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ON THE CHARACTERIZATION OF CERTAIN ADDITIVE MAPS IN PRIME $*$-RINGS

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Abstract. Let $\mathcal{A}$ be a noncommutative prime ring equipped with an involution '$*$', and let $\mathcal{Q}_{ms}(\mathcal{A})$ be the maximal symmetric ring of quotients of $\mathcal{A}$. Consider the additive maps $\mathcal{H}$ and $\mathcal{T}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$. We prove the following under some inevitable torsion restrictions. (a) If m and n are fixed positive integers such that $(m+n)\mathcal{T}(a^2) = m\mathcal{T}(a)a^* + na\mathcal{T}(a)$ for all $a \in \mathcal{A}$ and $(m+n)\mathcal{H}(a^2) = m\mathcal{H}(a)a^* + na\mathcal{T}(a)$ for all $a \in \mathcal{A}$, then $\mathcal{H} = 0$. (b) If $\mathcal{T}(aba) = a\mathcal{T}(b)a^*$ for all $a, b \in \mathcal{A}$, then $\mathcal{T} = 0$. Furthermore, we characterize Jordan left τ-centralizers in semiprime rings admitting an anti-automorphism τ. As applications, we find the structure of generalized Jordan $*$-derivations in prime rings and generalize as well as improve all the results of A. Abbasi, C. Abdioglu, S. Ali, M. R. Mozumder (2022).

Keywords: prime ring; involution; generalized (m,n)-Jordan $*$-centralizer

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1. Introduction

Throughout the paper, unless otherwise stated, $\mathcal{A}$ denotes a semiprime ring with the center $\mathcal{Z}(\mathcal{A})$. A ring $\mathcal{A}$ is called a *semiprime ring* if for any $x \in \mathcal{A}$, $x\mathcal{A}x = \{0\}$ implies that $x = 0$. A ring $\mathcal{A}$ is called a *prime ring* if for any $x, y \in \mathcal{A}$, $x\mathcal{A}y = \{0\}$ implies that either $x = 0$ or $y = 0$. A ring $\mathcal{A}$ is said to be n-torsion free, where n is a positive integer, if for any $x \in \mathcal{A}$, $nx = 0$ implies $x = 0$. If $\mathcal{A}$ is a prime ring and p is a prime number, then $\mathcal{A}$ is p-torsion free if and only if $\operatorname{char}(\mathcal{A}) \neq p$. We denote the maximal left ring of quotients of $\mathcal{A}$ by $\mathcal{Q}_{ml}(\mathcal{A})$, the symmetric Martindale quotient ring of $\mathcal{A}$ by $\mathcal{Q}_s(\mathcal{A})$, and the maximal symmetric ring of quotients of $\mathcal{A}$ by $\mathcal{Q}_{ms}(\mathcal{A})$. It is well known that $\mathcal{A} \subseteq \mathcal{Q}_s(\mathcal{A}) \subseteq \mathcal{Q}_{ms}(\mathcal{A}) \subseteq \mathcal{Q}_{ml}(\mathcal{A})$. The super rings $\mathcal{Q}_s(\mathcal{A})$, $\mathcal{Q}_{ms}(\mathcal{A})$ and $\mathcal{Q}_{ml}(\mathcal{A})$ are prime or semiprime according as $\mathcal{A}$ is prime or semiprime and they have the same centre $\mathcal{C}$, known as the extended

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centroid of $\mathcal{A}$. Moreover, $\mathcal{C} = \{\lambda \in \mathcal{Q}_s(\mathcal{A})\colon\ \lambda a = a\lambda$ for all $a \in \mathcal{A}\}$. Ring $\mathcal{A}$ is prime if and only if $\mathcal{C}$ is a field. For details one may refer to [3], [5], [6], [8], [9], [11].

An involution '$*$' on $\mathcal{A}$ is an anti-automorphism of order 1 or 2. It is well known that any anti-automorphism τ of $\mathcal{A}$ can be uniquely extended to an anti-automorphism of $\mathcal{Q}_{ms}(\mathcal{A})$ and hence can also be viewed as an anti-automorphism of $\mathcal{C}$. The anti-automorphism τ of $\mathcal{A}$ is said to be of the first kind if it acts as the identity map on $\mathcal{C}$ and of the second kind otherwise. For $x, y \in \mathcal{A}$, we denote $xy - yx$ by $[x, y]$, $xy + yx$ by $x \circ y$, and $xy - yx^*$ by ${}_*[x, y]$. For a fixed positive integer m, we let $\bar{a}_m = (a_1, a_2, \ldots, a_m) \in \mathcal{A}^m$, $\bar{a}_m^i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m) \in \mathcal{A}^{m-1}$ and $1 \leqslant i < j \leqslant m$, $\bar{a}_m^{ij} = \bar{a}_m^{ji} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_m) \in \mathcal{A}^{m-2}$. We write $\deg(x) = n$ if x is algebraic of minimal degree n over $\mathcal{C}$ and $\deg(x) = \infty$ otherwise. For a nonempty subset $\mathcal{M}$ of $\mathcal{A}$, we define $\deg(\mathcal{M}) = \sup\{\deg(y)\colon\ y \in \mathcal{M}\}$.

If $\mathcal{A}$ admits an involution '$*$', then an additive map $\Phi\colon\ \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is called a *Jordan $*$-derivation* if $\Phi(a^2) = \Phi(a)a^* + a\Phi(a)$ holds for all $a \in \mathcal{A}$. An additive map $\Psi\colon\ \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is called a *generalized Jordan $*$-derivation* if there exists a Jordan $*$-derivation $\Phi\colon\ \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ such that $\Psi(a^2) = \Psi(a)a^* + a\Phi(a)$ holds for all $a \in \mathcal{A}$. A Jordan $*$-derivation $\Phi\colon\ \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is said to be X-inner if there exists $x \in \mathcal{Q}_{ms}(\mathcal{A})$ such that $\Phi(a) = ax - xa^*$ for all $a \in \mathcal{A}$. In the present paper, a generalized Jordan $*$-derivation $\Psi\colon\ \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is called *generalized X-inner* if there exists $p \in \mathcal{Q}_{mr}(\mathcal{A})$ with $p\mathcal{A} \subseteq \mathcal{Q}_{ms}(\mathcal{A})$ and $q \in \mathcal{Q}_{ms}(\mathcal{A})$ such that $\Psi(a) = ap + qa^*$ for all $a \in \mathcal{A}$. An additive map $\mathcal{T}\colon\ \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ is called a right (or left) centralizer if $\mathcal{T}(ab) = a\mathcal{T}(b)$ (or $\mathcal{T}(ab) = \mathcal{T}(a)b$, respectively) for all $a, b \in \mathcal{A}$. Moreover, $\mathcal{T}\colon\ \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ is called a two sided centralizer if it is both a left as well as a right centralizer. If $\mathcal{A}$ is a semiprime ring with the extended centroid $\mathcal{C}$ and $\mathcal{T}\colon\ \mathcal{A} \to \mathcal{Q}_{mr}(\mathcal{A})$ is a two sided centralizer, then there exists an element $\lambda \in \mathcal{C}$ such that $\mathcal{T}(a) = \lambda a$ for all $a \in \mathcal{A}$, see [3], Theorem 2.3.2. An additive map $\mathcal{T}\colon\ \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ is called a right (or left) Jordan centralizer if $\mathcal{T}(a^2) = a\mathcal{T}(a)$ (or $\mathcal{T}(a^2) = \mathcal{T}(a)a$, respectively) for all $a \in \mathcal{A}$. If $\mathcal{A}$ is a semiprime ring with an anti-automorphism τ, then an additive map $\mathcal{T}\colon\ \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is called a Jordan left (or right) τ-centralizer if $\mathcal{T}(a^2) = \mathcal{T}(a)a^\tau$ (or $\mathcal{T}(a^2) = a^\tau\mathcal{T}(a)$, respectively) for all $a \in \mathcal{A}$. Vukman in [27] introduced the notion of an (m, n)-Jordan centralizer as follows: Let $\mathcal{A}$ be any ring and $m \geqslant 0$, $n \geqslant 0$ be fixed integers with $m + n \neq 0$. An additive map $\mathcal{T}\colon\ \mathcal{A} \to \mathcal{A}$ is called an (m, n)-Jordan centralizer if $(m + n)\mathcal{T}(a^2) = m\mathcal{T}(a)a + na\mathcal{T}(a)$ holds for all $a \in \mathcal{A}$. Fošner in [7] gave the notion of a generalized (m, n)-Jordan centralizer as follows: Let $\mathcal{A}$ be any ring and $m \geqslant 0$, $n \geqslant 0$ be fixed integers with $m + n \neq 0$. An additive map $\mathcal{G}\colon\ \mathcal{A} \to \mathcal{A}$ is called a generalized (m, n)-Jordan centralizer if there exists an (m, n)-Jordan centralizer $\mathcal{T}\colon\ \mathcal{A} \to \mathcal{A}$ such that $(m + n)\mathcal{G}(a^2) = m\mathcal{G}(a)a + na\mathcal{T}(a)$ holds for all $a \in \mathcal{A}$.

