Liuying Wu

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Czechoslovak Mathematical Journal, Vol. 74 (2024), No. 2, 535–548

Persistent URL: http://dml.cz/dmlcz/152456

Czechoslovak Mathematical Journal, 74 (149) (2024), 535–548

ON THE LEAST ALMOST-PRIME IN ARITHMETIC PROGRESSIONS

Liuying Wu, Shanghai

Received October 13, 2023. Published online May 16, 2024.

Abstract. Let $\mathcal{P}_2$ denote a positive integer with at most 2 prime factors, counted according to multiplicity. For integers a, q such that $(a,q)=1$, let $\mathcal{P}_2(q,a)$ denote the least $\mathcal{P}_2$ in the arithmetic progression $\{nq+a\}_{n=1}^{\infty}$. It is proved that for sufficiently large q, we have

$$\mathcal{P}_2(q,a) \ll q^{1.825}.$$

This result constitutes an improvement upon that of J. Li, M. Zhang and Y. Cai (2023), who obtained $\mathcal{P}_2(q,a) \ll q^{1.8345}$.

Keywords: almost-prime; arithmetic progression; linear sieve; Selberg's Λ^2-sieve

MSC 2020: 11N13, 11N35, 11N36

1. Introduction

The classical theorem of Dirichlet about primes states that any arithmetic progression $\{nq+a\}_{n=1}^{\infty}$ contains infinitely many prime numbers, where $(a,q)=1$. Then a natural question to ask is, how large is the least such prime, $\mathcal{P}(q,a)$ say? The generalized Riemann hypothesis implies that

$$\mathcal{P}(q,a) \ll (\varphi(q))^2(\log q)^2. \tag{1.1}$$

The most important result in this respect is due to Linnik, (see [12], [13]), who showed that

$$\mathcal{P}(q,a) \ll q^L$$

for some absolute constant L. Afterwards, a series of authors engaged in the estimation of the constant L. In 1992, Heath-Brown in [3] obtained $L \leqslant 5.5$, and by pushing Heath-Brown's ideals, Xylouris in [17] showed that $L \leqslant 5.18$, which is the best result up to now.

DOI: 10.21136/CMJ.2024.0459-23

Let $\mathcal{P}_2$ denote an almost-prime with at most 2 prime factors, counted according to multiplicity. Another approximation toward the bound (1.1) is to show that $\mathcal{P}_2(q,a) \ll q^2$, where $\mathcal{P}_2(q,a)$ denotes the least $\mathcal{P}_2$ in the arithmetic progressions $\{nq+a\}_{n=1}^{\infty}$. The first result in this circle is due to Levin (see [10]), who showed in 1965 that

$$\mathcal{P}_2(q,a) \ll q^{2.3696}.$$

Later, the constant 2.3696 was improved successively to $\frac{25}{11}+\varepsilon$ and $\frac{11}{5}$ by Richert, see [8] and by Halberstam and Richert, see [1], Chapter 9, respectively. In 1976, Motohashi in [15] obtained the exponent $2+\varepsilon$ subject to a certain unproved hypothesis. In 1978, Heath-Brown in [2] made the important improvement in this problem by showing that

$$\mathcal{P}_2(q,a) \ll q^{1.965}.$$

In 1982, Iwaniec in the paper [6] improved Heath-Brown's result by showing that $\mathcal{P}_2(q,a) \ll q^{1.845}$. Very recently, the exponent 1.845 was reduced by Li, Zhang and Cai (see [11]), they obtained $\mathcal{P}_2(q,a) \ll q^{1.8345}$.

As one of the concluding remarks in Theorem 13 of [6], Iwaniec suggested the possibility of improvements by using the more efficient weight of Laborde (see [9]) in place of Richert's weights of logarithmic type. We carry out his suggestion and extend the range of the summation over p in the weighted sum beyond $M = x^{1-\varepsilon}q^{-1}$, combining linear sieve results of Iwaniec (see [5]) with bilinear forms for the remainder term and the two-dimensional sieve of Selberg. In contrast to the previous result of [11], our improvement comes from the application of the weight of Laborde.

In this paper, we prove the following sharper result:

Theorem 1.1. *Suppose that a and q are positive integers such that $(a,q)=1$. Let $\mathcal{P}_2(q,a)$ be the least almost-prime $\mathcal{P}_2$ such that $\mathcal{P}_2 \equiv a \pmod{q}$. Then*

$$\mathcal{P}_2(q,a) \ll q^{1.825}.$$

2. Preliminaries

Throughout this paper, the letter p always stands for a prime number. We use ε to denote a sufficiently small positive number, and the value of ε may change from statement to statement. For any natural number n, we use $\varphi(n)$, $\mu(n)$ and $\tau(n)$ to denote Euler's function, Möbius' function, and the Dirichlet divisor function, respectively. Moreover, $\Omega(n)$ denotes the number of prime factors of n, counted with multiplicity.

By (m_1, m_2) and $[m_1, m_2]$ we denote the greatest common divisor and the least common multiple of m_1, m_2, respectively. We write $f = \mathcal{O}(g)$ or, equivalently, $f \ll g$ if $|f| \leqslant Cg$ for some positive number C. The number $\mathcal{P}_2$ always denotes a positive integer with at most 2 prime factors, counted with multiplicity. And we use $e(\alpha)$ to denote $\mathrm{e}^{2\pi\mathrm{i}\alpha}$.

