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PERTURBATIONS OF REAL PARTS OF EIGENVALUES OF BOUNDED LINEAR OPERATORS IN A HILBERT SPACE

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Abstract. Let A be a bounded linear operator in a complex separable Hilbert space $\mathcal{H}$, and S be a selfadjoint operator in $\mathcal{H}$. Assuming that $A-S$ belongs to the Schatten-von Neumann ideal $\mathcal{S}_p$ $(p>1)$, we derive a bound for $\sum\limits_k |\mathrm{R}\,\lambda_k(A)-\lambda_k(S)|^p$, where $\lambda_k(A)$ $(k=1,2,\dots)$ are the eigenvalues of A. Our results are formulated in terms of the "extended" eigenvalue sets in the sense introduced by T. Kato. In addition, in the case $p=2$ we refine the Weyl inequality between the real parts of the eigenvalues of A and the eigenvalues of its Hermitian component.

Keywords: Hilbert space; linear operator; eigenvalue; Kato theorem; Weyl inequality

MSC 2020: 47A10, 47A55, 47B10

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space with a scalar product $(\cdot,\cdot)$ and the unit operator I, and let $\mathcal{B}(\mathcal{H})$ be the set of linear bounded operators in $\mathcal{H}$. For an $A\in\mathcal{B}(\mathcal{H})$, A^* is the adjoint operator, $\|A\|$ is the operator norm, $\sigma(A)$ is the spectrum, $A_R=(A+A^*)/2$ and $A_I=(A-A^*)/(2\mathrm{i})$ $(\mathrm{i}^2=-1)$. By $\mathcal{S}_p$ $(p\geqslant 1)$ we denote the Schatten-von Neumann ideal of linear compact operators A in $\mathcal{H}$ with the finite norm $N_p(A):=\sqrt[p]{\operatorname{tr}(A^*A)^{p/2}}$.

In this paper we investigate the variation of real and imaginary parts of eigenvalues of a bounded linear operator under perturbation by a selfadjoint one. The perturbation theory of operators is very rich, the classical results can be found in [18], about the recent results, see [1], [2], [5], [12], [13], [16], [20], [21], [23] and the references, which are given therein, but to the best of our knowledge, perturbation bounds are derived mainly for the absolute values of eigenvalues, cf. [3], [4], [12], [13], [18], [19], etc. At the same time, the real and imaginary parts of eigenvalues, ex-

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cept the famous Weyl inequality (see [14], Section II.6) mainly for finite dimensional and compact operators have been investigated, cf. [6]–[10], [17], [22], etc.

Below we do not assume that the operators considered are compact. Let c_m $(m = 1, 2, \ldots)$ be a sequence of positive numbers defined by the recursive relation

$$c_1 = 1, \quad c_m = c_{m-1} + \sqrt{c_{m-1}^2 + 1} \quad (m = 2, 3, \ldots).$$

For a $p \in [2^m, 2^{m+1}]$ $(m = 1, 2, \ldots)$, put $b_p = c_m^t c_{m+1}^{1-t}$ with $t = 2 - 2^{-m}p$. In Section III.6 of [15], it is proved that $b_2 = 1$ and

$$b_p \leqslant \frac{p}{\mathrm{e}^{2/3} \ln 2} \leqslant 0.741p \quad (2 < p < \infty).$$

A sharper inequality

$$b_p \leqslant \frac{p\mathrm{e}^{1/3}}{2} \leqslant 0.699p \quad (2 < p < \infty)$$

is proved in [11], Lemma 9.5. In addition, in Section III.6 of [15] it is shown that

$$b_p = \cot(\pi/2p) \quad \text{for } p = 2^n,\ n = 1, 2, \ldots, \quad \text{and} \quad b_p = b_{p/(p-1)} \quad \text{for } 1 < p < 2.$$

Furthermore, let $C \in \mathcal{B}(\mathcal{H})$ have a nonempty discrete set. All its isolated eigenvalues having finite multiplicities are called *discrete eigenvalues* and their collection is denoted as $\sigma_{\mathrm{d}}(C)$. The essential spectrum of C is the set $\sigma_{\mathrm{ess}}(C) := \sigma(C) \setminus \sigma_{\mathrm{d}}(C)$.

