &= \int_0^{r_B}\int_0^{\infty} \frac{(t/s)^2 s^3}{(t/s+1)^{3-\gamma}s^{3-\gamma}}\frac{\mathrm{d}t}{t}\frac{\mathrm{d}s}{s}\\
&= \int_0^{r_B} s^{\gamma}\frac{\mathrm{d}s}{s}\int_0^{\infty}\frac{t^2}{(t+1)^{3-\gamma}}\frac{\mathrm{d}t}{t}\leqslant Cr_B^{\gamma}.
\end{aligned}
$$

Combining (2.10) with $\gamma>\theta$ and (TI), we can use Remark 2.4 and the same argument as in (2.9) to deduce that

$$
\begin{aligned}
&\sup_{\delta>0}\Big|\int_{X\setminus 4B} u_\varepsilon(x)\Big(\int_\delta^{1/\delta}(t\sqrt{\mathcal{L}}\mathcal{P}_t)^2(I-\mathcal{P}_{r_B})g(x)\frac{\mathrm{d}t}{t}\Big)\,\mathrm{d}\mu(x)\Big|\\
&\leqslant C\|g\|_{L^1(B)}\int_X\Big(\frac{r_B}{r_B+d(x,x_B)}\Big)^{\gamma}\frac{|u_\varepsilon(x)|}{\mu(B(x_B,r_B+d(x,x_B)))}\,\mathrm{d}\mu(x)\\
&\leqslant C\|g\|_{L^1(B)}\int_X\Big(\frac{r_B}{r_B+d(x,x_B)}\Big)^{\gamma}\frac{|u_\varepsilon(x)-u_\varepsilon(x_B)|+|u_\varepsilon(x_B)|}{\mu(B(x_B,r_B+d(x,x_B)))}\,\mathrm{d}\mu(x)\\
&\leqslant C\|g\|_{L^1(B)}\\
&\quad\times\Big\{\int_X\Big(\frac{r_B}{r_B+d(x,x_B)}\Big)^{\gamma}\Big(\frac{d(x,x_B)}{\varepsilon}\Big)^{\theta}\frac{\alpha(B(x_B,\varepsilon))\|u\|_{\mathrm{HL}^{2,\alpha}}}{\mu(B(x_B,r_B+d(x,x_B)))}\,\mathrm{d}\mu(x)+|u_\varepsilon(x_B)|\Big\}\\
&\leqslant C\|g\|_{L^1(B)}\Big\{\Big(\frac{r_B}{\varepsilon}\Big)^{\theta}\alpha(B(x_B,\varepsilon))\|u\|_{\mathrm{HL}^{2,\alpha}}+|u_\varepsilon(x_B)|\Big\}<\infty.
\end{aligned}
$$

This allows us to pass the limit inside the integral of the global part. Hence, one has

$$
\lim_{\delta\to 0+}\int_{X\setminus 4B} u_\varepsilon\Big(\int_\delta^{1/\delta}(t\sqrt{\mathcal{L}}\mathcal{P}_t)^2(I-\mathcal{P}_{r_B})g\frac{\mathrm{d}t}{t}\Big)\,\mathrm{d}\mu=\frac{1}{4}\int_{X\setminus 4B}u_\varepsilon(I-\mathcal{P}_{r_B})g\,\mathrm{d}\mu.
$$

Collecting the identity for the local part and the global part, we arrive at the desired result. □

Proposition 2.9. *Assume that the Dirichlet metric measure space $(X,d,\mu,\mathscr{E})$ satisfies* (D) *and admits* (P_2). *Suppose that the nonnegative ball function α satisfies* (De) *and* (TI). *If $u\in \mathrm{HL}^{2,\alpha}(X\times\mathbb{R}_+)$, then $u_\varepsilon\in L^{2,\alpha}(X)$ for every $\varepsilon>0$.*

Moreover, there exists a constant $C > 0$, independent of ε, such that

$$\|u_\varepsilon\|_{L^{2,\alpha}} \leqslant C\|u\|_{\mathrm{HL}^{2,\alpha}}.$$

Proof. Given a ball $B = B(x_B, r_B) \subset X$, by Lemma 2.8 for any $g \in L^2(B)$, it holds that

$$\left|\int_X u_\varepsilon (I - \mathcal{P}_{r_B}) g \,\mathrm{d}\mu\right| \leqslant C\alpha(B)[\mu(B)]^{1/2}\|u\|_{\mathrm{HL}^{2,\alpha}}\|g\|_{L^2(B)}.$$

