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IMAGES OF LOCALLY NILPOTENT DERIVATIONS OF BIVARIATE POLYNOMIAL ALGEBRAS OVER A DOMAIN

Xiaosong Sun, Beini Wang, Changchun

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Abstract. We study the LND conjecture concerning the images of locally nilpotent derivations, which arose from the Jacobian conjecture. Let R be a domain containing a field of characteristic zero. We prove that, when R is a one-dimensional unique factorization domain, the image of any locally nilpotent R-derivation of the bivariate polynomial algebra $R[x,y]$ is a Mathieu-Zhao subspace. Moreover, we prove that, when R is a Dedekind domain, the image of a locally nilpotent R-derivation of $R[x,y]$ with some additional conditions is a Mathieu-Zhao subspace.

Keywords: locally nilpotent derivation; Jacobian conjecture; LND conjecture; Mathieu-Zhao subspace

MSC 2020: 14R10, 13N15

1. Introduction

Throughout this paper, the symbol k denotes a field of characteristic zero, $k^{[n]} := k[x_1, x_2, \ldots, x_n]$ denotes the polynomial algebra in n variables over k, and our rings R are k-domains, i.e., domains containing k. The Jacobian conjecture (JC for short) in affine algebraic geometry asserts that any polynomial map of $k^{[n]}$ with nonzero constant determinant is invertible, cf. [3], [15].

The JC has close connections with some other notable problems. For example, it is related to some problems in combinatorics (cf. [20]); it is equivalent to the Dixmer conjecture on Weyl algebras (cf. [1], [4], [14]) and is equivalent to the Mathieu conjecture on integrals of complex valued finite functions over Lie groups, cf. [9].

In the process of in-depth study, people also found that the JC is closely related to the study of images of derivations and differential operators of polynomial algebras.

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More precisely, the Jacobian conjecture is attributed to the problem of whether the images of some derivations or differential operators are Mathieu-Zhao subspaces, see [17], [18], [19], [22].

Definition 1.1 ([21], [23]). Let B be a commutative k-algebra. A subspace M of B is called a *Mathieu-Zhao subspace* (MZ subspace for short) if for each pair $f, g \in B$ with $f^m \in M$ for all $m \geqslant 1$, we have $gf^m \in M$ for all $m \gg 0$, i.e., there exists some positive integer N such that $gf^m \in M$ for all $m \geqslant N$.

The notion of MZ subspaces was first introduced by Zhao in [21] for associative algebras named after the Mathieu conjecture, which is a natural generalization of the notion of ideal.

In 2011, van den Essen, Wright and Zhao in [18] proved that the two-dimensional Jacobian conjecture is equivalent to the statement that the image $\operatorname{Im} D$ is an MZ subspace for any divergence-zero k-derivation D of $k^{[2]}$ with $1 \in \operatorname{Im} D$. In 2017, the first author showed in [11] that $\operatorname{Im} D$ of a divergence-zero k-derivation D of $k^{[2]}$ need not be an MZ subspace, and in [16] van den Essen and the first author determined when the images of monomial preserving derivations of $k^{[n]}$ are MZ subspaces for any $n \geqslant 2$.

Since all locally nilpotent derivations of $k^{[n]}$ are of divergence zero, it is natural for Zhao to formulate in [24] the following conjecture.

Conjecture 1.2 (LND conjecture). *Let k be a field of characteristic zero, B a commutative k-algebra and D a locally nilpotent derivation of B. Then $\operatorname{Im} D$ is an MZ subspace of B.*

A more general conjecture called the *LNED conjecture* (proposed in [24]) asserts that the same conclusion holds if we replace $\operatorname{Im} D$ by the image $D(I)$ for any ideal I of B, and includes a variant of the conjecture, where derivations are replaced by $\mathcal{E}$-derivations.