Motivated by these definitions, we introduce the notions of (m,n)-Jordan $*$-centralizer and generalized (m,n)-Jordan $*$-centralizer as follows:

Definition 1.1. Let $\mathcal{A}$ be a semiprime ring with an involution '$*$', and let $m \geqslant 0$, $n \geqslant 0$ be fixed integers not both zero. An additive map $\mathcal{T}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is called an (m,n)-Jordan $*$-centralizer if $(m+n)\mathcal{T}(a^2) = m\mathcal{T}(a)a^* + na\mathcal{T}(a)$ for all $a \in \mathcal{A}$. Moreover, an additive map $\mathcal{H}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is called a generalized (m,n)-Jordan $*$-centralizer if there exists an (m,n)-Jordan $*$-centralizer $\mathcal{T}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ such that $(m+n)\mathcal{H}(a^2) = m\mathcal{H}(a)a^* + na\mathcal{T}(a)$ for all $a \in \mathcal{A}$.

Clearly, $(1,0)$-Jordan $*$-centralizer is a left Jordan $*$-centralizer, $(0,1)$-Jordan $*$-centralizer is a right Jordan centralizer.

Herstein in [8] proved that every Jordan derivation of a prime ring of characteristic different from 2 is a derivation. Lee and Lin in [12], Theorem 1.2 proved that every Jordan derivation of a prime ring $\mathcal{A}$ can be written as a sum of a derivation and a central valued additive map vanishing on square form elements and vice-versa. Šemrl studied the Jordan $*$-derivations and quadratic forms making use of bilinear forms, see [21] and [20]. Lee et al. in [13], Theorem 2.12, [15] and [16] obtained that any Jordan $*$-derivation on a noncommutative prime ring $\mathcal{A}$ with an involution '$*$' is X-inner unless $\dim_{\mathcal{C}} \mathcal{AC} = 4$ and $\mathrm{char}(\mathcal{A}) = 2$. Vukman in [27], Theorem 4 first characterized (m,n)-Jordan centralizers in prime rings and proved that every (m,n)-Jordan centralizer on a prime ring $\mathcal{A}$ is a two sided centralizer provided $\mathcal{Z}(\mathcal{A})$ is nonzero and $\mathcal{A}$ is $6mn(m+n)$-torsion free. Fošner [7], proved that any generalized (m,n)-Jordan centralizer on a $6mn(m+n)(m+2n)$-torsion free prime ring $\mathcal{A}$ is a two sided centralizer provided that $\mathcal{Z}(\mathcal{A})$ is nonzero. Vukman in [26], Theorem 1 proved that if $\mathcal{A}$ is a 2-torsion free semiprime ring and $\mathcal{T}\colon \mathcal{A} \to \mathcal{A}$ is an additive map such that $\mathcal{T}(aba) = a\mathcal{T}(b)a$ for all $a, b \in \mathcal{A}$, then $\mathcal{T}$ is a centralizer. Various other authors have investigated similar maps within the contexts of prime and semiprime rings, see [4], [28], [7], [10], [17], [22], [24], [23], [25], [29] and references therein. Motivated by these results on derivations and centralizers, in this paper, we will characterize generalized (m,n)-Jordan $*$-centralizers in prime rings and additive maps $\mathcal{T}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ satisfying $\mathcal{T}(aba) = a\mathcal{T}(b)a^*$ for all $a, b \in \mathcal{A}$. Specifically, we will explore conditions under which a prime ring $\mathcal{A}$ does not admit a nonzero generalized (m,n)-Jordan $*$-centralizer and an additive map $\mathcal{T}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ that satisfies $\mathcal{T}(aba) = a\mathcal{T}(b)a^*$ for all $a, b \in \mathcal{A}$, see Theorems 2.1 and 2.2.

Zalar in [30], characterized Jordan right centralizers in semiprime rings and proved that if $\mathcal{A}$ is a 2-torsion free semiprime ring and $\mathcal{T}\colon \mathcal{A} \to \mathcal{A}$ is a right Jordan centralizer, then there is $q \in \mathcal{Q}_{ml}(\mathcal{A})$ such that $\mathcal{T}(a) = aq$ for all $a \in \mathcal{A}$. Lee and Wong in [14], Theorem 1.2 completely characterized right Jordan centralizers in semiprime rings and obtained that every Jordan right centralizer $\mathcal{T}$ on a semiprime ring $\mathcal{A}$ is of the form $\mathcal{T}(a) = aq$ for all $a \in \mathcal{A}$, where $q \in \mathcal{Q}_{ml}(\mathcal{A})$. Very recently, Abbasi et al. in [1] characterized Jordan left $*$-centralizers in prime $*$-rings satisfying some identities involving skew Lie and Jordan products, and with the condition that '$*$' is not identity on $\mathcal{Z}(\mathcal{A})$. We also characterize Jordan left τ-centralizers in semiprime rings, and as an immediate corollaries we characterize generalized Jordan $*$-derivations in prime rings and generalize as well as improve all the results of [1].

2. Results

We begin with the following results which play a pivotal role in the proof of our main results. The first result holds for any arbitrary ring $\mathcal{A}$, see [16], Lemma 2.3.

Lemma 2.1. *Let $\mathcal{H}\colon \mathcal{A} \times \mathcal{A} \to \mathcal{G}$ be a bi-additive map and let $f, g\colon \mathcal{A} \to \mathcal{G}$ be additive maps, where $\mathcal{G}$ is an additive group. Suppose that $\mathcal{H}(a,b) = f(ab) + g(ba)$ for all $a, b \in \mathcal{A}$. Then*

$$\mathcal{H}(ad, bc) - \mathcal{H}(a, dbc) = \mathcal{H}(cad, b) - \mathcal{H}(ca, db)$$

holds for all $a, b, c, d \in \mathcal{A}$.

Lemma 2.2. *Let $\mathcal{A}$ be a prime PI-ring with an anti-automorphism τ. Then τ is of the first kind if and only if $\alpha^\tau = \alpha$ for all $\alpha \in \mathcal{Z}(\mathcal{A})$.*

Proof. By [6], Theorem C.1, $\dim_{\mathcal{C}} \mathcal{AC} < \infty$. Therefore, $\mathcal{AC} = \mathcal{Q}_{ml}(\mathcal{A})$, $\mathcal{Z}(\mathcal{A}) \neq \{0\}$ and any element in $\mathcal{AC}$ is of the form a/α for some $a \in \mathcal{A}$ and some nonzero $\alpha \in \mathcal{Z}(\mathcal{A})$. First suppose that $\alpha^\tau = \alpha$ for all $\alpha \in \mathcal{Z}(\mathcal{A})$. Then, τ can be uniquely extended to an anti-automorphism of $\mathcal{AC}$, denoted also by τ, by defining $(a/\alpha)^\tau = a^\tau/\alpha$ for $a \in \mathcal{A}$ and $0 \neq \alpha \in \mathcal{Z}(\mathcal{A})$. Therefore, τ is of the first kind. The converse part holds trivially. $\square$

Lemma 2.3. *Let $\mathcal{A}$ be a noncommutative prime ring with an anti-automorphism τ and let $\mathcal{J}$ be a nonzero ideal of $\mathcal{A}$. Suppose that $q, q_1 \in \mathcal{Q}_{ml}(\mathcal{A})$ are such that $q_1 a^\tau = aq$ for all $a \in \mathcal{J}$. Then $q = q_1 = 0$.*