This together with the L^2-duality argument implies that

$$\begin{aligned}\left(\fint_B |u_\varepsilon - \mathcal{P}_{r_B} u_\varepsilon|^2 \,\mathrm{d}\mu\right)^{1/2} &= [\mu(B)]^{-1/2} \sup_{\|g\|_{L^2(B)}\leqslant 1}\left|\int_X g(I - \mathcal{P}_{r_B}) u_\varepsilon \,\mathrm{d}\mu\right| \\ &= [\mu(B)]^{-1/2} \sup_{\|g\|_{L^2(B)}\leqslant 1}\left|\int_X u_\varepsilon (I - \mathcal{P}_{r_B}) g \,\mathrm{d}\mu\right| \\ &\leqslant C\alpha(B)\|u\|_{\mathrm{HL}^{2,\alpha}}.\end{aligned}$$

By Remark 2.4 and (TI), it holds that

$$\begin{aligned}&\left(\fint_B\fint_B |u(y, r_B + \varepsilon) - u(z, r_B + \varepsilon)|^2 \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z)\right)^{1/2} \\ &\quad\leqslant C\left\{\fint_B\fint_B \Big(\frac{d(y,z)}{r_B + \varepsilon}\Big)^{2\theta} [\alpha(B(y, r_B + \varepsilon))]^2 \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y)\right\}^{1/2} \|u\|_{\mathrm{HL}^{2,\alpha}} \\ &\quad\leqslant C\left\{\fint_B\fint_B \Big(\frac{2r_B}{r_B + \varepsilon}\Big)^{2\theta} [\alpha(B)]^2 \,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y)\right\}^{1/2} \|u\|_{\mathrm{HL}^{2,\alpha}} \leqslant C\alpha(B)\|u\|_{\mathrm{HL}^{2,\alpha}}.\end{aligned}$$

One may use the above two estimates and Proposition 2.6 to obtain

$$\begin{aligned}\left(\fint_B |u_\varepsilon - (u_\varepsilon)_B|^2 \,\mathrm{d}\mu\right)^{1/2} &\leqslant \left(\fint_B\fint_B |u(y,\varepsilon) - u(z,\varepsilon)|^2 \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z)\right)^{1/2} \\ &\leqslant 2\left(\fint_B |u_\varepsilon - \mathcal{P}_{r_B}(u_\varepsilon)|^2 \,\mathrm{d}\mu\right)^{1/2} \\ &\quad + \left(\fint_B\fint_B |\mathcal{P}_{r_B}(u_\varepsilon)(y) - \mathcal{P}_{r_B}(u_\varepsilon)(z)|^2 \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z)\right)^{1/2} \\ &\leqslant C\alpha(B)\|u\|_{\mathrm{HL}^{2,\alpha}},\end{aligned}$$

which means that

$$\frac{1}{\alpha(B)}\left(\fint_B |u_\varepsilon - (u_\varepsilon)_B|^2 \,\mathrm{d}\mu\right)^{1/2} \leqslant C\|u\|_{\mathrm{HL}^{2,\alpha}}.$$

Taking the supremum over all balls $B \subset X$, we get the desired estimate. $\square$

3. Proof of Theorem 1.4

3.1. From solution to trace. In this subsection, we shall seek traces of harmonic functions and get the desired estimate. Precisely, we will prove part (i) of Theorem 1.4.

Theorem 3.1. *Let* $(X, d, \mu, \mathscr{E})$ *be a complete Dirichlet metric measure space satisfying the doubling condition* (D), *and admitting an* L^2*-Poincaré inequality* (P_2). *Suppose that the nonnegative ball function* α *is decreasing* (De) *and translation invariant* (TI). *If* $u \in \mathrm{HL}^{2,\alpha}(X \times \mathbb{R}_+)$, *then there exists a function* $f \in L^{2,\alpha}(X)$ *such that* $u(x,t) = \mathcal{P}_t f(x)$. *Moreover, there exists a constant* $C > 0$, *independent of* u, *such that*

$$\|f\|_{L^{2,\alpha}} \leqslant C\|u\|_{\mathrm{HL}^{2,\alpha}}.$$

Proof. To prove this theorem, we will follow the argument as in [27]. Assume that u belongs to $\mathrm{HL}^{2,\alpha}(X \times \mathbb{R}_+)$. Fix a ball $B = B(x_B, r_B) \subset X$. For any integer $m \geqslant 0$ and $0 < \varepsilon < 1$, it follows from Proposition 2.9 that