Let $\mathcal{A}$ be a finite sequence of integers, $\mathcal{P}$ a set of primes. For a given real number $z \geqslant 2$, let

$$S(\mathcal{A}, z) = \sum_{\substack{n \in \mathcal{A} \\ (n, P(z))=1}} 1,$$

where

$$P(z) = \prod_{\substack{p<z \\ p \in \mathcal{P}}} p.$$

For $d \mid P(z)$, put

$$\mathcal{A}_d = \{n \in \mathcal{A} \colon\ n \equiv 0 \pmod d\}.$$

Let further

$$V(z) = \prod_{\substack{p<z \\ p \in \mathcal{P}}} \Big(1 - \frac{\omega(p)}{p}\Big),$$

where ω is a multiplicative function supported on square-free integers, such that $0 \leqslant \omega(d) < d$ and

$$|\mathcal{A}_d| = \frac{\omega(d)}{d} X + r(\mathcal{A}, d)$$

for some remainders $r(\mathcal{A}, d)$ sufficiently small and X is independent of d.

Lemma 2.1. *Let $0 < \varepsilon < \frac{1}{3}$, $M > 1$, $N > 1$, $D = MN$. Assume that for some absolute constant $A \geqslant 1$*

$$\frac{V(w_1)}{V(w_2)} \leqslant \frac{\log w_2}{\log w_1}\Big(1 + \frac{A}{\log w_1}\Big), \quad w_2 > w_1 \geqslant 2.$$

Then for all $2 \leqslant z \leqslant D^{1/2}$ and $s = \log D/\log z$, we have

$$S(\mathcal{A}, z) \geqslant XV(z)\{f(s) - E\} - R^- \quad \text{and} \quad S(\mathcal{A}, z) \leqslant XV(z)\{F(s) + E\} + R^+,$$

where $f(s)$ and $F(s)$ are determined by the differential-difference equations

$$\begin{cases} F(s) = \dfrac{2\mathrm{e}^{\gamma}}{s}, \ f(s) = 0, & 1 \leqslant s \leqslant 2, \\ (sF(s))' = f(s-1), \ (sf(s))' = F(s-1), & s \geqslant 2. \end{cases}$$

The error term E is

$$E = \mathcal{O}(\varepsilon + \varepsilon^{-8}(\log D)^{-1/3})$$

and the remainder terms are

$$R^{\pm} = \sum_{l \leqslant \exp(8\varepsilon^{-3})} \sum_{\substack{m<M \\ m \mid P(z)}} \sum_{\substack{n<N \\ n \mid P(z)}} a^{\pm}_{m,l}(M,N,\varepsilon) b^{\pm}_{n,l}(M,N,\varepsilon) r(\mathcal{A}, mn)$$

with some coefficients $a^{\pm}_{m,l}$ and $b^{\pm}_{n,l}$ bounded by 1 in absolute value and dependent at most on parameters implied in the notation.

P r o o f. See Theorem 1 of [5]. □

Lemma 2.2. *We have*

$$F(s) = \frac{2\mathrm{e}^{\gamma}}{s}, \quad 0 < s \leqslant 3;$$
$$F(s) = \frac{2\mathrm{e}^{\gamma}}{s}\Big(1 + \int_3^s \frac{\log(t-2)}{t-1}\,\mathrm{d}t\Big), \quad 3 < s \leqslant 5;$$
$$f(s) = \frac{2\mathrm{e}^{\gamma}\log(s-1)}{s}, \quad 2 < s \leqslant 4;$$
$$f(s) = \frac{2\mathrm{e}^{\gamma}}{s}\Big(\log(s-1) + \int_4^s \int_3^{t-1} \frac{\log(u-2)}{(t-1)(u-1)}\,\mathrm{d}u\,\mathrm{d}t\Big), \quad 4 < s \leqslant 6,$$

where $\gamma = 0.577\ldots$ is the Euler constant.

P r o o f. This lemma can be found on pages 126–127 of [16]. □

Lemma 2.3. *For all real numbers $0 < \varepsilon < 1$, all integers $1 \leqslant a < b$ and $m, q \geqslant 1$, we have*

$$\sum_{a < v \leqslant b} e\Big(m\frac{\overline{v}}{q}\Big) \ll (m,q)^{1/2} q^{1/2+\varepsilon} + (m,q)\frac{b-a}{q},$$

where $\overline{v}$ is defined by $v\overline{v} \equiv 1 \pmod{q}$.

P r o o f. This can be deduced from Lemma 1 of [4]. □

Lemma 2.4. *For any nonnegative numbers a_n and real x_n we have*

$$\Big|\sum_{n=1}^{Q} a_n \psi(x_n)\Big| \ll \frac{1}{H}\sum_{n=1}^{Q} a_n + \sum_{h=1}^{H} \frac{1}{h}\Big|\sum_{n=1}^{Q} a_n e(hx_n)\Big|$$

for all $H \geqslant 1$, the constant implied in the symbol $\ll$ being absolute.