In this paper, we focus on the LND conjecture. In 2017, Zhao in [24] showed that the LND conjecture holds for the univariate polynomial algebra $k^{[1]}$. For the polynomial algebra $k^{[n]}$ ($n \geqslant 2$), it was shown by van den Essen, Wright and Zhao (see [18]) using the technique of Newton polytopes that the images of locally finite derivations of $k^{[2]}$ are MZ subspaces. In 2020, Liu and the first author in [8] showed that the images of linear locally nilpotent derivations of $k^{[3]}$ are MZ subspaces using the technique of integrals, and in 2021, they showed in [12] that for rank-two or homogeneous rank-three locally nilpotent derivations D of $k^{[3]}$, $\operatorname{Im} D$ are MZ subspaces by improving some results on local slice constructions. In 2023, the authors showed in [13] that the images of rank-one locally nilpotent derivations of $k^{[n]}$ and rank-two locally nilpotent derivations of $k^{[3]}$ acting on principal ideals are MZ subspaces.

In conclusion, up to now, only a few results on the LND conjecture are known even for B being a polynomial algebra over a field.

In this paper, we are concerned with the case, where B is a polynomial algebra over a domain R containing a field of characteristic zero. In Section 2, we recall some notions and results concerning locally nilpotent derivations. In Section 3, we prove that, when R is a one-dimensional unique factorization domain, the image of any locally nilpotent R-derivation of the bivariate polynomial algebra $R[x,y]$ is an MZ subspace. Furthermore, we prove that, when R is a Dedekind domain, then the image of a locally nilpotent R-derivation D of $R[x,y]$ is an MZ subspace under some additional conditions on the kernels or local slices of D.

2. Preliminaries

In this section, we set up the notations, and recall some notions and results which will be used in the paper. For more details, we refer the readers to [7], Chapters 1, 2.

Let R be a domain containing a field of characteristic zero. Recall that R is a greatest common divisor ring (GCD-ring) if the intersection of any two principal ideals of R is again principal. We say that R satisfies the ascending chain condition (ACC) on principal ideals, if every infinite chain $(b_1) \subseteq (b_2) \subseteq (b_3) \subseteq \ldots$ of principal ideals of R stabilizes. It is well-known that a unique factorization domain (UFD) is just a domain which is a GCD-ring satisfying ACC on principal ideals. Note that every UFD or commutative noetherian ring satisfies ACC on principal ideals.

Let B be a commutative R-algebra. A derivation D of B is a map $D\colon B \to B$ that satisfies:

(i) $D(a+b) = Da + Db$ for all $a, b \in B$,

(ii) $D(ab) = D(a)b + aD(b)$ for all $a, b \in B$.

The derivation D is called a *locally nilpotent derivation* if for each $a \in B$ there exists $m \geqslant 1$ such that $D^m a = 0$. The kernel of D is defined to be $\ker D := \{a \in A\colon D(a) = 0\}$. And D is called an *R-derivation* if $R \subseteq \ker D$.

For brief, we use the following notations:

(1) $\mathrm{Der}_R(B) = \{D\colon D \text{ is an } R\text{-derivation of } B\}$,

(2) $\mathrm{LND}_R(B) = \{D \in \mathrm{Der}_R(B)\colon D \text{ is locally nilpotent}\}$.

Let S be a multiplicatively closed subset of $B \setminus \{0\}$ and $A = \ker D$. Consider $S^{-1}D\colon S^{-1}B \to S^{-1}B$, the natural extension of D to $S^{-1}B$. Then

(i) $S^{-1}D$ is locally nilpotent if and only if D is locally nilpotent and $S \subseteq A$,

(ii) if $S \subseteq A$ then $\ker S^{-1}D = S^{-1}A$ and $(S^{-1}A) \cap B = A$.

As a matter of notation, if $S = \{r^i\}_{i\geqslant 0}$ for some $r \in B$, then B_r denotes $S^{-1}B$ and D_r denotes the induced derivation $S^{-1}D$ on B_r. Likewise, if $S = B \setminus I$ for some prime ideal I of B, then B_I denotes $S^{-1}B$ and D_I denotes the induced derivation $S^{-1}D$ on B_I.

A derivation D of B is irreducible if there exists no non-unit $b \in B$ such that $DB \subseteq bB$. For a derivation D_1 of B and $a \in B$, one has aD_1 locally nilpotent if and only if $a \in \ker D_1$ and D_1 is locally nilpotent, cf. [15], Corollary 1.3.34.