$$\begin{aligned}\int_{2^m B} |u_\varepsilon - (u_\varepsilon)_{2^m B}|^2 \,\mathrm{d}\mu &\leqslant [\alpha(2^m B)]^2 \mu(2^m B)\|u_\varepsilon\|^2_{L^{2,\alpha}} \\ &\leqslant C[\alpha(2^m B)]^2 \mu(2^m B)\|u\|^2_{\mathrm{HL}^{2,\alpha}},\end{aligned}$$

which means that the family $\{u_\varepsilon - (u_\varepsilon)_{2^m B}\}_{0<\varepsilon<1}$ is uniformly bounded in $L^2(2^m B)$. Hence, there exist a sequence $\varepsilon_k \to 0$ ($k \to \infty$) and a function $g_m \in L^2(2^m B)$ such that $u_{\varepsilon_k} - (u_{\varepsilon_k})_{2^m B} \to g_m$ (weak convergence) with

$$(g_m)_{2^m B} = \lim_{k\to\infty} [u_{\varepsilon_k} - (u_{\varepsilon_k})_{2^m B}]_{2^m B} = 0.$$

Now we define

$$\delta(B, 2^m B) = \lim_{k\to\infty} [(u_{\varepsilon_k})_B - (u_{\varepsilon_k})_{2^m B}] \quad \text{and} \quad f = g_m - \delta(B, 2^m B).$$

Evidently, the function f is well defined on $X = \bigcup_{m=0}^{\infty} 2^m B$.

We next verify $f \in L^{2,\alpha}(X)$. There holds by the Hölder inequality and Proposition 2.9 that

$$\begin{aligned}\fint_B |g_0|^2 \,\mathrm{d}\mu &= \lim_{k\to\infty} \fint_B g_0[u_{\varepsilon_k} - (u_{\varepsilon_k})_B] \,\mathrm{d}\mu \\ &\leqslant \Big(\fint_B |g_0|^2 \,\mathrm{d}\mu\Big)^{1/2} \limsup_{k\to\infty} \Big(\fint_B |u_{\varepsilon_k} - (u_{\varepsilon_k})_B|^2 \,\mathrm{d}\mu\Big)^{1/2} \\ &\leqslant C\Big(\fint_B |g_0|^2 \,\mathrm{d}\mu\Big)^{1/2} \alpha(B)\|u\|_{\mathrm{HL}^{2,\alpha}}.\end{aligned}$$

Notice that $\delta(B,B)=0$, $f=g_0$ on B, and thus, $f_B=(g_0)_B=0$. One has

$$f-f_B=g_0 \tag{3.1}$$

on B, and

$$\left(\fint_B |f-f_B|^2\,\mathrm{d}\mu\right)^{1/2}=\left(\fint_B |g_0|^2\,\mathrm{d}\mu\right)^{1/2}\leqslant C\alpha(B)\|u\|_{\mathrm{HL}^{2,\alpha}},$$

where the constant $C>0$ is harmless. This implies that

$$\|f\|_{L^{2,\alpha}}\leqslant C\|u\|_{\mathrm{HL}^{2,\alpha}}. \tag{3.2}$$

Finally, let us prove

$$u(x,t)=\mathcal{P}_tf(x)+h$$

for a constant h. Note that $\partial_tu(x,t)$ is also an $\mathcal{L}_+$-harmonic function. From (2.1) and Proposition 2.6, we see

$$\partial_tu(x,t)=\lim_{k\to\infty}\partial_tu(x,t+\varepsilon_k)=\lim_{k\to\infty}\partial_t\mathcal{P}_t(u_{\varepsilon_k})(x).$$

For any integer $m\geqslant 1$, by the conservation property $\mathcal{P}_t1\equiv 1$, one writes

$$\begin{aligned}\partial_t\mathcal{P}_t(u_{\varepsilon_k})(x)=&\int_{X\setminus B(x,2^m)}\partial_tp_t(x,y)[u_{\varepsilon_k}(y)-(u_{\varepsilon_k})_{B(x,2^m)}]\,\mathrm{d}\mu(y)\\&+\int_{B(x,2^m)}\partial_tp_t(x,y)[u_{\varepsilon_k}(y)-(u_{\varepsilon_k})_{B(x,2^m)}-f(y)]\,\mathrm{d}\mu(y)\\&+\int_{B(x,2^m)}\partial_tp_t(x,y)f(y)\,\mathrm{d}\mu(y).\end{aligned}$$