Proof. First assume that $q_1 = q$. Then by the given hypothesis, we have $baq = bqa^\tau = q(ab)^\tau = abq$, that is, $[a,b]q = 0$ for all $a, b \in \mathcal{J}$, from which it can be easily deduced that $q = 0$. Now suppose that $q_1 a^\tau = aq$ for all $a \in \mathcal{J}$. Then $(bq)a^\tau = q_1(ab)^\tau = a(bq)$ for all $a, b \in \mathcal{J}$. By the above, $bq = 0$ for all $b \in \mathcal{J}$. Hence, $q = 0$, which further gives $q_1 = 0$. $\square$

Lemma 2.4. *Let $\mathcal{A}$ be a noncommutative prime ring with an involution '$*$' and let $m \geqslant 1$, $n \geqslant 1$ be some fixed integers such that $\mathcal{A}$ is mn-torsion free. Suppose that $\mathcal{T}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is an (m,n)-Jordan $*$-centralizer. Then $\mathcal{T} = 0$ unless $\dim_{\mathcal{C}} \mathcal{AC} = 4$, $\mathcal{A}$ is not $(m+n)$-torsion free and '$*$' is of the first kind.*

Proof. By the hypothesis given, we have

$$(m+n)\mathcal{T}(a^2) = m\mathcal{T}(a)a^* + na\mathcal{T}(a) \tag{2.1}$$

for all $a \in \mathcal{A}$. Linearizing it, we get

$$(m+n)\mathcal{T}(ab+ba) = na\mathcal{T}(b) + nb\mathcal{T}(a) + m\mathcal{T}(a)b^* + m\mathcal{T}(b)a^* \tag{2.2}$$

for all $a, b \in \mathcal{A}$. We proceed by considering the following cases:

Case 1: $\mathcal{A}$ is $(m+n)$-torsion free.

Subcase 1.1: $\deg(\mathcal{A}) > 4$. Consider the bi-additive map $\Phi\colon \mathcal{A}^2 \to \mathcal{Q}_{ms}(\mathcal{A})$ given by

$$\Phi(a,b) = (m+n)\mathcal{T}(ab+ba) = (m+n)\mathcal{T}(ab) + (m+n)\mathcal{T}(ba). \tag{2.3}$$

By Lemma 2.1, it follows that Φ satisfies the relation

$$\Phi(a_1a_4, a_2a_3) - \Phi(a_1, a_4a_2a_3) = \Phi(a_3a_1a_4, a_2) - \Phi(a_3a_1, a_4a_2)$$

for all $\bar{a}_4 \in \mathcal{A}^4$. Using (2.2), we get the functonal identity

$$\sum_{i=1}^{4} a_i E_i(\bar{a}_4^i) + \sum_{j=1}^{4} H_j(\bar{a}_4^j)a_j^* = 0 \tag{2.4}$$

for all $\bar{a}_4 \in \mathcal{A}^4$, where

$$\begin{aligned} E_1(\bar{a}_4^1) &= na_4\mathcal{T}(a_2a_3) - n\mathcal{T}(a_4a_2a_3), \quad & E_2(\bar{a}_4^2) &= na_3\mathcal{T}(a_1a_4) - n\mathcal{T}(a_3a_1a_4),\\ E_3(\bar{a}_4^3) &= na_1\mathcal{T}(a_4a_2) - na_1a_4\mathcal{T}(a_2), \quad & E_4(\bar{a}_4^4) &= na_2\mathcal{T}(a_3a_1) - na_2a_3\mathcal{T}(a_1), \end{aligned}$$

and

$$\begin{aligned} H_1(\bar{a}_4^1) &= m\mathcal{T}(a_2a_3)a_4^* - m\mathcal{T}(a_4a_2a_3), \quad & H_2(\bar{a}_4^2) &= m\mathcal{T}(a_1a_4)a_3^* - m\mathcal{T}(a_3a_1a_4),\\ H_3(\bar{a}_4^3) &= m\mathcal{T}(a_4a_2)a_1^* - m\mathcal{T}(a_2)a_4^*a_1^*, \quad & H_4(\bar{a}_4^4) &= m\mathcal{T}(a_3a_1)a_2^* - m\mathcal{T}(a_1)a_3^*a_2^*. \end{aligned}$$

To prove our result we focus on the solutions of E_3. In view of [2], Theorem 3.5, there exist bi-additive maps f_1, f_2 and $f_3\colon \mathcal{A}^2 \to \mathcal{Q}_{ml}(\mathcal{A})$ such that

$$a_1\{n\mathcal{T}(a_4a_2) - na_4\mathcal{T}(a_2)\} = \sum_{\substack{i=1\\ i\neq 3}}^{4} f_i(\bar{a}_4^{3i})a_i^*$$

for all $\bar{a}_4 \in \mathcal{A}^4$. Again by [2], Theorem 3.5, we infer that there exist unique additive maps g' and $h'\colon \mathcal{A} \to \mathcal{Q}_{ml}(\mathcal{A})$ such that for all $a, b \in \mathcal{A}$, we have

$$n\mathcal{T}(ab) - na\mathcal{T}(b) = g'(a)b^* + h'(b)a^*. \tag{2.5}$$

Since $\mathcal{A}$ is n-torsion free, $n1$ is invertible in $\mathcal{C}$, where 1 denotes the unity of $\mathcal{Q}_{ml}(\mathcal{A})$. Therefore, from (2.5), we have

$$\mathcal{T}(ab) = a\mathcal{T}(b) + g(a)b^* + h(b)a^*, \tag{2.6}$$

where g and h are also additive maps from $\mathcal{A}$ to $\mathcal{Q}_{ml}(\mathcal{A})$ given by $g(a) = g'(a)/(n1)$ and $f(a) = f'(a)/(n1)$. Hence, we have

$$\begin{aligned}(m+n)\mathcal{T}(ab+ba) = (m+n)a\mathcal{T}(b) + (m+n)b\mathcal{T}(a) + (m+n)g(a)b^* \\ + (m+n)g(b)a^* + (m+n)h(a)b^* + (m+n)h(b)a^*\end{aligned} \tag{2.7}$$

for all $a, b \in \mathcal{A}$. From (2.2) and (2.7), we find that

$$\begin{aligned}ma\mathcal{T}(b) + mb\mathcal{T}(a) + ((m+n)g(a) + (m+n)h(a) - m\mathcal{T}(a))b^* \\ +((m+n)g(b) + (m+n)h(b) - m\mathcal{T}(b))a^* = 0\end{aligned} \tag{2.8}$$

for all $a, b \in \mathcal{A}$. According to [2], Theorem 3.5, there exists $q_1 \in \mathcal{Q}_{ml}(\mathcal{A})$ such that $m\mathcal{T}(a) = a^*q_1$ for all $a \in \mathcal{A}$. Therefore, $\mathcal{T}(a) = q'a^*$ for all $a \in \mathcal{A}$, where $q' = q_1/(m1)$. Using this in (2.2), we find that

$$(q'a^* - aq')b^* + (q'b^* - bq')a^* = 0$$

for all $a, b \in \mathcal{A}$. By [2], Theorem 3.3, we get $aq' = q'a^*$ for all $a \in \mathcal{A}$. Applying Lemma 2.3, we deduce that $q' = 0$ and hence $\mathcal{T} = 0$.

Subcase 1.2: $\deg(\mathcal{A}) \leqslant 4$. In this case by [19], $\mathcal{Z}(\mathcal{A}) \neq \{0\}$ and $\mathcal{Q}_{ms}(\mathcal{A}) = \mathcal{A}\mathcal{C}$. Let α be a nonzero fixed element of $\mathcal{Z}(\mathcal{A})$ such that $\alpha^* = \alpha$. Then setting $a = \alpha$ in (2.2), we get $\mathcal{T}(\alpha^2) = \alpha\mathcal{T}(\alpha)$. Therefore, using (2.2), we have

$$(m+n)\mathcal{T}(\alpha^2 a + a\alpha^2) = m\alpha^2\mathcal{T}(a) + n\alpha a\mathcal{T}(\alpha) + m\alpha\mathcal{T}(\alpha)a^* + n\alpha^2\mathcal{T}(a)$$

for all $a \in \mathcal{A}$. On the other hand, we have

$$\begin{aligned}(m+n)\mathcal{T}(\alpha^2 a + a\alpha^2) &= (m+n)\mathcal{T}(\alpha(\alpha a) + (\alpha a)\alpha) \\ &= n\alpha\mathcal{T}(\alpha a) + n\alpha a\mathcal{T}(\alpha) + m\alpha\mathcal{T}(\alpha)a^* + m\alpha\mathcal{T}(\alpha a)\end{aligned} \tag{2.9}$$

for all $a \in \mathcal{A}$. Comparing the last two relations, we conclude that $\mathcal{T}(\alpha a) = \alpha\mathcal{T}(a)$ for all $a \in \mathcal{A}$. Now substituting α for b in (2.2), we get

$$\mathcal{T}(a) = qa^* + ap$$

for all $a \in \mathcal{A}$, where $q = m\mathcal{T}(\alpha)/((m+n)\alpha) \in \mathcal{AC}$ and $p = n\mathcal{T}(\alpha)/((m+n)\alpha) \in \mathcal{AC}$. Utilizing this in (2.1), we find that $ma(ap - pa^*) + n(qa^* - aq)a^* = 0$ for all $a \in \mathcal{A}$. Now $mp = nq$. Thus, from the previous expression, we get ${}_*[a, {}_*[a, q]] = 0$ for all $a \in \mathcal{A}$, which is a $*$-GPI and hence it holds for all $a \in \mathcal{AC}$. Applying [18], Theorem 2.5, we infer that $q = 0$. Consequently, $\mathcal{T} = 0$.