P r o o f. This is Lemma 6 in [6]. □

3. Weighted sieve

The sequence of interest in this paper is

$$\mathcal{A} = \{n \leqslant x\colon\ n \equiv a \pmod q\},$$

where $(a,q)=1$. Then

$$|\mathcal{A}_d| = \Bigl\{n \leqslant \frac{x}{d}\colon\ nd \equiv a \pmod q\Bigr\} = \frac{x}{dq} + r(\mathcal{A},d) \tag{3.1}$$

for $(d,q)=1$, where

$$r(\mathcal{A},d) = \psi\Bigl(\frac{x}{dq} - a\frac{\bar{d}}{q}\Bigr) - \psi\Bigl(-a\frac{\bar{d}}{q}\Bigr) \quad \text{and} \quad \psi(t) = [t] - t + \frac{1}{2}. \tag{3.2}$$

In this paper, we set $x = q^{\theta}$, $\theta = 1.825$ and

$$\mathcal{P} = \{p\colon\ p \nmid q\}, \quad M = x^{1-3\varepsilon}q^{-1}, \quad N = x^{1/2-4\varepsilon}q^{-3/4}, \quad D = MN, \quad z = D^{1/6}.$$

Here it is worth noting that $M \approx x^{0.452-3\varepsilon}$. For $1 \leqslant b \leqslant c \leqslant 6$, we consider the expression

$$W(\mathcal{A},b,c) = W_1(\mathcal{A},b,c) - W_2(\mathcal{A},b,c),$$

where

$$\begin{aligned} W_1(\mathcal{A},b,c) = S(\mathcal{A},z) - \frac{1}{2c-b-1}\Biggl((c-b)&\sum_{z\leqslant p<D^{b/6}} S(\mathcal{A}_p,z) \\ &+ 6\int_{1/6}^{(b+1)/12}\Bigl(\sum_{D^s\leqslant p\leqslant D^{(b+1)/6-s}} S(\mathcal{A}_p,D^s)\Bigr)\,\mathrm{d}s \\ &+ \sum_{D^{1/6}\leqslant p\leqslant D^{(b+1)/12}}\Bigl(b+1-12\frac{\log p}{\log D}\Bigr)S(\mathcal{A}_p,p) \\ &+ \sum_{D^{b/6}\leqslant p<M}\Bigl(c-6\frac{\log p}{\log D}\Bigr)S(\mathcal{A}_p,z)\Biggr), \end{aligned}$$

$$W_2(\mathcal{A},b,c) = \frac{1}{2c-b-1}\sum_{M\leqslant p<D^{c/6}}\Bigl(c-6\frac{\log p}{\log D}\Bigr)S(\mathcal{A}_p,z).$$

Lemma 3.1. *If $b+c+1 = 6\log x/\log D$ and $3c - 6\log x/\log D > 0$, then we have*

$$\sum_{\mathcal{P}_2\in\mathcal{A}} 1 \geqslant W(\mathcal{A},b,c) + o\Bigl(\frac{x}{\varphi(q)\log x}\Bigr).$$

Proof. See Theorem 1 in [9] for details. $\square$

In this paper, we choose $b = 4.86553$, $c = 5.22308$ and prove that

$$W(\mathcal{A}, b, c) \gg \frac{x}{\varphi(q)\log D}$$

in Section 5. By the choice of b and c, it is easy to verify that

$$D^{b/6} < M < D^{c/6}.$$

4. The evaluation of sifting functions

To evaluate $W_1(\mathcal{A}, b, c)$, we write

$$W_1(\mathcal{A}, b, c) = S(\mathcal{A}, z) - \frac{1}{2c-b-1}(S_1 + S_2 + S_3 + S_4),$$

where

$$\begin{aligned}
S_1 &= (c-b) \sum_{z \leqslant p < D^{b/6}} S(\mathcal{A}_p, z),\\
S_2 &= 6\int_{1/6}^{(b+1)/12} \Big(\sum_{D^s \leqslant p \leqslant D^{(b+1)/6-s}} S(\mathcal{A}_p, D^s)\Big)\,\mathrm{d}s,\\
S_3 &= \sum_{z \leqslant p \leqslant D^{(b+1)/12}} \Big(b+1-12\frac{\log p}{\log D}\Big) S(\mathcal{A}_p, p),\\
S_4 &= \sum_{D^{b/6} \leqslant p < M} \Big(c - 6\frac{\log p}{\log D}\Big) S(\mathcal{A}_p, z).
\end{aligned}$$

Lemma 4.1. *Let $\varepsilon > 0$, $x^{2/5} < q \leqslant x^{2/3-6\varepsilon}$, $M = x^{1-3\varepsilon}q^{-1}$, $N = x^{1/2-4\varepsilon}q^{-3/4}$, $|a_m| \leqslant 1$, $|b_n| \leqslant 1$. We then have*

$$\sum_{\substack{m \leqslant M\\ (mn,q)=1}} \sum_{\substack{n \leqslant N\\ (mn,q)=1}} a_m b_n r(\mathcal{A}, mn) \ll \frac{x^{1-\varepsilon}}{\varphi(q)},$$

where $r(\mathcal{A}, d)$ is defined by (3.2).