To estimate the global part, we deduce from (PUB) and Proposition 2.9 that

$$\begin{aligned}&\left|\int_{X\setminus B(x,2^m)}\partial_tp_t(x,y)[u_{\varepsilon_k}(y)-(u_{\varepsilon_k})_{B(x,2^m)}]\,\mathrm{d}\mu(y)\right|\\&\leqslant\int_{X\setminus B(x,2^m)}\frac{C}{(t+d(x,y))}\frac{|u_{\varepsilon_k}(y)-(u_{\varepsilon_k})_{B(x,2^m)}|}{\mu(B(x,t+d(x,y)))}\,\mathrm{d}\mu(y)\\&\leqslant C\sum_{i=m+1}^{\infty}2^{-i}\fint_{B(x,2^i)}|u_{\varepsilon_k}(y)-(u_{\varepsilon_k})_{B(x,2^m)}|\,\mathrm{d}\mu(y)\\&\leqslant C\sum_{i=m+1}^{\infty}2^{-i}\fint_{B(x,2^i)}(|u_{\varepsilon_k}(y)-(u_{\varepsilon_k})_{B(x,2^i)}|+|(u_{\varepsilon_k})_{B(x,2^i)}-(u_{\varepsilon_k})_{B(x,2^m)}|)\,\mathrm{d}\mu(y)\\&\leqslant C\sum_{i=m+1}^{\infty}\frac{i}{2^i}\alpha(B(x,1))\|u_{\varepsilon_k}\|_{L^{2,\alpha}}\leqslant C\frac{m}{2^m}\alpha(B(x,1))\|u\|_{\mathrm{HL}^{2,\alpha}}\to 0,\quad m\to\infty,\end{aligned}$$

where the penultimate inequality is due to the fact that for any $j \in \mathbb{N}$,

$$\begin{aligned}|(u_{\varepsilon_k})_{B(x,2^j)} - (u_{\varepsilon_k})_{B(x,2^{j+1})}| &\leqslant \fint_{B(x,2^j)} |u_{\varepsilon_k} - (u_{\varepsilon_k})_{B(x,2^{j+1})}| \,\mathrm{d}\mu \\ &\leqslant C\biggl(\fint_{B(x,2^{j+1})} |u_{\varepsilon_k} - (u_{\varepsilon_k})_{B(x,2^{j+1})}|^2 \,\mathrm{d}\mu\biggr)^{1/2} \\ &\leqslant C\alpha(B(x,2^j))\|u_{\varepsilon_k}\|_{L^{2,\alpha}} \leqslant C\alpha(B(x,1))\|u_{\varepsilon_k}\|_{L^{2,\alpha}}.\end{aligned}$$

For the middle term, note that

$$\begin{aligned}\biggl|\int_{B(x,2^m)} \partial_t p_t(x,y)\,\mathrm{d}\mu(y)\biggr| &= \biggl|\int_{X\setminus B(x,2^m)} \partial_t p_t(x,y)\,\mathrm{d}\mu(y)\biggr| \\ &\leqslant \int_{X\setminus B(x,2^m)} \frac{C}{(t+d(x,y))} \frac{\mathrm{d}\mu(y)}{\mu(B(x,t+d(x,y)))} \\ &\leqslant C\sum_{i=m+1}^{\infty} 2^{-i} \fint_{B(x,2^i)} \mathrm{d}\mu(y) \leqslant C2^{-m},\end{aligned}$$

and

$$|\delta(B(x,1),B(x,2^m))| \leqslant \lim_{k\to\infty} |(u_{\varepsilon_k})_{B(x,1)} - (u_{\varepsilon_k})_{B(x,2^m)}| \leqslant Cm\alpha(B(x,1))\|u\|_{\mathrm{HL}^{2,\alpha}},$$

which yields

$$\begin{aligned}&\biggl|\lim_{k\to\infty}\int_{B(x,2^m)} \partial_t p_t(x,y)[u_{\varepsilon_k}(y) - (u_{\varepsilon_k})_{B(x,2^m)} - f(y)]\,\mathrm{d}\mu(y)\biggr| \\ &\qquad = \biggl|\int_{B(x,2^m)} \partial_t p_t(x,y)\delta(B(x,1),B(x,2^m))\,\mathrm{d}\mu(y)\biggr| \\ &\qquad \leqslant Cm2^{-m}\alpha(B(x,1))\|u\|_{\mathrm{HL}^{2,\alpha}} \to 0, \quad m\to\infty.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\partial_t u(x,t) &= \lim_{k\to\infty} \partial_t \mathcal{P}_t(u_{\varepsilon_k})(x) \\ &= \lim_{m\to\infty}\lim_{k\to\infty}\int_{X\setminus B(x,2^m)} \partial_t p_t(x,y)[u_{\varepsilon_k}(y) - (u_{\varepsilon_k})_{B(x,2^m)}]\,\mathrm{d}\mu(y) \\ &\quad + \lim_{m\to\infty}\lim_{k\to\infty}\int_{B(x,2^m)} \partial_t p_t(x,y)[u_{\varepsilon_k}(y) - (u_{\varepsilon_k})_{B(x,2^m)} - f(y)]\,\mathrm{d}\mu(y) \\ &\quad + \lim_{m\to\infty}\int_{B(x,2^m)} \partial_t p_t(x,y) f(y)\,\mathrm{d}\mu(y) \\ &= \int_X \partial_t p_t(x,y) f(y)\,\mathrm{d}\mu(y) = \partial_t \mathcal{P}_t f(x),\end{aligned}$$