Case 2: $\mathcal{A}$ is not $(m+n)$-torsion free. In this case, it can be easily seen that $(m+n)1 = 0$. Therefore, from (2.2), we have

$$na\mathcal{T}(b) + nb\mathcal{T}(a) + m\mathcal{T}(a)b^* + m\mathcal{T}(b)a^* = 0 \tag{2.10}$$

for all $a, b \in \mathcal{A}$.

Subcase 2.1: $\deg(\mathcal{A}) > 2$. According to [2], Theorem 3.5, there exists $q_1 \in \mathcal{Q}_{ml}(\mathcal{A})$ such that $m\mathcal{T}(a) = aq_1$, that is, $\mathcal{T}(a) = aq$ for all $a \in \mathcal{A}$, where $q = q_1/(m1)$. On the other hand there exists $q_2 \in \mathcal{Q}_{ml}(\mathcal{A})$ such that $n\mathcal{T}(a) = q_2a^*$, that is, $\mathcal{T}(a) = q'a^*$ for all $a \in \mathcal{A}$, where $q' = q_2/(n1)$. Therefore, $aq = q'a^*$ for all $a \in \mathcal{A}$. Invoking Lemma 2.3, we conclude that $q = 0$.

Subcase 2.2: $\deg(\mathcal{A}) = 2$ and '$*$' is of the second kind. By Lemma 2.2, there exists $\alpha \in \mathcal{Z}(\mathcal{A})$ such that $\alpha^* \neq \alpha$. Now, replacing b by αb in (2.10), we get

$$na\mathcal{T}(\alpha b) + n\alpha b\mathcal{T}(a) + m\alpha^*\mathcal{T}(a)b^* + m\mathcal{T}(\alpha b)a^* = 0 \tag{2.11}$$

for all $a, b \in \mathcal{A}$. Also from (2.11), we get

$$n\alpha a\mathcal{T}(b) + n\alpha b\mathcal{T}(a) + m\alpha\mathcal{T}(a)b^* + m\alpha\mathcal{T}(b)a^* = 0 \tag{2.12}$$

for all $a, b \in \mathcal{A}$. Subtracting (2.12) from (2.11), we get

$$na(\mathcal{T}(\alpha b) - \alpha\mathcal{T}(b)) + m(\alpha^* - \alpha)\mathcal{T}(a)b^* + m(\mathcal{T}(\alpha b) - \alpha\mathcal{T}(b))a^* = 0 \tag{2.13}$$

for all $a, b \in \mathcal{A}$. Now, substituting αa in place of a in (2.13) and using it again, we find that

$$(\mathcal{T}(\alpha a^*) - \alpha\mathcal{T}(a^*))b + (\mathcal{T}(\alpha b^*) - \alpha\mathcal{T}(b^*))a = 0$$

for all $a, b \in \mathcal{A}$. In view of [2], Theorem 2.4, it follows that $\mathcal{T}(\alpha a) = \alpha\mathcal{T}(a)$ for all $a \in \mathcal{A}$. Hence, from (2.11), we have

$$n\alpha a\mathcal{T}(b) + n\alpha b\mathcal{T}(a) + m\alpha^*\mathcal{T}(a)b^* + m\alpha\mathcal{T}(b)a^* = 0 \tag{2.14}$$

for all $a, b \in \mathcal{A}$. From (2.12) and (2.14), we obtain $\mathcal{T}(a)b = 0$ for all $a, b \in \mathcal{A}$. Consequently, $\mathcal{T} = 0$ and this completes the proof. □

Corollary 2.1. *Let $\mathcal{A}$ be a noncommutative prime PI-ring with an anti-automorphism τ and $m \geqslant 1$, $n \geqslant 1$ be some fixed integers such that $\mathcal{A}$ is mn-torsion free. Suppose $\alpha^\tau = \alpha$ for all $\alpha \in \mathcal{Z}(\mathcal{A})$ and let $\mathcal{T}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ be an additive map such that*

$$(m+n)\mathcal{T}(a^2) = m\mathcal{T}(a)a^\tau + na\mathcal{T}(a) \tag{2.15}$$

for all $a \in \mathcal{A}$. Then $\mathcal{T} = 0$ unless $\dim_{\mathcal{C}} \mathcal{AC} = 4$ and $\mathcal{A}$ is not $(m+n)$-torsion free.

Proof. Linearizing (2.15), we have

$$(m+n)\mathcal{T}(ab+ba) = m\mathcal{T}(a)b^\tau + m\mathcal{T}(b)a^\tau + na\mathcal{T}(b) + nb\mathcal{T}(a) \tag{2.16}$$

for all $a, b \in \mathcal{A}$. Let $\alpha \in \mathcal{Z}(\mathcal{A})$. Then taking $a = \alpha$ in (2.15), we obtain $\mathcal{T}(\alpha^2) = \alpha\mathcal{T}(\alpha)$. Therefore, using (2.16), we have

$$\begin{aligned}(m+n)\mathcal{T}(\alpha^2 a + a\alpha^2) &= (m+n)(\mathcal{T}((\alpha a)\alpha + \alpha(\alpha a)))\\ &= n\alpha\mathcal{T}(\alpha a) + m\alpha\mathcal{T}(\alpha)a^\tau + n\alpha a\mathcal{T}(\alpha) + m\alpha\mathcal{T}(\alpha a)\end{aligned}$$

for all $a \in \mathcal{A}$. On the other hand, we have

$$(m+n)\mathcal{T}(\alpha^2 a + a\alpha^2) = m\alpha\mathcal{T}(\alpha)a^\tau + m\alpha^2\mathcal{T}(a) + n\alpha^2\mathcal{T}(a) + n\alpha a\mathcal{T}(a)$$

for all $a \in \mathcal{A}$. Comparing the last two relations, we infer that $\mathcal{T}(\alpha a) = \alpha\mathcal{T}(a)$ for all $a \in \mathcal{A}$. Thus, $\mathcal{T}$ is a $\mathcal{Z}(\mathcal{A})$-linear map.