Following [19] (see also [3]), by an *enumeration of the discrete eigenvalues of* C we mean a sequence $\lambda_1(C), \lambda_2(C), \ldots$, whose terms consist of all the eigenvalues of C each counted as often as its multiplicity. By an *extended enumeration of the eigenvalues of* C we mean a sequence $\lambda_1'(C), \lambda_2'(C), \ldots$, whose terms consist of all the eigenvalues of C each counted as often as its multiplicity and, in addition, may contain boundary points of the essential spectrum $\sigma_{\mathrm{ess}}(C)$ of C. (Not all the boundary points are required to be present, and those present can be repeated arbitrarily often.)

Besides, an enumeration of the real parts of the eigenvalues of a bounded operator C is the sequence $\operatorname{Re} \lambda_1(C), \operatorname{Re} \lambda_2(C), \ldots$, whose terms consist of all the real parts of the eigenvalues of C each counted as often as its multiplicity. The extended enumeration of the real parts of eigenvalues of C is a sequence $\operatorname{Re} \lambda_1'(C), \operatorname{Re} \lambda_2'(C), \ldots$, whose terms consist of all real parts of the eigenvalues of C each counted as often as its multiplicity and which, in addition, may contain the real parts of boundary points of $\sigma_{\mathrm{ess}}(C)$.

Similarly we define the extended enumeration $\operatorname{Im} \lambda_1'(C), \operatorname{Im} \lambda_2'(C), \ldots$ of the imaginary parts of eigenvalues of C.

Now we are in a position to formulate the main result of the paper.

Theorem 1.1. *Let $S \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator and let $A \in \mathcal{B}(\mathcal{H})$ be such that*

$$A - S \in \mathcal{S}_p \quad (1 < p < \infty). \tag{1.1}$$

Let $\operatorname{Re}\lambda_1'(A), \operatorname{Re}\lambda_2'(A), \ldots$ *be the extended enumeration of the real parts of eigenvalues of A and $\lambda_1'(S), \lambda_2'(S), \ldots$ be the extended enumeration of eigenvalues of S. Then there exists a permutation π of natural numbers, such that*

$$\left[\sum_{k=1}^{\infty} |\operatorname{Re}\lambda_k'(A) - \lambda_{\pi(k)}'(S)|^p\right]^{1/p} \leqslant N_p(E_R) + 2b_p N_p(E_I) \leqslant (1 + 2b_p) N_p(E),$$

where $E = A - S$, $E_R = (E + E^*)/2$ and $E_I = (E - E^*)/(2\mathrm{i})$.

The proof of this theorem is presented in the next section. In addition, in Section 3 in a particular case we refine the above mentioned Weyl inequality.

Put $B = \mathrm{i}A$. Taking into account that $B_I = A_R$ and $\operatorname{Im}\lambda_k(B) = \operatorname{Re}\lambda_k(A)$, we can assert the following.

Corollary 1.2. *Let $Z \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator and let $B \in \mathcal{B}(\mathcal{H})$ be such that*

$$G := B - \mathrm{i}Z \in \mathcal{S}_p \quad (1 < p < \infty).$$

Then there exists a permutation π of natural numbers, such that

$$\left[\sum_{k=1}^{\infty} |\operatorname{Im}\lambda_k'(B) - \lambda_{\pi(k)}'(Z)|^p\right]^{1/p} \leqslant N_p(G_I) + 2b_p N_p(G_R) \leqslant (1 + 2b_p) N_p(G).$$

2. Proof of Theorem 1.1

Recall that $V \in \mathcal{B}(\mathcal{H})$, satisfying the condition $\lim\limits_{n\to\infty} \|V^n\|^{1/n} = 0$, is called a *quasi-nilpotent operator*, and $D \in \mathcal{B}(\mathcal{H})$, satisfying the condition $D^*D = DD^*$, is called a *normal operator*.

Lemma 2.1. *Let A satisfy the condition*

$$A_I \in \mathcal{S}_p \quad (1 < p < \infty). \tag{2.1}$$

Then there are a normal operator D and a compact quasi-nilpotent operator V, such that

$$A = D + V \quad (\sigma(A) = \sigma(D)). \tag{2.2}$$

P r o o f. For two orthogonal projections P_1, P_2 in $\mathcal{H}$ we write $P_1 < P_2$ if $P_1\mathcal{H} \subset P_2\mathcal{H}$. A set $\mathcal{P}$ of orthogonal projections in $\mathcal{H}$ containing at least two orthogonal projections is called a *chain*, if from $P_1, P_2 \in \mathcal{P}$ with $P_1 \neq P_2$ it follows that either $P_1 < P_2$ or $P_1 > P_2$. For two chains $\mathcal{P}_1$, $\mathcal{P}_2$ we write $\mathcal{P}_1 < \mathcal{P}_2$ if from $P \in \mathcal{P}_1$ it follows that $P \in \mathcal{P}_2$. In this case we say that $\mathcal{P}_1$ precedes $\mathcal{P}_2$. The chain that precedes only itself is called a *maximal chain*. For more details about maximal chains see [15], Section 8.1 in [11], and the references given therein.