which means that there exists a function $h(x)$ such that

$$u(x,t) = \mathcal{P}_t f(x) + h(x). \tag{3.3}$$

It remains to prove that h is a constant. Since $f \in L^{2,\alpha}(X)$, it holds for each $x \in B$ and all $t > 0$ that

$$
\begin{aligned}
(3.4)\qquad & |P_t((f-f_B)\chi_{X\setminus 2B})(x)| \\
&\leqslant C\int_{X\setminus 2B} \frac{t}{t+d(x,y)} \frac{|f(y)-f_B|}{\mu(B(x,t+d(x,y)))}\,\mathrm{d}\mu(y) \\
&\leqslant C\sum_{m=1}^{\infty}\int_{2^{m+1}B\setminus 2^m B} \frac{t}{2^m r_B}\frac{|f(y)-f_B|}{\mu(B(x_B,2^m r_B))}\,\mathrm{d}\mu(y) \\
&\leqslant C\sum_{m=1}^{\infty}\frac{t}{2^m r_B}\fint_{2^{m+1}B}(|f(y)-f_{2^{m+1}B}|+|f_{2^{m+1}B}-f_B|)\,\mathrm{d}\mu(y) \\
&\leqslant C\frac{t}{r_B}\alpha(B)\|f\|_{L^{2,\alpha}}\sum_{m=1}^{\infty}\frac{m}{2^m} \leqslant C\frac{t}{r_B}\alpha(B)\|f\|_{L^{2,\alpha}}.
\end{aligned}
$$

Let $L_0^2(B)$ denote the class of all $L^2(B)$-functions having mean value zero. Then for any $g \in L_0^2(B)$, applying the fact $h = u_{\varepsilon_k} - \mathcal{P}_{\varepsilon_k} f$, the weak convergence $u_{\varepsilon_k} - (u_{\varepsilon_k})_B \to g_0$ and the identity (3.1), one has

$$
\begin{aligned}
\left|\int_X gh\,\mathrm{d}\mu\right| &= \left|\lim_{k\to\infty}\int_X g(u_{\varepsilon_k}-\mathcal{P}_{\varepsilon_k}f)\,\mathrm{d}\mu\right| \\
&= \left|\lim_{k\to\infty}\int_X g[u_{\varepsilon_k}-(u_{\varepsilon_k})_B-g_0]\,\mathrm{d}\mu + \lim_{k\to\infty}\int_X g[(u_{\varepsilon_k})_B+g_0-\mathcal{P}_{\varepsilon_k}f]\,\mathrm{d}\mu\right| \\
&= \left|\lim_{k\to\infty}\int_X g(f-f_B-\mathcal{P}_{\varepsilon_k}f)\,\mathrm{d}\mu\right| \\
&= \left|\lim_{k\to\infty}\int_X g[(f-f_B)\chi_{2B}-\mathcal{P}_{\varepsilon_k}((f-f_B)\chi_{2B})-\mathcal{P}_{\varepsilon_k}((f-f_B)\chi_{X\setminus 2B})]\,\mathrm{d}\mu\right| \\
&= \left|\lim_{k\to\infty}\int_X g[(f-f_B)\chi_{2B}-\mathcal{P}_{\varepsilon_k}((f-f_B)\chi_{2B})]\,\mathrm{d}\mu\right| \\
&\leqslant \lim_{k\to\infty}\|g\|_{L^2}\|(f-f_B)\chi_{2B}-\mathcal{P}_{\varepsilon_k}((f-f_B)\chi_{2B})\|_{L^2}=0,
\end{aligned}
$$

where the penultimate equality is due to (3.4), and the last equality is due to $\mathcal{P}_{\varepsilon_k}((f-f_B)\chi_{2B}) \to (f-f_B)\chi_{2B}$ in $L^2(X)$.