Proof. This is Theorem 5 in [6]. □

We appeal to Lemma 2.1 to estimate $S(\mathcal{A}, z)$. We have

$$S(\mathcal{A}, z) \geqslant \frac{x}{q} V(z)(f(6) + \mathcal{O}(\varepsilon + \varepsilon^{-8}(\log D)^{-1/3})) - R^-, \tag{4.1}$$

where

$$V(z) = \prod_{\substack{p<z\\ p\nmid q}} \Big(1 - \frac{1}{p}\Big) = \frac{q}{\varphi(q)} \prod_{p<z} \Big(1-\frac{1}{p}\Big) = \frac{q}{\varphi(q)} \frac{\mathrm{e}^{-\gamma}}{\log z}\Big(1 + \mathcal{O}\Big(\frac{1}{\log z}\Big)\Big) \tag{4.2}$$

by Mertens' prime number theorem (see [14]), and

$$R^- = \sum_{l \leqslant \exp(8\varepsilon^{-3})} \sum_{\substack{m<M \\ (mn,q)=1}} \sum_{\substack{n<N \\ (mn,q)=1}} a_m^-(l) b_n^-(l) r(\mathcal{A}, mn).$$

By Lemma 4.1, one has

$$R^- \ll \frac{x^{1-\varepsilon}}{\varphi(q)}. \tag{4.3}$$

By (4.1)–(4.3), Lemma 2.2 and numerical integration, we get

$$\begin{aligned} S(\mathcal{A}, z) &\geqslant \frac{6x}{\varphi(q)\log D} \mathrm{e}^{-\gamma}\{f(6)+o(1)\} + \mathcal{O}\Big(\frac{x^{1-\varepsilon}}{\varphi(q)}\Big) \\ &= (1+o(1)) \frac{2x}{\varphi(q)\log D}\Big(\log 5 + \int_4^6 \int_3^{t-1} \frac{\log(u-2)}{(t-1)(u-1)} \,\mathrm{d}u \,\mathrm{d}t\Big) \\ &\geqslant 3.36839 \frac{x(1+o(1))}{\varphi(q)\log D}. \end{aligned} \tag{4.4}$$

Lemma 4.2. *For sufficiently large x, we have*

$$S_1 \leqslant 2.40596 \frac{x(1+o(1))}{\varphi(q)\log D}.$$

Proof. For each $S(\mathcal{A}_p, z)$, by Lemma 2.1 we have

$$\begin{aligned} S(\mathcal{A}_p, z) \leqslant{}& \frac{x}{pq} V(z)\Big(F\Big(\frac{\log(D/p)}{\log z}\Big) + \mathcal{O}\Big(\varepsilon + \varepsilon^{-8}\Big(\log \frac{D}{p}\Big)^{-1/3}\Big)\Big) \\ &+ \sum_{l \leqslant \exp(8\varepsilon^{-3})} \sum_{\substack{m<M/p \\ (mn,q)=1}} \sum_{\substack{n<N \\ (mn,q)=1}} a_m^+(l) b_n^+(l) r(\mathcal{A}, pnm), \end{aligned} \tag{4.5}$$

where $|a_m^+(l)| \leqslant 1$, $|b_n^+(l)| \leqslant 1$. By Lemma 4.1, we have

$$\begin{aligned} &\sum_{\substack{z \leqslant p < D^{b/6} \\ p \nmid q}} \sum_{l \leqslant \exp(8\varepsilon^{-3})} \sum_{\substack{m<M/p \\ (mn,q)=1}} \sum_{\substack{n<N \\ (mn,q)=1}} a_m^+(l) b_n^+(l) r(\mathcal{A}, pnm) \\ &\quad = \sum_{l \leqslant \exp(8\varepsilon^{-3})} \sum_{\substack{z \leqslant k < M \\ (kn,q)=1}} \sum_{\substack{n<N \\ (kn,q)=1}} c_k^+(l) b_n^+(l) r(\mathcal{A}, kn) \ll \frac{x^{1-\varepsilon}}{\varphi(q)}, \end{aligned} \tag{4.6}$$

where

$$c_k^+(l) = \sum_{\substack{z \leqslant p < D^{b/6} \\ pm=k}} a_m^+(l) \ll 1.$$

Therefore, by (4.5) and (4.6) we obtain that

$$\sum_{z\leqslant p<D^{b/6}} S(\mathcal{A}_p, z) \leqslant \frac{x}{q}V(z) \sum_{z\leqslant p<D^{b/6}} \frac{1}{p}\Big(F\Big(\frac{\log(D/p)}{\log z}\Big) + \mathcal{O}\Big(\varepsilon + \varepsilon^{-8}\Big(\log\frac{D}{p}\Big)^{-1/3}\Big)\Big) + \mathcal{O}\Big(\frac{x^{1-\varepsilon}}{\varphi(q)}\Big). \tag{4.7}$$

From Mertens' prime number theorem (see [14]), we have

$$\sum_{z\leqslant p<D^{b/6}} \frac{1}{p}\Big(\log\frac{D}{p}\Big)^{-1/3} \ll \frac{1}{(\log D)^{1/3}} = o(1).$$