Now, it is well known that if $\mathcal{A}$ is a prime PI-ring, then $\mathcal{AC} = \mathcal{Q}_{ml}(\mathcal{A})$, $\mathcal{Z}(\mathcal{A}) \neq \{0\}$ and any element in $\mathcal{AC}$ is of the form a/α, for some $a \in \mathcal{A}$ and some nonzero $\alpha \in \mathcal{Z}(\mathcal{A})$. Clearly, τ can be uniquely extended to an anti-automorphism of $\mathcal{AC}$, denoted also by τ, by defining $(a/\alpha)^\tau = a^\tau/\alpha$ for $a \in \mathcal{A}$ and $0 \neq \alpha \in \mathcal{Z}(\mathcal{A})$. Moreover, $\mathcal{T}$ can also be uniquely extended to a map $\widetilde{\mathcal{T}}\colon \mathcal{AC} \to \mathcal{AC}$, by defining $\widetilde{\mathcal{T}}(a/\alpha) = \mathcal{T}(a)/\alpha$. A simple computation shows that $\widetilde{\mathcal{T}}$ is well-defined and

$$(m+n)\widetilde{\mathcal{T}}(a^2) = m\widetilde{\mathcal{T}}(a)a^\tau + na\widetilde{\mathcal{T}}(a)$$

for all $a \in \mathcal{AC}$. Now $\mathcal{A}$ is a prime PI-ring, so $\mathcal{AC}$ is a central simple $\mathcal{C}$- algebra and is also finite dimensional over $\mathcal{C}$. Therefore, by [13], Proposition 2.11, it follows that there exists an invertible element $x \in \mathcal{AC}$ such that for $a \in \mathcal{AC}$, the map $a \mapsto x^{-1}a^\tau x$ is an involution on $\mathcal{AC}$. Let '$*$' denotes such an involution, that is, for all $a \in \mathcal{AC}$, $a^* = x^{-1}a^\tau x$. Then

$$(m+n)\widetilde{\mathcal{T}}(a^2)x = m\widetilde{\mathcal{T}}(a)xa^* + na\widetilde{\mathcal{T}}(a)x$$

for all $a \in \mathcal{AC}$. Therefore, the map $a \mapsto \widetilde{\mathcal{T}}(a)x$ is an (m,n)-Jordan $*$-centralizer. Hence, by Lemma 2.4, $\widetilde{\mathcal{T}}(a)x = 0$ for all $a \in \mathcal{AC}$, which yields $\widetilde{\mathcal{T}} = 0$. Consequently, $\mathcal{T} = 0$. $\square$

The following result gives a characterization of Jordan left τ-centralizers in semiprime rings.

Proposition 2.1. *Let $\mathcal{A}$ be a semiprime ring with an anti-automorphism τ. Then a map $\mathcal{F}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is a Jordan left τ-centralizer if and only if there exists $q \in \mathcal{Q}_{mr}(\mathcal{A})$ such that $q\mathcal{A} \subseteq \mathcal{Q}_{ms}(\mathcal{A})$ and $\mathcal{F}(a) = qa^{\tau}$ for all $a \in \mathcal{A}$. Moreover, if $\mathcal{A}$ contains an element which is invertible in $\mathcal{Q}_{ms}(\mathcal{A})$, then $q \in \mathcal{Q}_{ms}(\mathcal{A})$.*

P r o o f. According to the given hypothesis, $\mathcal{F}(a^2) = \mathcal{F}(a)a^{\tau}$ for all $a \in \mathcal{A}$. Replacing a by $a^{\tau^{-1}}$ in the last relation, we have $\mathcal{F}((a^2)^{\tau^{-1}}) = \mathcal{F}(a^{\tau^{-1}})a$ for all $a \in \mathcal{A}$. Now define the map $\mathcal{H}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ by the rule $\mathcal{H}(a) = \mathcal{F}(a^{\tau^{-1}})$. Then it is easy to see that $\mathcal{H}(a^2) = \mathcal{H}(a)a$ for all $a \in \mathcal{A}$. Therefore, $\mathcal{H}$ is a left Jordan centralizer. Hence, in view of [14], Theorem 1.2, it follows that there exists $q \in \mathcal{Q}_{mr}(\mathcal{A})$ such that $\mathcal{H}(a) = qa$ for all $a \in \mathcal{A}$. Consequently, $\mathcal{F}(a) = qa^{\tau}$ for all $a \in \mathcal{A}$. Conversely suppose that there exists $q \in \mathcal{Q}_{mr}(\mathcal{A})$ such that $q\mathcal{A} \subseteq \mathcal{Q}_{ms}(\mathcal{A})$. Then it can be easily seen that the map $\mathcal{F}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ given by $\mathcal{F}(a) = qa^{\tau}$ is a Jordan left τ-centralizer.

Now, let $x \in \mathcal{A}$ be such that $x^{-1} \in \mathcal{Q}_{ms}(\mathcal{A})$. Then, it is obvious to see that $qx \in \mathcal{Q}_{ms}(\mathcal{A})$ implies that $q \in \mathcal{Q}_{ms}(\mathcal{A})$. □

The following result gives a characterization of generalized Jordan $*$-derivations in prime rings.

Corollary 2.2. *Let $\mathcal{A}$ be a noncommutative prime ring with an involution '$*$'. Then any generalized Jordan $*$-derivation of $\mathcal{A}$ is generalized X-inner except when char$(\mathcal{A}) = 2$ and $\dim_{\mathcal{C}} \mathcal{AC} = 4$.*

P r o o f. Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ be the generalized Jordan $*$-derivation with $d\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ as associated Jordan $*$-derivation. Define the map $\mathcal{G}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ by $\mathcal{G}(a) = (\mathcal{F} - d)(a)$. Then for all $a \in \mathcal{A}$, we have

$$\mathcal{G}(a^2) = \mathcal{F}(a^2) - d(a^2) = (\mathcal{F}(a)a^* - ad(a)) - (d(a)a^* - ad(a)) = (\mathcal{F} - d)(a)a^* = \mathcal{G}(a)a^*.$$

Thus, $\mathcal{G}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is a Jordan left $*$-centralizer. Hence, in view of Proposition 2.1, we infer that there exists $q_1 \in \mathcal{Q}_{mr}(\mathcal{A})$ with $q_1\mathcal{A} \subseteq \mathcal{Q}_{ms}(\mathcal{A})$ such that $\mathcal{G}(a) = q_1 a^*$ for all $a \in \mathcal{A}$. Therefore, $\mathcal{F}(a) = d(a) + q_1 a^*$ for all $a \in \mathcal{A}$. Also by [13], Theorem 2.12, it follows that we can find $q \in \mathcal{Q}_{ms}(\mathcal{A})$ such that $d(a) = qa^* - aq$ holds for all $a \in \mathcal{A}$. Hence, $\mathcal{F}(a) = (q_1 + q)a^* - aq = pa^* - aq$, where $p = q_1 + q \in \mathcal{Q}_{mr}(\mathcal{A})$ and $p\mathcal{A} \subseteq \mathcal{Q}_{ms}(\mathcal{A})$. □

Now, let $\mathcal{A}$ be any ring with an (anti)-automorphism σ. For $x, y \in \mathcal{A}$ and k being any nonnegative integer, we define ${}_{\sigma}[x, y]_k$ inductively as: ${}_{\sigma}[x, y]_0 = y$, ${}_{\sigma}[x, y]_1 = xy - yx^{\sigma}$ and ${}_{\sigma}[x, y]_k = {}_{\sigma}[x, {}_{\sigma}[x, y]_{k-1}]_1$. Similarly, ${}_{\sigma}x \circ_k y$ is defined as: ${}_{\sigma}x \circ_0 y = y$,

${}_{\sigma}x \circ {}_{1}y = xy + yx^{\sigma}$ and ${}_{\sigma}x \circ {}_{k}y = {}_{\sigma}x \circ {}_{1}({}_{\sigma}x \circ {}_{k-1}y)$. Now ${}_{\sigma}[x, y]_k$ can be viewed as $(L_x - R_{x^{\sigma}})^k(y)$, where R_x and $L_{x^{\sigma}}$ are the right and left translations induced by x and x^{σ} on y, respectively. Since L_x and $R_{x^{\sigma}}$ commute, therefore we have the following remark.

Remark 2.1. We have

$$
{}_{\sigma}[x, y]_k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} x^i y (x^{\sigma})^{k-i}.
$$

Similarly,

$$
{}_{\sigma}x \circ {}_{k}y = \sum_{i=0}^{k} \binom{k}{i} x^i y (x^{\sigma})^{k-i}.
$$

The following corollary generalizes and improves (removal of torsion restriction) all the results of [1] to any anti-automorphism τ and any positive integer k.

Corollary 2.3. *Let $k \geqslant 1$ be a fixed integer and let $\mathcal{A}$ be a prime ring with an anti-automorphism τ such that $\alpha^{\tau} \neq \alpha$ for some $\alpha \in \mathcal{Z}(\mathcal{A})$. Suppose that $\mathcal{F} : \mathcal{A} \to \mathcal{Q}_{mr}(\mathcal{A})$ is a Jordan left τ-centralizer such that any one among the following conditions holds:*

(i) ${}_{\tau}[a, \mathcal{F}(a)]_k \in \mathcal{C}$ *for all* $a \in \mathcal{A}$*;*

(ii) $\mathcal{F}({}_{\tau}[a, a^{\tau}]_k) \in \mathcal{C}$ *for all* $a \in \mathcal{A}$*;*

(iii) ${}_{\tau}a \circ_k \mathcal{F}(a) \in \mathcal{C}$ *for all* $a \in \mathcal{A}$*;*

(iv) $\mathcal{F}({}_{\tau}x \circ_k x^{\tau}) \in \mathcal{C}$ *for all* $a \in \mathcal{A}$*.*

Then there exists $\lambda \in \mathcal{C}$ such that $\mathcal{F}(a) = \lambda a^{\tau}$ for all $a \in \mathcal{A}$.