Since $(L_0^2(B))^* = L^2(B)/\{\text{constant functions on } B\}$, one concludes $h \equiv$ constant. This together with (3.3) and (3.2) implies

$$
u(x,t) = \mathcal{P}_t(f+h)(x) \quad\text{and}\quad \|f+h\|_{L^{2,\alpha}} = \|f\|_{L^{2,\alpha}} \leqslant C\|u\|_{\mathrm{HL}^{2,\alpha}},
$$

which completes the proof of Theorem 3.1. □

3.2. From trace to solution. In this subsection, we will show Theorem 1.4 (ii), i.e., every Morrey function f induces a Carleson type measure $t|\nabla \mathcal{P}_t f|^2 \,\mathrm{d}\mu\,\mathrm{d}t$. In order to estimate the space derivation part $t|D_x \mathcal{P}_t f|^2 \,\mathrm{d}\mu\,\mathrm{d}t$, we shall introduce the following Proposition 3.2, which establishes a Caccioppoli inequality for the Poisson semigroup.

Proposition 3.2. *Assume that the Dirichlet metric measure space* $(X, d, \mu, \mathscr{E})$ *satisfies* (D) *and admits* (P_2). *Suppose that* $g \in L^2(X)$. *Then for any ball* $B = B(x_B, r_B) \subset X$, *it holds that*

$$\int_0^{r_B} \int_B |tD_x \mathcal{P}_t g|^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t} \leqslant C \int_0^{2r_B} \int_{2B} (|t^2 \partial_t^2 \mathcal{P}_t g|^2 + |\mathcal{P}_t g|^2) \,\mathrm{d}\mu \frac{\mathrm{d}t}{t}.$$

P r o o f. Although the proof is similar to that of [25], Proposition 5.2, we include it for reader's convenience. By the L^2-boundedness of the Riesz transform $D_x \mathcal{L}^{-1/2}$ and the spectral theory, one has

$$\begin{aligned}
\int_\varepsilon^{2r_B} \int_{2B} t|D_x \mathcal{P}_t g|^2 \,\mathrm{d}\mu\,\mathrm{d}t &\leqslant \int_0^\infty \int_X t|D_x \mathcal{P}_t g|^2 \,\mathrm{d}\mu\,\mathrm{d}t \\
&\leqslant C \int_0^\infty \int_X |t\sqrt{\mathcal{L}} \mathcal{P}_t g|^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t} \\
&\leqslant C \int_X |g|^2 \,\mathrm{d}\mu < \infty.
\end{aligned}$$

We take a Lipschitz function φ on X with $\operatorname{supp}\varphi \subset 2B$ such that $\varphi = 1$ on B and $|D_x \varphi| \leqslant C/r_B$, and for each $\varepsilon \in (0, r_B)$, take a smooth function $\phi_\varepsilon(t)$ on $\mathbb{R}$ such that $\operatorname{supp}\phi_\varepsilon \subset (\varepsilon, 2r_B)$, $\phi_\varepsilon(t) = 1$ on $(2\varepsilon, r_B)$, $|\partial_t \phi_\varepsilon(t)| \leqslant C/\varepsilon$ for $t \in (\varepsilon, 2\varepsilon)$, $|\partial_t \phi_\varepsilon(t)| \leqslant C/r_B$ for $t > r_B$. By repeating the Caccioppoli argument, we arrive at

$$\begin{aligned}
\int_0^\infty \int_X |tD_x \mathcal{P}_t g|^2 (\varphi\phi_\varepsilon)^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t} &= \int_0^\infty \int_X \langle D_x \mathcal{P}_t g, D_x(t(\varphi\phi_\varepsilon)^2 \mathcal{P}_t g)\rangle \,\mathrm{d}\mu\,\mathrm{d}t \\
&\quad - 2\int_0^\infty \int_X \langle D_x \mathcal{P}_t g, D_x(\varphi\phi_\varepsilon)\rangle t\varphi\phi_\varepsilon \mathcal{P}_t g \,\mathrm{d}\mu\,\mathrm{d}t \\
&\leqslant \int_0^\infty \int_X \langle \mathcal{L}\mathcal{P}_t g, t(\varphi\phi_\varepsilon)^2 \mathcal{P}_t g\rangle \,\mathrm{d}\mu\,\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^\infty \int_X t|D_x \mathcal{P}_t g|^2 (\varphi\phi_\varepsilon)^2 \,\mathrm{d}\mu\,\mathrm{d}t \\
&\quad + 2\int_0^\infty \int_X t|D_x(\varphi\phi_\varepsilon)|^2 |\mathcal{P}_t g|^2 \,\mathrm{d}\mu\,\mathrm{d}t,
\end{aligned}$$