By partial summation and prime number theorem, we get

$$\begin{aligned}\sum_{z\leqslant p<D^{b/6}} \frac{1}{p}F\Big(\frac{\log(D/p)}{\log z}\Big) &= \int_z^{D^{b/6}} \frac{1}{t}\frac{1}{\log t}F\Big(\frac{\log(D/t)}{\log z}\Big)\,\mathrm{d}t + o(1)\\ &= \int_{1/6}^{b/6} F(6-6\varrho)\frac{\mathrm{d}\varrho}{\varrho} + o(1).\end{aligned} \tag{4.8}$$

From (4.5)–(4.8), Mertens' prime number theorem, Lemma 2.2 and numerical integration, we have

$$\begin{aligned}S_1 &\leqslant \frac{6x}{\varphi(q)\log D}\mathrm{e}^{-\gamma}\Big((c-b)\int_{1/6}^{b/6} F(6-6\varrho)\frac{\mathrm{d}\varrho}{\varrho} + o(1)\Big) + \mathcal{O}\Big(\frac{x^{1-\varepsilon}}{\varphi(q)}\Big)\\ &= (1+o(1))\frac{12(c-b)x}{\varphi(q)\log D}\Big(\int_{1/6}^{b/6} \frac{\mathrm{d}\varrho}{\varrho(6-6\varrho)} + \int_{1/6}^{1/2} \frac{\mathrm{d}\varrho}{\varrho(6-6\varrho)}\int_3^{6-6\varrho} \frac{\log(u-2)}{u-1}\,\mathrm{d}u\Big)\\ &\leqslant 2.40596\frac{x(1+o(1))}{\varphi(q)\log D}.\end{aligned}$$

□

By arguments similar to the evaluation of S_1, we get:

Lemma 4.3.

$$\begin{aligned}S_2 &\leqslant \frac{6x}{\varphi(q)\log D}\mathrm{e}^{-\gamma}\Big(\int_{1/6}^{(b+1)/12}\Big(\int_s^{(b+1)/6-s} F\Big(\frac{1-t}{s}\Big)\frac{\mathrm{d}t}{t}\Big)\frac{\mathrm{d}s}{s} + o(1)\Big)\\ &= (1+o(1))\frac{12x}{\varphi(q)\log D}\Big(\int_{1/6}^{(b+1)/12} \mathrm{d}s\int_s^{(b+1)/6-s} \frac{1}{t(1-t)}\,\mathrm{d}t\\ &\quad + \int_{1/6}^{1/4} \mathrm{d}s\int_s^{1-3s} \frac{\mathrm{d}t}{t(1-t)}\int_3^{(1-t)/s} \frac{\log(u-2)}{u-1}\,\mathrm{d}u\Big)\\ &\leqslant 5.47585\frac{x(1+o(1))}{\varphi(q)\log D},\end{aligned}$$

$$
\begin{aligned}
S_3 &\leqslant \frac{6x}{\varphi(q)\log D}\mathrm{e}^{-\gamma}\Big(\int_{1/6}^{(b+1)/12}\Big(\frac{b+1}{6}-2t\Big)F\Big(\frac{1-t}{t}\Big)\frac{\mathrm{d}t}{t^2}+o(1)\Big)\\
&=(1+o(1))\frac{12x}{\varphi(q)\log D}\Big(\int_{1/6}^{(b+1)/12}\Big(\frac{b+1}{6}-2t\Big)\frac{1}{t(1-t)}\,\mathrm{d}t\\
&\quad+\int_{1/6}^{1/4}\Big(\frac{b+1}{6}-2t\Big)\frac{1}{t(1-t)}\,\mathrm{d}t\int_{3}^{(1-t)/t}\frac{\log(u-2)}{u-1}\,\mathrm{d}u\Big)\\
&\leqslant 7.16879\frac{x(1+o(1))}{\varphi(q)\log D},\\
S_4 &\leqslant \frac{6x}{\varphi(q)\log D}\mathrm{e}^{-\gamma}\Big(\int_{b/6}^{(4\theta-4)/6\theta-7}(c-6t)F(6-6t)\frac{\mathrm{d}t}{t}+o(1)\Big)\\
&=(1+o(1))\frac{12x}{\varphi(q)\log D}\int_{b/6}^{(4\theta-4)/(6\theta-7)}\frac{c-6t}{t(6-6t)}\,\mathrm{d}t\\
&\leqslant 0.09536\frac{x(1+o(1))}{\varphi(q)\log D}.
\end{aligned}
$$

To deal with $W_2(\mathcal{A},b,c)$, we follow arguments from [7] to estimate the main term, and we use the Kloosterman sums approach for the error term.

Lemma 4.4. *We have*

$$W_2(\mathcal{A},b,c)\leqslant 0.05951\frac{x(1+o(1))}{\varphi(q)\log D}.$$

Proof. Let $y=D^{c/6}$ and put

$$T(\mathcal{A},c)=\sum_{M\leqslant p<y}\Big(1-\frac{\log p}{\log y}\Big)S(\mathcal{A}_p,z).$$

We have

$$
\begin{aligned}
T(\mathcal{A},c) &\leqslant \sum_{M\leqslant n<y}\Big(1-\frac{\log n}{\log y}\Big)\sum_{\substack{m\in\mathcal{A}\\ m\equiv 0 \ (\mathrm{mod}\ n)\\ (m,P(z))=1}}1\\
&=\sum_{M\leqslant n<y}\Big(1-\frac{\log n}{\log y}\Big)\sum_{\substack{m\in\mathcal{A}\\ m\equiv 0 \ (\mathrm{mod}\ n)}}\sum_{d\mid(m,P(z))}\mu(d).
\end{aligned}
\tag{4.9}
$$