Proof. (i) Suppose ${}_{\tau}[a, \mathcal{F}(a)]_k \in \mathcal{C}$ for all $a \in \mathcal{A}$. Then by Remark 2.1,

$$
\sum_{i=0}^{k} (-1)^i \binom{k}{i} a^i \mathcal{F}(a) (a^{\tau})^{k-i} \in \mathcal{C}
$$

for all $a \in \mathcal{A}$. Now $0 \neq \alpha \in \mathcal{Z}(\mathcal{A})$ and $\mathcal{A}$ is prime, so $\alpha^{-1} \in \mathcal{C}$. Hence, by Proposition 2.1, there exists $\lambda \in \mathcal{Q}_{ms}(\mathcal{A})$ such that $\mathcal{F}(a) = \lambda a^{\tau}$ for all $a \in \mathcal{A}$. Therefore,

$$
\sum_{i=0}^{k} (-1)^i \binom{k}{i} a^i \lambda a^{\tau} (a^{\tau})^{k-i} \in \mathcal{C}
$$

for all $a \in \mathcal{A}$. Now setting $a = \alpha$, we get $\lambda \alpha^{\tau} (\alpha^{\tau} - \alpha)^k \in \mathcal{C}$. Consequently, $\lambda \in \mathcal{C}$.

(ii) Proceeding in the same way as in (i) we get the desired result.

(iii) By Proposition 2.1, there is $\lambda \in \mathcal{Q}_{ms}(\mathcal{A})$ such that $\mathcal{F}(a) = \lambda a^\tau$ for all $a \in \mathcal{A}$. Hence, using Remark 2.1, we find that

$$\sum_{i=0}^{k} \binom{k}{i} a^i \lambda a^\tau (a^\tau)^{k-i} \in \mathcal{C} \tag{2.17}$$

for all $a \in \mathcal{A}$. Now putting $a = \alpha$ in (2.17), we get $\lambda \alpha^\tau (\alpha^\tau + \alpha)^k \in \mathcal{C}$. Consequently, $\lambda \in \mathcal{C}$ unless $\alpha^\tau = -\alpha$. So, henceforth, we assume that $\operatorname{char}(\mathcal{A}) \neq 2$ and $\alpha^\tau = -\alpha$. Substituting αa for a in (2.17), we get

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} a^i \lambda a^\tau (a^\tau)^{k-i} \in \mathcal{C}. \tag{2.18}$$

From (2.17) and (2.18), we find that

$$\sum_{\substack{i=0\\ i\text{even}}}^{k} \binom{k}{i} a^i \lambda a^\tau (a^\tau)^{k-i} \in \mathcal{C}$$

for all $a \in \mathcal{A}$. Finally, putting $a = \alpha$ in the last relation, we get $2^{k-1}\lambda \in \mathcal{C}$. Consequently, $\lambda \in \mathcal{C}$.

(iv) Proceed in the same way as in (iii). □

One may observe that the last result also holds when τ is an automorphism. The following result provides a characterization of generalized (m,n)-Jordan $*$-centralizers in prime rings.

Theorem 2.1. *Let $\mathcal{A}$ be a noncommutative prime ring with an involution '$*$' and let $m \geqslant 1$, $n \geqslant 1$ be some fixed integers such that $\mathcal{A}$ is mn-torsion free. Suppose that $\mathcal{H}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is a generalized (m,n)-Jordan $*$-centralizer with $\mathcal{T}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ as associated (m,n)-Jordan $*$-centralizer. Then $\mathcal{H} = 0$ unless $\dim_{\mathcal{C}} \mathcal{AC} = 4$, $\mathcal{A}$ is not $(m+n)$-torsion free and '$*$' is of the first kind.*

Proof. According to Lemma 2.4, $\mathcal{T} = 0$. Therefore, for all $a, b \in \mathcal{A}$, we have

$$(m+n)\mathcal{H}(a^2) = m\mathcal{H}(a)a^* \tag{2.19}$$

and

$$(m+n)\mathcal{H}(ab+ba) = m\mathcal{H}(a)b^* + m\mathcal{H}(b)a^*. \tag{2.20}$$

We divide the proof in two cases.

Case 1: $\deg(\mathcal{A}) \leqslant 4$. In this case choose $0 \neq \alpha^* = \alpha \in \mathcal{Z}(\mathcal{A})$. Putting αa in place of b in (2.20), we have

$$2(m+n)\mathcal{H}(\alpha a^2) = m\mathcal{H}(\alpha a)a^* + m\alpha\mathcal{H}(a)a^* \tag{2.21}$$

for all $a \in \mathcal{A}$. Next substituting a^2 for a and α for b in (2.20), we have

$$2(m+n)\mathcal{H}(\alpha a^2) = m\alpha\mathcal{H}(a^2) + m\mathcal{H}(\alpha)(a^*)^2 \tag{2.22}$$

for all $a \in \mathcal{A}$. Since $\mathcal{A}$ is m-torsion free, comparing (2.21) and (2.22), we find that

$$\mathcal{H}(\alpha a)a^* + \alpha\mathcal{H}(a)a^* = \alpha\mathcal{H}(a^2) + \mathcal{H}(\alpha)(a^*)^2$$

for all $a \in \mathcal{A}$. Therefore,

$$2(m+n)\mathcal{H}(\alpha a)a^* + 2(m+n)\alpha\mathcal{H}(a)a^* = 2(m+n)\alpha\mathcal{H}(a^2) + 2(m+n)\mathcal{H}(\alpha)(a^*)^2$$

for all $a \in \mathcal{A}$. Now using (2.19), we see that

$$-2n\alpha\mathcal{H}(a)a^* = 2(m+n)(\mathcal{H}(\alpha a) - \mathcal{H}(\alpha)a^*)a^* \tag{2.23}$$

for all $a \in \mathcal{A}$. Also putting $b = \alpha$ in (2.20), we get

$$2(m+n)\mathcal{H}(\alpha a) = m\alpha\mathcal{H}(a) + m\mathcal{H}(\alpha)a^*$$

for all $a \in \mathcal{A}$. Therefore, from (2.23), we have

$$(2n+m)\alpha\mathcal{H}(a)a^* = (2n+m)\mathcal{H}(\alpha)(a^*)^2$$

for all $a \in \mathcal{A}$. If $\mathcal{A}$ is $(2n+m)$-torsion free, then replacing a by $a+\alpha$ in the last relation, we find that $\mathcal{H}(a) = qa^*$ for all $a \in \mathcal{A}$, where $q = \alpha^{-1}\mathcal{H}(\alpha)$. Utilizing this in (2.19), we infer that $q = 0$. Consequently, $\mathcal{H} = 0$.

Now, assume that $\mathcal{A}$ is not $(2n+m)$-torsion free. In this situation, from (2.19) we have

$$\mathcal{G}(a^2) = 2\mathcal{G}(a)a \tag{2.24}$$

for all $a \in \mathcal{A}$, where $\mathcal{G}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ is a map given by $\mathcal{G}(a) = \mathcal{H}(a^*)$. Linearizing (2.24), we get

$$\mathcal{G}(ab+ba) = 2\mathcal{G}(a)b + 2\mathcal{G}(b)a \tag{2.25}$$

for all $a, b \in \mathcal{A}$. Let $\beta \in \mathcal{Z}(\mathcal{A})$. Then from (2.25), we have

$$\mathcal{G}(\beta a) = \beta\mathcal{G}(a) + \mathcal{G}(\beta)a \tag{2.26}$$

for all $a \in \mathcal{A}$. Therefore, putting βa in the place of a in (2.25), we get

$$\mathcal{G}(\beta)(ab+ba)+\beta\mathcal{G}(ab+ba)=2\beta\mathcal{G}(a)b+2\mathcal{G}(\beta)ba+2\beta\mathcal{G}(b)a \tag{2.27}$$

for all $a, b \in \mathcal{A}$. Hence, $\mathcal{G}(\beta)[a,b]=0$ for all $a, b \in \mathcal{A}$, which further entails $\mathcal{G}(\beta)=0$. As a result in view of (2.26), we conclude that $\mathcal{G}$ is $\mathcal{Z}(\mathcal{A})$-linear.