Following Definition 8.1 from [11] we say that $A \in \mathcal{B}(\mathcal{H})$ is a $\mathcal{P}$*-triangular operator*, if $A = D + V$, where D is a normal operator such that $PD = DP$ for all $P \in \mathcal{P}$ and V is a compact quasi-nilpotent operator in $\mathcal{H}$, such that $PVP = VP$ $(P \in \mathcal{P})$, and, in addition, the essential spectrum of A lies on an unclosed Jordan curve.

Due to [11], Theorem 8.8 under condition (2.1) A is a $\mathcal{P}$-triangular operator, but due to Corollary 8.2 from [11], for any $\mathcal{P}$-triangular operator, the equality $\sigma(A) = \sigma(D)$ is valid. This proves the lemma. $\square$

Recall that the non-real spectrum of A under condition (2.1) consists of the isolated eigenvalues, see [14], Lemma I.4.1. So the essential spectrum of A lies on the real axis, if (2.1) holds.

To prove Theorem 1.1 we need the well-known Kato theorem, see [19], Theorem II (the generalized Mirsky theorem). It asserts that for bounded selfadjoint operators S and $\widetilde{S}$ with $S - \widetilde{S}$ belonging to $\mathcal{S}_p$ $(1 \leqslant p < \infty)$, there exists a permutation π of natural numbers, such that

$$\sum_{k=1}^{\infty} |\operatorname{Im} \lambda'_k(S) - \lambda'_{\pi(k)}(\widetilde{S})|^p \leqslant N_p^p(\widetilde{S} - S). \tag{2.3}$$

Furthermore, according to (1.1) we have $A_I = E_I \in \mathcal{S}_p$. So (2.1) holds and (2.2) is valid. Therefore, $N_p(V_I) = N_p(A_I - D_I) \leqslant N_p(A_I) + N_p(D_I)$. But thanks to the Weyl inequalities, $N_p(D_I) \leqslant N_p(A_I)$, see [11], Lemma 8.7. So

$$N_p(V_I) \leqslant N_p(A_I) + N_p(D_I) \leqslant 2N_p(A_I). \tag{2.4}$$

Due to the Macaev theorem (see [15], Theorem III.6.2), one has $N_p(V_R) \leqslant \hat{b}_p N_p(V_I)$, where $\hat{b}_p$ depends on p, only. According to Theorem III.6.3 from [15], $\hat{b}_p \leqslant b_p$, see also [11], Theorem 9.4. Hence, (2.4) implies

$$N_p(V_R) \leqslant b_p N_p(V_I) \leqslant 2b_p N_p(A_I). \tag{2.5}$$

Taking into account that the spectra of A and D coincide and D is normal, we can write $\operatorname{Re} \lambda_k(A) = \lambda_k(D_R)$. Since D_R and S are selfadjoint, due to the above-mentioned Kato theorem there exists a permutation π of natural numbers, such that

$$\sum_{k=1}^{\infty} |\lambda'_k(D_R) - \lambda'_{\pi(k)}(S)|^p \leqslant N_p^p(D_R - S).$$

Besides, $\lambda'_1(D_R) = \operatorname{Re}\lambda'_k(A)$ $(k = 1, 2, \ldots)$. So according to (2.2),

$$\begin{aligned}\left[\sum_{k=1}^{\infty}|\operatorname{Re}\lambda'_k(D_R) - \lambda'_{\pi(k)}(S)|^p\right]^{1/p} &\leqslant N_p(D_R - S) = N_p(A_R - V_R - S) \\ &\leqslant N_p(A_R - S) + N_p(V_R).\end{aligned} \tag{2.6}$$

Taking into account (2.5), we arrive at the following lemma.

Lemma 2.2. *Under the condition* (1.1) *there exists a permutation π of natural numbers, such that*

$$\left[\sum_{k=1}^{\infty} |\operatorname{Re}\lambda'_k(A) - \lambda'_{\pi(k)}(S)|^p\right]^{1/p} \leqslant N_p(A_R - S) + N_p(V_R) \leqslant N_p(A_R - S) + b_p N_p(V_I),$$

where V is the nilpotent part of A.