Lemma 2.1. *Let B be a k-domain and $D = aD_1 \in \mathrm{LND}_k(B)$. If $\operatorname{Im} D_1$ is an MZ subspace of B, then so is $\operatorname{Im} D$.*

Proof. Note that $\operatorname{Im} D \subseteq \operatorname{Im} D_1$ and $\operatorname{Im} D = a \operatorname{Im} D_1$. Let $f \in B$ be such that $f^m \in \operatorname{Im} D (\subseteq \operatorname{Im} D_1)$ for all $m \geqslant 1$. Write $f = af_1$. For any $g \in B$, since $\operatorname{Im} D_1$ is an MZ subspace, we have $gf_1f^m \in \operatorname{Im} D_1$ for all $m \gg 0$, and thus $gf^{m+1} = agf_1f^m \in a \operatorname{Im} D_1 = \operatorname{Im} D$ for all $m \gg 0$. Therefore, $\operatorname{Im} D$ is an MZ subspace. $\square$

The following lemma, which states a property of moments of polynomial functions, will be used in the next section. The research of moments is also closely related to some problems in representation theory and the theory of differential equations.

Lemma 2.2 ([6])**.** *Let $a, b \in k$. If $f \in k^{[1]} = k[t]$ is such that $\int_a^b f(t)^m \,\mathrm{d}t = 0$ for all $m \geqslant 1$, then $f = 0$.*

3. Images of locally nilpotent derivations of bivariate polynomial algebras over a k-domain

In this section, we focus on the LND conjecture, where B is a bivariate polynomial algebra $R[x, y]$ over a k-domain R.

Lemma 3.1. *Let R be a k-domain, $B = R^{[2]} = R[x, y]$, and let $D \in \mathrm{LND}_R(B)$ be irreducible. Then there exists some $Q \in B$ such that $DQ \in R$.*

Proof. Let $K = \mathrm{frac(R)}$, and extend D to the locally nilpotent derivation D_K on $K[x, y]$. By Rentschler's theorem (see [7], Theorem 4.1), there exist $P, Q' \in K[x, y]$ and $f(P) \in K[P]$ such that $K[x, y] = K[P, Q']$ and $D_K = f(P)\partial/\partial Q'$ relative to the coordinate system P, Q'. Since D is irreducible, D_K is also irreducible, implying that $\deg_P f(P) = 0$. Therefore, $D_K Q' \in K^*$. Choose nonzero $b \in R$ such that $Q := bQ' \in B$. Then $DQ = D_K Q = bD_K Q' \in R$. $\square$

Let R be a k-domain. Given $F \in B = R^{[2]} = R[x,y]$, write $F_x = \partial_x(F)$ and $F_y = \partial_y(F)$. Define a derivation $\Delta_F := F_y\partial x - F_x\partial y$. If R is also a GCD-ring, we have the following further description on the derivations $D \in \mathrm{LND}_R(B)$.

Lemma 3.2 ([7], Section 4.4.1 or [5]). *Let R be a GCD-ring containing a field of characteristic zero, let $B = R^{[2]} = R[x,y]$, and let $K = \mathrm{frac(R)}$. If $D \in \mathrm{LND}_R(B)$ is irreducible, then there exist some $P, Q \in B$ such that $K[P,Q] = K[x,y]$, $\ker D = R[P]$ and $DQ \in R$. Furthermore, $D = \Delta_P$.*

For a k-domain B and $D \in \mathrm{LND}_k(B)$, let $A = \ker D$. An element $b \in B$ with $Db \neq 0$ and $D^2 b = 0$ is called a local slice of D. Any nonzero $D \in \mathrm{LND}_k(B)$ has a local slice. A local slice b of D is called minimal if $A[b]$ is a maximal element in $\{A[b']\colon\ b' \text{ is a local slice of } D\}$ with respect to inclusion. Denote by $\min(D)$ the set of all minimal local slices of D. If B satisfies ACC on principal ideals, then any nonzero $D \in \mathrm{LND}(B)$ has a minimal local slice, see [7], Section 2.2.