Now $\mathcal{A}$ is a PI-ring. Hence, $\mathcal{Q}_{ml}(\mathcal{A})=\mathcal{AC}$ is a finite dimensional central simple algebra and any element of $\mathcal{AC}$ is of the form a/β, where $a \in \mathcal{A}$ and $\beta \in \mathcal{Z}(\mathcal{A})-\{0\}$. Moreover, $\mathcal{G}$ can be uniquely extended to a map $\overline{\mathcal{G}}\colon \mathcal{AC}\to\mathcal{AC}$ by defining $\overline{\mathcal{G}}(a/\beta)=\mathcal{G}(a)/\beta$. A simple computation shows that $\overline{\mathcal{G}}\colon \mathcal{AC}\to\mathcal{AC}$ is a well-defined, $\mathcal{C}$-linear map satisfying $\overline{\mathcal{G}}(a^2)=2\overline{\mathcal{G}}(a)a$ for all $a \in \mathcal{A}$. By the Wedderburn-Artin theorem, $\mathcal{AC}\cong\mathcal{M}_k(\mathbb{D})$ for some division ring $\mathbb{D}$ and $k \leqslant n=\sqrt{\dim_{\mathcal{C}}\mathcal{AC}}$. Let $\mathbb{K}$ be a separable maximal subfield of $\mathbb{D}$. Then $\mathcal{AC}\otimes_{\mathcal{C}}\mathbb{K}\cong\mathcal{M}_n(\mathbb{K})$ and the map $x\in\mathcal{AC}\mapsto x\otimes 1\in\mathcal{AC}\otimes\mathbb{K}$ gives a ring embedding. By the universal property of tensor product, there exists a unique additive map $\widetilde{\mathcal{G}}$ from $\mathcal{AC}\otimes\mathbb{K}$ into itself which maps $a\otimes\nu$ to $\overline{\mathcal{G}}(a)\otimes\nu$ for all $a\in\mathcal{AC}$ and $\nu\in\mathbb{K}$. Since $\mathcal{AC}\otimes_{\mathcal{C}}\mathbb{K}\cong\mathcal{M}_n(\mathbb{K})$, where $n=\sqrt{\dim_{\mathcal{C}}\mathcal{AC}}>1$, we can view $\widetilde{\mathcal{G}}$ as a $\mathbb{K}$-linear map from $\mathcal{M}_n(\mathbb{K})$ to itself satisfying

$$\widetilde{\mathcal{G}}(a^2)=2\widetilde{\mathcal{G}}(a)a \tag{2.28}$$

and

$$\widetilde{\mathcal{G}}(ab+ba)=2\widetilde{\mathcal{G}}(a)b+2\widetilde{\mathcal{G}}(b)a. \tag{2.29}$$

Let e_{ij} be the standard matrix unit. Then setting $a=e_{ii}$ in (2.28), it follows that $\widetilde{\mathcal{G}}(e_{ii})=0$. Next putting $a=e_{ii}$ and $b=e_{ij}$ in (2.29), we infer that $\widetilde{\mathcal{G}}(e_{ij})=0$ for $i\neq j$. Therefore, $\widetilde{\mathcal{G}}(a)=0$ for all $a\in\mathcal{AC}\otimes_{\mathcal{C}}\mathbb{K}$, with the help of which it can be easily deduced that $\mathcal{H}=0$.

Case 2: $\deg(\mathcal{A})>4$. Consider the bi-additive map $\Phi\colon \mathcal{A}^2\to\mathcal{Q}_{ms}(\mathcal{A})$ given by

$$\Phi(a,b)=(m+n)\mathcal{H}(ab+ba)=(m+n)\mathcal{H}(ab)+(m+n)\mathcal{H}(ba). \tag{2.30}$$

By Lemma 2.1, it follows that Φ satisfies the relation

$$\Phi(a_1a_4,a_2a_3)-\Phi(a_1,a_4a_2a_3)=\Phi(a_3a_1a_4,a_2)-\Phi(a_3a_1,a_4a_2)$$

for all $\bar{a}_4\in\mathcal{A}^4$. Using (2.20), we get the relation

$$\begin{aligned}(\mathcal{H}(a_2a_3)a_4^*-\mathcal{H}(a_4a_2a_3))a_1^*+(\mathcal{H}(a_1a_4)a_3^*-\mathcal{H}(a_3a_1a_4))a_2^*\\+(\mathcal{H}(a_4a_2)a_1^*-\mathcal{H}(a_2)a_4^*a_1^*)a_3^*+(\mathcal{H}(a_3a_1)a_2^*-\mathcal{H}(a_1)a_3^*a_2^*)a_4^*=0\end{aligned}$$

for all $\bar{a}_4 \in \mathcal{A}^4$. Applying [2], Theorem 3.3, we see that

$$\mathcal{H}(ab) = \mathcal{H}(b)a^*$$

for all $a, b \in \mathcal{A}$. Thus, $\mathcal{H}$ is a Jordan left $*$-centralizer. According to Proposition 2.1, there exists $q \in \mathcal{Q}_{mr}(\mathcal{A})$ such that $\mathcal{H}(a) = qa^*$ for all $a \in \mathcal{A}$. Hence, from (2.20), we have $q(ab + ba) = 0$ for all $a, b \in \mathcal{A}$. Consequently, $q = 0$ and this completes the proof. $\square$

In particular, when $n = m = 1$, we have the following result.

Corollary 2.4. *Let $\mathcal{A}$ be a noncommutative prime ring with an involution '$*$'. Assume that $\mathcal{T}, \mathcal{G}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ are additive maps such that $2\mathcal{T}(a^2) = \mathcal{T}(a)a^* + a\mathcal{T}(a)$ and $2\mathcal{G}(a^2) = \mathcal{G}(a)a^* + a\mathcal{T}(a)$ holds for all $a \in \mathcal{A}$. Then $\mathcal{G} = 0$ unless char$(\mathcal{A}) = 2$, $\dim_{\mathcal{C}} \mathcal{AC} = 4$ and '$*$' is of the first kind.*

The following example illustrates that Theorem 2.1 does not hold if $\dim_{\mathcal{C}} \mathcal{AC} = 4$, $\mathcal{A}$ is not $(m + n)$-torsion free and '$*$' is of the first kind.

Example 2.1. Consider the ring $M_2(\mathbb{K})$ of all 2×2 matrices over any field $\mathbb{K}$ of characteristic 2 with the involution '$*$' as the usual transpose map. Take $m = n = 1$ and define a map $\mathcal{H} = \mathcal{T}\colon M_2(\mathbb{K}) \to M_2(\mathbb{K})$ by $\mathcal{T}\left(\left[\begin{smallmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \alpha_1+\alpha_4 & \alpha_2+\alpha_3 \\ \alpha_2+\alpha_3 & \alpha_1+\alpha_4 \end{smallmatrix}\right]$. Then it can be easily seen that $(m + n)\mathcal{T}(a^2) = m\mathcal{T}(a)a^* + na\mathcal{T}(a)$ for all $a \in M_2(\mathbb{K})$. Note that $\mathcal{H} \neq 0$ and $M_2(\mathbb{K})$ is a prime ring.

Theorem 2.2. *Let $\mathcal{A}$ be a noncommutative prime ring with an involution '$*$', and let $\mathcal{T}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ be an additive map such that*

$$\mathcal{T}(aba) = a\mathcal{T}(b)a^* \tag{2.31}$$

for all $a, b \in \mathcal{A}$. Then $\mathcal{T} = 0$.

P r o o f. Replacing a by $a + c$ in (2.31), we have

$$\mathcal{T}(abc + cba) = a\mathcal{T}(b)c^* + c\mathcal{T}(b)a^* \tag{2.32}$$

for all $a, b, c \in \mathcal{A}$. We proceed by considering the following two cases.