Let $\{\lambda_d\}$ be Selberg's upper bound sieve of dimension two of level $\xi=N^2=x^{1-8\varepsilon}q^{-3/2}$, then we have $\lambda_d=0$ for $d>\xi$, $\lambda_1=1$ such that

$$\sum_{d\mid n}\mu(d)\leqslant\sum_{d\mid n}\lambda_d$$

and $|\lambda_d| \leqslant 3^{\Omega(d)}$ for all $d \geqslant 1$. Now by (4.9), we have

$$
\begin{aligned}
T(\mathcal{A}, c) &\leqslant \sum_{M \leqslant n < y} \Big(1 - \frac{\log n}{\log y}\Big) \sum_{\substack{m \in \mathcal{A} \\ m \equiv 0 \pmod{n}}} \sum_{d \mid (m, P(z))} \lambda_d \\
&= \sum_{\substack{d < \xi \\ d \mid P(z)}} \lambda_d \sum_{M \leqslant n < y} \Big(1 - \frac{\log n}{\log y}\Big) \sum_{\substack{m \in \mathcal{A} \\ m \equiv 0 \pmod{[d,n]}}} 1 \\
&= \sum_{\substack{d < \xi \\ d \mid P(z)}} \lambda_d \sum_{M \leqslant n < y} \Big(1 - \frac{\log n}{\log y}\Big) \sum_{\substack{l \leqslant x/[d,n] \\ l \equiv a\overline{[d,n]} \pmod{q}}} 1 \\
&= \sum_{\substack{d < \xi \\ d \mid P(z)}} \lambda_d \sum_{M \leqslant n < y} \Big(1 - \frac{\log n}{\log y}\Big) \frac{x}{[d,n]q} \\
&\quad + \sum_{\substack{d < \xi \\ d \mid P(z)}} \lambda_d \sum_{M \leqslant n < y} \Big(1 - \frac{\log n}{\log y}\Big) \psi\Big(\frac{x}{[d,n]q} - a\frac{\overline{[d,n]}}{q}\Big) \\
&\quad - \sum_{\substack{d < \xi \\ d \mid P(z)}} \lambda_d \sum_{M \leqslant n < y} \Big(1 - \frac{\log n}{\log y}\Big) \psi\Big(-a\frac{\overline{[d,n]}}{q}\Big) \\
&= T_1(\mathcal{A}, c) + T_2(\mathcal{A}, x, c) - T_2(\mathcal{A}, 0, c),
\end{aligned}
\tag{4.10}
$$

where $\psi(t) = [t] - t + \frac{1}{2}$.

We estimate the sum $T_2(\mathcal{A}, \chi, c)$, where $\chi = x$ or 0 first. By a splitting argument, it is clear that

(4.11)

$$
T_2(\mathcal{A}, \chi, c) \ll (\log y) \max_{\substack{M \leqslant L \leqslant y/2 \\ L < L_1 \leqslant 2L}} \sum_{d < \xi} 3^{\Omega(d)} \Big| \sum_{L \leqslant n < L_1} \Big(1 - \frac{\log n}{\log y}\Big) \psi\Big(\frac{\chi}{[d,n]q} - a\frac{\overline{[d,n]}}{q}\Big) \Big|.
$$

For $L < L_1 \leqslant 2L$, by Lemma 2.4 we have

$$
\begin{aligned}
S(\chi, q, L, d) &= \sum_{L \leqslant n < L_1} \Big(1 - \frac{\log n}{\log y}\Big) \psi\Big(\frac{\chi}{[d,n]q} - a\frac{\overline{[d,n]}}{q}\Big) \\
&= \sum_{\alpha v \mid d} \mu(\alpha) \sum_{L/(\alpha v) \leqslant n < L_1/(\alpha v)} \Big(1 - \frac{\log \alpha v n}{\log y}\Big) \psi\Big(\frac{\chi}{dn\alpha q} - a\frac{\overline{dn\alpha}}{q}\Big) \\
&\ll \sum_{\alpha v \mid d} \Big(\frac{L}{H\alpha v} + \sum_{h=1}^{H} \frac{1}{h} |S_h(\chi, q, L, \alpha, v, d)|\Big),
\end{aligned}
\tag{4.12}
$$

where

$$S_h(\chi, q, L, \alpha, v, d) = \sum_{L/(\alpha v)\leqslant n<L_1/(\alpha v)} \Big(1-\frac{\log \alpha v n}{\log y}\Big) e\Big(\frac{h\chi}{dn\alpha q} - ah\frac{\overline{dn\alpha}}{q}\Big).$$

By partial summation and Lemma 2.3, we have

$$S_h(\chi, q, L, \alpha, v, d) \ll \Big(1+\frac{hxv}{dLq}\Big)\Big((h,q)^{1/2}q^{1/2} + (h,q)\frac{L}{\alpha v q}\Big)q^{\varepsilon}. \tag{4.13}$$