and hence,

$$\begin{aligned}
&\int_0^\infty \int_X |tD_x\mathcal{P}_t g|^2 (\varphi\phi_\varepsilon)^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t} \\
&\quad\leqslant 2\int_0^\infty \int_X |\partial_t^2 \mathcal{P}_t g||t(\varphi\phi_\varepsilon)^2 \mathcal{P}_t g| \,\mathrm{d}\mu \,\mathrm{d}t + 4\int_0^\infty \int_X t|D_x(\varphi\phi_\varepsilon)|^2 |\mathcal{P}_t g|^2 \,\mathrm{d}\mu \,\mathrm{d}t \\
&\quad\leqslant C\int_0^{2r_B} \int_{2B} |t^2\partial_t^2 \mathcal{P}_t g||\mathcal{P}_t g| \,\mathrm{d}\mu \frac{\mathrm{d}t}{t} + C\int_0^{2r_B} \int_{2B} \frac{t^2}{r_B^2} |\mathcal{P}_t g|^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t} \\
&\quad\leqslant C\int_0^{2r_B} \int_{2B} (|t^2\partial_t^2 \mathcal{P}_t g|^2 + |\mathcal{P}_t g|^2) \,\mathrm{d}\mu \frac{\mathrm{d}t}{t}.
\end{aligned}$$

This together with the monotonic convergence theorem implies that

$$\begin{aligned}
&\int_0^{r_B} \int_B |tD_x\mathcal{P}_t g|^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t} = \lim_{\varepsilon\to 0} \int_{2\varepsilon}^{r_B} \int_B |tD_x\mathcal{P}_t g|^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t} \\
&\quad\leqslant \limsup_{\varepsilon\to 0} \int_0^\infty \int_X |tD_x\mathcal{P}_t g|^2 (\varphi\phi_\varepsilon)^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t} \leqslant C\int_0^{2r_B} \int_{2B} (|t^2\partial_t^2 \mathcal{P}_t g|^2 + |\mathcal{P}_t g|^2) \,\mathrm{d}\mu \frac{\mathrm{d}t}{t},
\end{aligned}$$

which completes the proof. $\square$

Theorem 3.3. *Let $(X, d, \mu, \mathscr{E})$ be a complete Dirichlet metric measure space satisfying the doubling condition* (D) *and admitting an L^2-Poincaré inequality* (P_2). *Suppose that the nonnegative ball function α is decreasing* (De). *If $f \in L^{2,\alpha}(X)$, then $u(x,t) = \mathcal{P}_t f(x) \in \mathrm{HL}^{2,\alpha}(X \times \mathbb{R}_+)$, and there exists a constant $C > 0$, independent of f, such that*

$$\|u\|_{\mathrm{HL}^{2,\alpha}} \leqslant C\|f\|_{L^{2,\alpha}}.$$

Proof. Assume that f belongs to $L^{2,\alpha}(X)$. It suffices to show that

$$\Big(\int_0^{r_B} \fint_B |t\nabla\mathcal{P}_t f|^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t}\Big)^{1/2} \leqslant C\alpha(B)\|f\|_{L^{2,\alpha}},$$

where the constant $C > 0$ is harmless.

By the conservation property $\mathcal{P}_t 1 \equiv 1$ and the Minkowski inequality, we have

$$\begin{aligned}
\Big(\int_0^{r_B} \fint_B |t\nabla\mathcal{P}_t f|^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t}\Big)^{1/2} &= \Big(\int_0^{r_B} \fint_B |t\nabla\mathcal{P}_t (f - f_B)|^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t}\Big)^{1/2} \\
&\leqslant \sum_{k=0}^\infty \Big(\int_0^{r_B} \fint_B |t\nabla\mathcal{P}_t f_k|^2 \,\mathrm{d}\mu \frac{\mathrm{d}t}{t}\Big)^{1/2},
\end{aligned}$$

where $f_0 = (f - f_B)\chi_{4B}$ and $f_k = (f - f_B)\chi_{4^{k+1}B\setminus 4^k B}$ for any $k \in \mathbb{N}$.