P r o o f of Theorem 1.1. Since S is selfadjoint, we have $E_I = A_I$ and $E_R = A_R - S$. In addition, due to (2.5) and the inequalities $N_p(E_R) \leqslant N_p(E)$ and $N_p(E_I) \leqslant N_p(E)$, we obtain

$$\begin{aligned}N_p(A_R - S) + b_p N_p(V_I) &\leqslant N_p(A_R - S) + 2b_p N_p(A_I) = N_p(E_R) + 2b_p N_p(E_I) \\ &\leqslant (1 + 2b_p) N_p(E).\end{aligned}$$

This and Lemma 2.2 prove the theorem. □

3. Some applications

3.1. The case $p = 2$. Let the condition (1.1) be fulfilled with $p = 2$, i.e., $E_I = A_I$ is a Hilbert-Schmidt operator. In this case Lemma 9.3 from [11], page 149 yields

$$N_2(V_I) = g_I(A) \quad (A_I \in \mathcal{S}_2), \tag{3.1}$$

where

$$g_I(A) = \left[N_2^2(A_I) - \sum_{k=1}^{\infty}(\operatorname{Im}\lambda_k(A))^2\right]^{1/2} \leqslant N_2(A_I).$$

Making use of Lemma 2.2 and the equality $b_2 = 1$, we arrive at the corollary.

Corollary 3.1. *Let $S \in \mathcal{B}(\mathcal{H})$ be selfadjoint and $A \in \mathcal{B}(\mathcal{H})$ be such that $A - S \in \mathcal{S}_2$. Then there exists a permutation π of natural numbers, such that*

$$\left[\sum_{k=1}^{\infty} |\mathrm{Re}\,\lambda'_k(A) - \lambda'_{\pi(k)}(S)|^2\right]^{1/2} \leqslant N_2(A_R - S) + \left[N_2^2(A_I) - \sum_{k=1}^{\infty} (\mathrm{Im}\,\lambda_k(A))^2\right]^{1/2}.$$

In particular, with $S = A_R$, we have

$$\sum_{k=1}^{\infty} (\mathrm{Im}\,\lambda_k(A))^2 + \sum_{k=1}^{\infty} |\mathrm{Re}\,\lambda'_k(A) - \lambda'_{\pi(k)}(A_R)|^2 \leqslant N_2^2(A_I). \tag{3.2}$$

Inequality (3.2) *refines the Weyl inequality*

$$\sum_{k=1}^{\infty} (\mathrm{Im}\,\lambda_k(A))^2 \leqslant N_2^2(A_I),$$

cf. [14], *Section* II.6.

3.2. Compact operators. Assume that

$$A \in \mathcal{S}_p \quad \text{and} \quad S = S^* \in \mathcal{S}_p \quad (1 < p < \infty). \tag{3.3}$$

Since

$$\sum_{k=1}^{\infty} |\lambda'_{\pi(k)}(S)|^p = \sum_{k=1}^{\infty} |\lambda_{\pi(k)}(S)|^p = \sum_{k=1}^{\infty} |\lambda_k(S)|^p = N_p^p(S)$$

and

$$\sum_{k=1}^{\infty} |\mathrm{Re}\,\lambda'_k(A)|^p = \sum_{k=1}^{\infty} |\mathrm{Re}\,\lambda_k(A)|^p,$$

from Theorem 1.1 we have

$$\left[\sum_{k=1}^{\infty} |\mathrm{Re}\,\lambda_k(A)|^p\right]^{1/p} \geqslant N_p(S) - N_p(A_R - S) - 2b_p N_p(A_I).$$

Hence, with $S = A_R$ we obtain

$$\left[\sum_{k=1}^{\infty} |\mathrm{Re}\,\lambda_k(A)|^p\right]^{1/p} \geqslant N_p(A_R) - 2b_p N_p(A_I). \tag{3.4}$$

Hence, we get our next result.

Corollary 3.2. *Let $A \in \mathcal{S}_p$ $(1 < p < \infty)$. Then A has at least one non-imaginary eigenvalue, provided $N_p(A_R) > 2b_p N_p(A_I)$.*

Replacing A by $B = \mathrm{i}A$, we can assert that any $B \in \mathcal{S}_p$ has at least one non-real eigenvalue, provided $N_p(B_I) > 2b_p N_p(B_R)$.

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