Case 1: $\mathcal{A}$ is not a PI-ring. Replacing b by a and c by b in (2.32), we arrive at

$$\mathcal{T}(a^2b + ba^2) = a\mathcal{T}(a)b^* + b\mathcal{T}(a)a^* \tag{2.33}$$

for all $a, b \in \mathcal{A}$. Substituting $a^2b + ba^2$ for b in (2.31) and using (2.33), we get

$$\mathcal{T}(a^3ba + aba^3) = a^2\mathcal{T}(a)b^*a^* + ab\mathcal{T}(a)(a^*)^2 \tag{2.34}$$

for all $a, b \in \mathcal{A}$. On the other hand, replacing b by aba in (2.33), we have

$$\mathcal{T}(a^3ba + aba^3) = a\mathcal{T}(a)a^*b^*a^* + aba\mathcal{T}(a)a^* \tag{2.35}$$

for all $a, b \in \mathcal{A}$. Comparing (2.34) and (2.35), we find that

$$a(a\mathcal{T}(a) - \mathcal{T}(a)a^*)b^*a^* + ab(\mathcal{T}(a)a^* - a\mathcal{T}(a))a^* = 0 \tag{2.36}$$

for all $a, b \in \mathcal{A}$. A complete linearizaton of (2.36) gives

$$\begin{aligned}\sum_{\sigma \in S_4} a_{\sigma(1)}\{(a_{\sigma(2)}\mathcal{T}(a_{\sigma(3)}) - \mathcal{T}(a_{\sigma(2)})a^*_{\sigma(3)})b^* \\ +b(\mathcal{T}(a_{\sigma(2)})a^*_{\sigma(3)} - a_{\sigma(2)}\mathcal{T}(a_{\sigma(3)}))\}a^*_{\sigma(4)} = 0\end{aligned}$$

for all $a_1, a_2, a_3, a_4, b \in \mathcal{A}$. Now $\mathcal{A}$ is not a PI-ring, therefore, by [2], Theorem 3.3, we have

$$\sum_{\substack{\sigma \in S_4 \\ \sigma(4)=4}} a_{\sigma(1)}\{(a_{\sigma(2)}\mathcal{T}(a_{\sigma(3)}) - \mathcal{T}(a_{\sigma(2)})a^*_{\sigma(3)})b^* + b(\mathcal{T}(a_{\sigma(2)})a^*_{\sigma(3)} - a_{\sigma(2)}\mathcal{T}(a_{\sigma(3)}))\} = 0$$

for all $a_1, a_2, a_3, b \in \mathcal{A}$. Again by [2], Theorem 3.3, we have

$$\sum_{\substack{\sigma \in S_4 \\ \sigma(1)=1 \\ \sigma(4)=4}} (a_{\sigma(2)}\mathcal{T}(a_{\sigma(3)}) - \mathcal{T}(a_{\sigma(2)})a^*_{\sigma(3)})b^* + b(\mathcal{T}(a_{\sigma(2)})a^*_{\sigma(3)} - a_{\sigma(2)}\mathcal{T}(a_{\sigma(3)}) = 0$$

for all $a_2, a_3, b \in \mathcal{A}$. Applying Lemma 2.3, it follows that

$$a_2\mathcal{T}(a_3) + a_3\mathcal{T}(a_2) - \mathcal{T}(a_2)a_3^* - \mathcal{T}(a_3)a_2^* = 0$$

for all $a_2, a_3 \in \mathcal{A}$. Now in the light of [2], Theorem 3.5, we infer that there exists $q_1 \in \mathcal{Q}_{ml}(\mathcal{A})$ such that $\mathcal{T}(a) = q_1a^*$ for all $a \in \mathcal{A}$. On the other hand there exists $q_2 \in \mathcal{Q}_{ml}(\mathcal{A})$ such that $\mathcal{T}(a) = aq_2$ for all $a \in \mathcal{A}$. Therefore, $q_1a^* = q_2a$ for all $a \in \mathcal{A}$. Invoking Lemma 2.3, we conclude that $q_1 = q_2 = 0$.

Case 2: $\mathcal{A}$ is a PI-ring. In this case by [19], $\mathcal{Z}(\mathcal{A}) \neq \{0\}$. Let $0 \neq \alpha^* = \alpha \in \mathcal{Z}(\mathcal{A})$. Then setting $c = \alpha$ in (2.32), we get

$$\mathcal{T}(\alpha(ab + ba)) = \alpha a\mathcal{T}(b) + \alpha\mathcal{T}(b)a^*$$

for all $a \in \mathcal{A}$. Now by [3], Proposition 2.2.3, there exists a nonzero ideal $\mathcal{J}$ of $\mathcal{A}$ such that $\alpha^{-1}\mathcal{J} \subseteq \mathcal{A}$. Hence, setting $a = \alpha^{-1}a$ in the above relation, we get

$$\mathcal{T}(ab + ba) = a\mathcal{T}(b) + \mathcal{T}(b)a^* \tag{2.37}$$

for all $a \in \mathcal{J}$ and $b \in \mathcal{A}$. First assume that $\mathrm{char}(\mathcal{A}) \neq 2$. Then replacing b by α and a by $\alpha^{-1}a$ in (2.37), we find that

$$\mathcal{T}(a) = qa^* + aq$$

for all $a \in \mathcal{J}$, where $q = (2\alpha)^{-1}\mathcal{T}(\alpha) \in \mathcal{AC}$. Therefore, from (2.37), we have $(aq - qa^*)b^* = b(aq - qa^*)$ for all $a \in \mathcal{J}$ and $b \in \mathcal{A}$. Now using Lemma 2.3, we conclude that $aq = qa^*$ for all $a \in \mathcal{J}$. Again by Lemma 2.3, we get $q = 0$.

Next suppose that $\operatorname{char}(\mathcal{A}) = 2$. Then putting $b = \alpha$ in (2.37), we see that $a\mathcal{T}(\alpha) + \mathcal{T}(\alpha)a^* = 0$ for all $a \in \mathcal{J}$, whence by Lemma 2.3, $\mathcal{T}(\alpha) = 0$. Therefore, putting $b = \alpha$ in (2.31), we get $\mathcal{T}(\alpha a^2) = 0$ for all $a \in \mathcal{A}$. Substituting $a+b$ for a in the last relation, we obtain $\mathcal{T}(\alpha ab) = \mathcal{T}(\alpha ba)$ for all $a, b \in \mathcal{A}$. Now replacing a by $\alpha^{-1}a$ in the previous relation, we find that $\mathcal{T}(ab) = \mathcal{T}(ba)$ for all $a \in \mathcal{J}$ and $b \in \mathcal{A}$. Thus, from (2.37), we have $a\mathcal{T}(b) + \mathcal{T}(b)a^* = 0$ for all $a \in \mathcal{J}$ and $b \in \mathcal{A}$. Applying Lemma 2.3, we conclude that $\mathcal{T} = 0$. This completes the proof. $\square$

The following example shows that Theorems 2.1 and 2.2 do not hold for semiprime rings and hence the condition of primeness is essential.

Example 2.2. Consider the semiprime ring $\mathcal{R} = \mathbb{H} \times \mathbb{F}$, where $\mathbb{H}$ is the ring of real quaternions and $\mathbb{F}$ is any field of charactersitic different from 2. Define the maps $*\colon \mathcal{R} \to \mathcal{R}$ by $(q, \alpha)^* = (\bar{q}, \alpha)$ and $\mathcal{T}\colon \mathcal{R} \to \mathcal{Q}_{ms}(\mathcal{R})$ by $\mathcal{T}(q, \alpha) = (0, \alpha)$, where $\bar{q}$ denotes the conjugate of q. Then it can be easily seen that '$*$' is an involution on $\mathcal{R}$ and $\mathcal{T}$ is a nonzero additive map. Moreover, $2\mathcal{T}(a^2) = \mathcal{T}(a)a^* + a\mathcal{T}(a)$ for all $a \in \mathcal{R}$ and $\mathcal{T}(aba) = a\mathcal{T}(b)a^*$ for all $a, b \in \mathcal{R}$.

In the view of Theorem 2.2, we have an unanswered question given below.

Question. Let $\mathcal{A}$ be a noncommutative prime ring with an involution '$*$'. Characterize additive maps $\mathcal{T}\colon \mathcal{A} \to \mathcal{Q}_{ms}(\mathcal{A})$ which satisfy any one of the following:
(i) $\mathcal{T}(a^3) = a\mathcal{T}(a)a^*$ for all $a \in \mathcal{A}$,
(ii) $\mathcal{T}(a^3) = \mathcal{T}(a^2)a^* + a^2\mathcal{T}(a)$ for all $a \in \mathcal{A}$?

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