Take any $b \in \min(D)$. Let $B_0 = A[b]$ and $s = b/(Db)$. Then we have $B_{Db} = A_{Db}[s] = (B_0)_{Db}$. Thus, for any $f \in B$, there exists some $i \in \mathbb{N}$ such that $(Dr)^i f \in B_0$. Put

$$s(f) := \min\{i \in \mathbb{Z}\colon\ (Dr)^i f \in B_0\}.$$

One can obtain by definition the following lemma.

Lemma 3.3. *Use the notations above and set $D_0 := D|_{B_0}$. Then*

$$s(f) \leqslant 0 \text{ if and only if } f \in B_, \quad s(f) < 0 \text{ if and only if } f \in rB_0 = \mathrm{Im}\, D_0$$

and

$$s(fg) \leqslant s(f) + s(g), \quad s(f+g) \leqslant \max\{s(f), s(g)\},$$
$$s(r^j f) = s(f) - j, \quad j \in \mathbb{N}.$$

Lemma 3.4 ([12]). *Let B be a k-domain, $D \in \mathrm{LND}_k(B)$ and let $A = \ker D$. Take a local slice b of D. Let $B_0 = A[b]$ and $f = \sum_i a_i b^i \in B_0$, where $a_i \in A$. If $f \in \mathrm{Im}\, D$, then*

$$b \int_0^1 \sum_i a_i b^i t^i \,\mathrm{d}t \in A + (Db)B.$$

Theorem 3.5. *Let $B = R^{[2]} = R[x,y]$, where R is a one-dimensional UFD containing a field of characteristic zero, and let $D \in \mathrm{LND}_R(B)$. Then $\mathrm{Im}\, D$ is an MZ subspace of B.*

P r o o f. We may assume that D is irreducible due to Lemma 2.1. By Lemma 3.2, there exist $P, Q \in B$ such that $K[P,Q] = K[x,y]$, where $K = \mathrm{frac(R)}$, $A = \ker D = R[P]$ and $DQ \in R$, and we have $D = \Delta_P$. Note that Q is a local slice of D. We may assume that $Q \in \min(D)$. In fact, if $Q \notin \min(D)$, we can take $Q_1 \in \min(D)$ such that $A[Q] \subseteq A[Q_1]$. Thus, $Q = aQ_1 + b$ for some $a, b \in A$. Then $aDQ_1 = DQ \in R$, and thus $a \in R$, $D(Q_1) \in R$. Since $a \in R$, we have also $K[P,Q] = K[P,Q_1]$. Therefore, we may replace Q by the minimal local slice Q_1.

Let $B_0 = R[P,Q]$ and $r := DQ \in R$. Since R is a UFD, we have $r = r_0 p_1^{s_1} p_2^{s_2} \dots p_t^{s_t}$, where $r_0 \in R$ is a unit, $p_1, p_2, \dots, p_t \in R$ are distinct prime elements, and $s_1, s_2, \dots, s_t$ are positive integers. Thus,

$$rR = \langle p_1 \rangle^{s_1} \langle p_2 \rangle^{s_2} \dots \langle p_t \rangle^{s_t},$$

where $\langle p_1 \rangle, \dots, \langle p_t \rangle \subseteq R$ are prime ideals, and thus maximal ideals since R is one-dimensional, and $\langle p_i \rangle$, $\langle p_j \rangle$ are coprime whenever $i \neq j$. Let $I_k = \langle p_k \rangle$, $k = 1, 2, \dots, t$.

Claim 3.6. *Let $f \in B$ be such that $f^m \in \operatorname{Im} D$ for all $m \geqslant 1$. Then we have $s(f^l) < 0$ for some $l \geqslant 1$.*

It is trivial for the case $s(f) < 0$ due to the fact that $s(fg) \leqslant s(f) + s(g)$. Below, we consider two cases: $s(f) = 0$ (Case 1) and $s(f) > 0$ (Case 2). It turns out that Case 2 cannot occur in fact.