From (4.12) and (4.13), we get

$$\begin{aligned} S(\chi, q, L, d) &\ll \sum_{\alpha v\mid d}\Big(\frac{L}{H\alpha v} + \sum_{h=1}^{H}\frac{1}{h}\Big(1+\frac{hxv}{dLq}\Big)\Big((h,q)^{1/2}q^{1/2} + (h,q)\frac{L}{\alpha v q}\Big)q^{\varepsilon}\Big) \\ &\ll \sum_{\alpha v\mid d}\Big(\frac{L}{H\alpha v} + \Big(1+\frac{Hxv}{dLq}\Big)\Big(q^{1/2} + \frac{L}{\alpha v q}\Big)q^{\varepsilon}\log 2H\Big). \end{aligned} \tag{4.14}$$

On taking $H = [Ldx^{-1+\varepsilon}q/(\alpha v)] + 1$, it follows from (4.14) that

$$S(\chi, q, L, d) \ll \sum_{\alpha v\mid d}(x^{1-\varepsilon}q^{-1}d^{-1} + q^{1/2} + x^{\varepsilon}\alpha^{-1}). \tag{4.15}$$

Finally, from (4.11) and (4.15), we find that

$$\begin{aligned} T_2(\mathcal{A}, \chi, c) &\ll (\log y)\max_{\substack{M\leqslant L\leqslant y/2\\ L<L_1\leqslant 2L}}\sum_{d<\xi}3^{\Omega(d)}|S(\chi, q, L, d)| \\ &\ll (\log y)\max_{M\leqslant L\leqslant y/2}\sum_{d<\xi}3^{\Omega(d)}\sum_{\alpha v\mid d}(x^{1-\varepsilon}q^{-1}d^{-1} + q^{1/2} + x^{\varepsilon}\alpha^{-1}) \\ &\ll (\log y)\Big(x^{1-\varepsilon}q^{-1}\sum_{d<\xi}\frac{3^{\Omega(d)}}{d}\sum_{\alpha\mid d}\tau(\alpha) + (q^{1/2}+x^{\varepsilon})\sum_{d<\xi}3^{\Omega(d)}\sum_{\alpha\mid d}\tau(\alpha)\Big) \\ &\ll (\log y)\Big(x^{1-\varepsilon}q^{-1}\sum_{d<\xi}\frac{\tau^4(d)}{d} + (q^{1/2}+x^{\varepsilon})\sum_{d<\xi}\tau^4(d)\Big) \\ &\ll (\log y)\Big(x^{1-\varepsilon}q^{-1}(\log\xi)^{16} + (q^{1/2}+x^{\varepsilon})\xi(\log\xi)^{15}\Big) \ll x^{1-\varepsilon}q^{-1}. \end{aligned} \tag{4.16}$$

Now we consider the sum $T_1(\mathcal{A}, c)$. We rewrite it as

$$T_1(\mathcal{A}, c) = \sum_{\substack{d<\xi\\ d\mid P(z)}}\lambda_d\sum_{M\leqslant n<y}\Big(1-\frac{\log n}{\log y}\Big)\frac{x}{[d,n]q} = \frac{x}{q}\sum_{\substack{d<\xi\\ d\mid P(z)}}\frac{\lambda_d}{d}\sum_{M\leqslant n<y}\Big(1-\frac{\log n}{\log y}\Big)\frac{(d,n)}{n}. \tag{4.17}$$

For the inner sum of (4.17), we have

(4.18)

$$\begin{aligned}\sum_{M\leqslant n<y}\Big(1-\frac{\log n}{\log y}\Big)\frac{(d,n)}{n} &= \sum_{v|d}\sum_{\substack{M/v\leqslant n<y/v\\(n,d/v)=1}}\Big(1-\frac{\log vn}{\log y}\Big)\frac{1}{n}\\ &= \sum_{\alpha v|d}\frac{\mu(\alpha)}{\alpha}\sum_{M/\alpha v\leqslant n<y/\alpha v}\Big(1-\frac{\log \alpha vn}{\log y}\Big)\frac{1}{n}\\ &= \sum_{\alpha v|d}\frac{\mu(\alpha)}{\alpha}\int_M^y\Big(1-\frac{\log t}{\log y}\Big)\frac{\mathrm{d}t}{t}+\mathcal{O}\Big(M^{-1}\sum_{v|d}v\tau\Big(\frac{\mathrm{d}}{v}\Big)\Big).\end{aligned}$$

We also have

$$\int_M^y\Big(1-\frac{\log t}{\log y}\Big)\frac{\mathrm{d}t}{t}=\frac{1}{2\log y}\Big(\log\frac{y}{M}\Big)^2, \tag{4.19}$$

$$\omega_1(d):=\sum_{\alpha v|d}\frac{\mu(\alpha)}{\alpha}=\sum_{\alpha|d}\frac{\mu(\alpha)}{\alpha}\tau\Big(\frac{d}{\alpha}\Big)=\prod_{p|d}\Big(2-\frac{1}{p}\Big) \tag{4.20}$$

and

(4.21)

$$\begin{aligned}\sum_{\substack{d<\xi\\d|P(z)}}\frac{|\lambda_d|}{d}\sum_{v|d}v\tau\Big(\frac{d}{v}\Big) &\ll \sum_{d<\xi}3^{\Omega(d)}\sum_{v|d}\frac{\tau(v)}{v}\ll\sum_{d<\xi}3^{\Omega(d)}\prod_{p|d}\Big(1+\frac{2}{p}\Big)\\ &\ll\sum_{d<\xi}3^{\Omega(d)}(\log\log d)^2\ll(\log\log\xi)^2\sum_{d<\xi}\tau^2(d)\ll\xi(\log\xi)^4.\end{aligned}$$