For the term f_0, we apply the L^2-boundedness of the Riesz transform $D_x\mathcal{L}^{-1/2}$ and the spectral theory to deduce that

$$\begin{aligned}\int_0^{r_B} \fint_B |t\nabla\mathcal{P}_t f_0|^2 \,\mathrm{d}\mu\frac{\mathrm{d}t}{t} &\leqslant \frac{1}{\mu(B)}\int_0^\infty \int_X |t\nabla\mathcal{P}_t f_0|^2\,\mathrm{d}\mu\frac{\mathrm{d}t}{t}\\ &\leqslant \frac{C}{\mu(B)}\int_0^\infty \int_X |t\sqrt{\mathcal{L}}\mathcal{P}_t f_0|^2\,\mathrm{d}\mu\frac{\mathrm{d}t}{t}\\ &\leqslant \frac{C}{\mu(B)}\int_{4B} |f-f_B|^2\,\mathrm{d}\mu \leqslant C[\alpha(B)]^2\|f\|^2_{L^{2,\alpha}}.\end{aligned}$$

For the term f_k, $k\in\mathbb{N}$, it can be seen that $f_k\in L^2(X)$. Therefore, we can use Proposition 3.2 to deduce that

$$\int_0^{r_B}\int_B |t\nabla\mathcal{P}_t f_k|^2\,\mathrm{d}\mu\frac{\mathrm{d}t}{t} \leqslant C\int_0^{2r_B}\int_{2B}(|t\partial_t\mathcal{P}_t f_k|^2 + |t^2\partial_t^2\mathcal{P}_t f_k|^2 + |\mathcal{P}_t f_k|^2)\,\mathrm{d}\mu\frac{\mathrm{d}t}{t}.$$

It follows that for each $x\in 2B$ and $0<t<2r_B$,

$$\begin{aligned}&|t\partial_t\mathcal{P}_t f_k| + |t^2\partial_t^2\mathcal{P}_t f_k(x)| + |\mathcal{P}_t f_k(x)|\\ &\quad\leqslant C\int_{4^{k+1}B\setminus 4^kB}\frac{t}{t+d(x,y)}\frac{|f(y)-f_B|}{\mu(B(x,t+d(x,y)))}\,\mathrm{d}\mu(y)\\ &\quad\leqslant C\frac{t}{4^k r_B}\fint_{4^{k+1}B}|f-f_B|\,\mathrm{d}\mu(y)\\ &\quad\leqslant C\frac{t}{4^k r_B}\fint_{4^{k+1}B}|f-f_{4^{k+1}B}|\,\mathrm{d}\mu(y) + C\frac{t}{4^k r_B}|f_{4^{k+1}B}-f_B| \leqslant C\frac{kt}{4^k r_B}\alpha(B)\|f\|_{L^{2,\alpha}}.\end{aligned}$$

Hence, one has

$$\begin{aligned}\int_0^{r_B}\fint_B |t\nabla\mathcal{P}_t f_k|^2\,\mathrm{d}\mu\frac{\mathrm{d}t}{t} &\leqslant C\int_0^{2r_B}\fint_{2B}\Big(\frac{kt}{4^k r_B}\alpha(B)\|f\|_{L^{2,\alpha}}\Big)^2\,\mathrm{d}\mu\frac{\mathrm{d}t}{t}\\ &\leqslant C\frac{k^2}{4^{2k}}[\alpha(B)]^2\|f\|^2_{L^{2,\alpha}}.\end{aligned}$$

Combining the above estimates of f_k, we conclude

$$\begin{aligned}\Big(\int_0^{r_B}\fint_B |t\nabla\mathcal{P}_t f|^2\,\mathrm{d}\mu\frac{\mathrm{d}t}{t}\Big)^{1/2} &\leqslant \sum_{k=0}^\infty\Big(\int_0^{r_B}\fint_B |t\nabla\mathcal{P}_t f_k|^2\,\mathrm{d}\mu\frac{\mathrm{d}t}{t}\Big)^{1/2}\\ &\leqslant C\sum_{k=0}^\infty \frac{k}{4^k}\alpha(B)\|f\|_{L^{2,\alpha}} \leqslant C\alpha(B)\|f\|_{L^{2,\alpha}},\end{aligned}$$

which means

$$\frac{1}{\alpha(B)}\Big(\int_0^{r_B}\fint_B |t\nabla\mathcal{P}_t f|^2\,\mathrm{d}\mu\frac{\mathrm{d}t}{t}\Big)^{1/2} \leqslant C\|f\|_{L^{2,\alpha}}.$$

Taking the supremum over all balls $B\subset X$, we complete the proof of Theorem 3.3. $\square$

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