From (4.17) and (4.18)–(4.21), we have

$$T_1(\mathcal{A},c)=\frac{x}{2q\log y}\log^2\frac{y}{M}\sum_{\substack{d<\xi\\d|P(z)}}\frac{\omega_1(d)}{d}\lambda_d+\mathcal{O}\Big(\frac{x}{qM}\xi\log^4\xi\Big). \tag{4.22}$$

Noting that the function $\omega_1(d)$ is multiplicative and it satisfies the two-dimensional sieve assumptions, we conclude from Selberg's Λ^2-sieve that (one can see it in [1], Lemma 6.1)

$$\sum_{\substack{d<\xi\\d|P(z)}}\frac{\omega_1(d)}{d}\lambda_d=\frac{1}{G(\xi,z)}=\frac{V_1(z)}{\sigma(s)}\Big(1+\mathcal{O}\Big(\frac{1}{\log z}\Big)\Big)\quad\text{for } z\leqslant\xi, \tag{4.23}$$

where

$$s=\frac{\log\xi}{\log z},\quad \sigma(s)=\frac{s^2}{8e^{2\gamma}}\quad\text{for } 0<s\leqslant 2 \tag{4.24}$$

and

$$V_1(z)=\prod_{p<z}\Big(1-\frac{\omega_1(p)}{p}\Big)=\prod_{p<z}\Big(1-\frac{1}{p}\Big)^2=\frac{\mathrm{e}^{-2\gamma}}{\log^2 z}\Big(1+\mathcal{O}\Big(\frac{1}{\log z}\Big)\Big). \tag{4.25}$$

By (4.22)–(4.25), we finally obtain

$$T_1(\mathcal{A}, c) = \frac{4x}{q \log y}\Big(\frac{\log(y/M)}{\log \xi}\Big)^2\Big(1 + \mathcal{O}\Big(\frac{1}{\log z}\Big)\Big) + \mathcal{O}\Big(\frac{x}{qM}\xi \log^4 \xi\Big). \tag{4.26}$$

For $\xi = N^2$ we have $z \leqslant \xi \leqslant z^2$, and (4.23) holds. Then from (4.10), (4.16) and (4.26), we obtain

$$\begin{aligned} T(\mathcal{A}, c) &\leqslant \frac{x}{q \log y}\Big(\frac{\log(y/M)}{\log N}\Big)^2 + \mathcal{O}\Big(\frac{x}{qM}\xi \log^4 \xi\Big) + \mathcal{O}\Big(\frac{x^{1-\varepsilon}}{q}\Big) \\ &= \frac{x}{q \log D}\frac{\log D}{\log y}\Big(\frac{\log(y/M)}{\log N}\Big)^2 + o\Big(\frac{x}{q \log D}\Big) \\ &= \frac{6x}{cq \log D}\Big(\Big(\frac{6\theta c - 7c - 24\theta + 24}{12\theta - 18}\Big)^2 + o(1)\Big) \\ &\leqslant 0.05218\frac{x(1 + o(1))}{\varphi(q) \log D}. \end{aligned} \tag{4.27}$$

Now, by (4.27) we have

$$W_2(\mathcal{A}, b, c) = \frac{c}{2c - b - 1}T(\mathcal{A}, c) \leqslant 0.05951\frac{x(1 + o(1))}{\varphi(q) \log D},$$

and the proof of Lemma 4.4 is completed. □

5. Proof of Theorem 1.1

We conclude from Lemmas 4.2 and 4.3 that

$$S_1 + S_2 + S_3 + S_4 \leqslant 15.14596\frac{x(1 + o(1))}{\varphi(q) \log D}. \tag{5.1}$$

By (4.4), (5.1) and Lemma 4.4 we get

$$\begin{aligned} W(\mathcal{A}, b, c) &= W_1(\mathcal{A}, b, c) - W_2(\mathcal{A}, b, c) \\ &= S(\mathcal{A}, z) - \frac{1}{2c - b - 1}(S_1 + S_2 + S_3 + S_4) - W_2(\mathcal{A}, b, c) \\ &\geqslant \Big(3.36839 - \frac{15.14596}{2 \times 5.22308 - 4.86553 - 1} - 0.05951\Big)\frac{x(1 + o(1))}{\varphi(q) \log D} \\ &\geqslant 0.002\frac{x(1 + o(1))}{\varphi(q) \log D}. \end{aligned} \tag{5.2}$$

Now, by Lemma 3.1 and (5.2), Theorem 1.1 is proved. □

Acknowledgement. We wish to thank the referee for a thorough reading of the paper and helpful remarks.

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Author's address: Liuying Wu, School of Mathematical Science, Tongji University, Zhiyuan building, no.1239, Siping Road, Shanghai, 200092, P. R. China, e-mail: `liuyingw@tongji.edu.cn`.