Case 1: Let $s(f) = 0$, i.e., $f \in B_0 \setminus rB_0$. Let $f = \sum\limits_i \sum\limits_j r_{ij} P^i Q^j$, $r_{ij} \in R$. We want to show that $s(f^l) < 0$, i.e., $f^l \in rB_0$ for some $l \geqslant 1$, which is equivalent to showing $f^l \in rR[P,Q]$. Since $rR = \langle p_1 \rangle^{s_1} \langle p_2 \rangle^{s_2} \dots \langle p_t \rangle^{s_t} = I_1^{s_1} I_2^{s_2} \dots I_t^{s_t}$, it suffices to show that $r_{ij} \in I_k$, $k = 1, 2, \dots, t$.

Let I be one of I_k, $k = 1, 2, \dots, t$, say $I = I_1$. We consider the ring homomorphism

$$\pi\colon\ R[x,y] \to R/I[x,y], \quad \sum_i \sum_j a_{ij} x^i y^j \mapsto \sum_i \sum_j \bar{a}_{ij} x^i y^j,$$

where $\bar{a}_{ij}$ is the coset of a_{ij} in R/I. Note that R/I is a field due to the fact that I is a maximal ideal of R.

We now show that $\overline{P} \neq 0$ and $\overline{Q} \neq 0$ in $R/I[x,y]$. In fact, if $\overline{P} = 0$, i.e., all the coefficients of P belong to I, then all the coefficients of P^{s_1} belong to I^{s_1}. Thus, there exists $b \in I_2^{s_2} \dots I_t^{s_t} \setminus I_1^{s_1} I_2^{s_2} \dots I_t^{s_t} = I_2^{s_2} \dots I_t^{s_t} \setminus rR$ such that all the coefficients of $P^{s_1} b$ belong to rR. It follows that $rd = P^{s_1} b \in R[P] = \ker D$ for some $d \in B$. Since D is a locally nilpotent derivation of B, we have that $\ker D$ is factorially closed due to [7], Principle 1 and thus $d, r \in \ker D = R[P]$. Let $d = \sum\limits_i d_i P^i$, $d_i \in R$. Then

$$P^{s_1} b = rd = r \sum_i d_i P^i.$$

Since P is transcendental over R, we have $P^{s_1}b = rd_{s_1}P^{s_1}$. As B is a domain, $b = rd_{s_1} \in rR$, a contradiction. Therefore, $\overline{P} \neq 0$.

If $\overline{Q} = 0$, i.e., all the coefficients of Q belong to I, then there exists $b' \in I_1^{s_1-1}I_2^{s_2}\ldots I_t^{s_t} \setminus I_1^{s_1}I_2^{s_2}\ldots I_t^{s_t} = I_1^{s_1-1}I_2^{s_2}\ldots I_t^{s_t} \setminus rR$ such that all the coefficients of Qb' belong to rR. It follows that $Qb' = rh$ for some $h \in B$. Since R is a UFD, we have that B is a UFD and $b' = rm$ for some $m \in R$, which contradicts the fact $b' \notin R$. Therefore, $\overline{Q} \neq 0$.

Let $u := \deg \overline{P}(x,y)$ and $v_0 := \deg \overline{Q}(x,y)$. Since

$$r = DQ = \Delta_P(Q) = \begin{vmatrix} P_x & Q_x \\ P_y & Q_y \end{vmatrix},$$

we have in $R/I[x,y]$ that $0 = \bar{r} = \begin{vmatrix} \overline{P}_x & \overline{Q}_x \\ \overline{P}_y & \overline{Q}_y \end{vmatrix}$, i.e., $\overline{P}_x\overline{Q}_y - \overline{P}_y\overline{Q}_x = 0$, which implies that $\overline{P}(x,y)$ and $\overline{Q}(x,y)$ are algebraically dependent over R/I. Note that R/I is a field, by [10], Theorem 1, there is some $e \in R/I[x,y]$ such that $\overline{P}(x,y) = \xi(e)$ and $\overline{Q}(x,y) = \eta(e)$ for some $\xi(t), \eta(t) \in R/I[t]$.

If $u \mid v_0$, then $\deg(\overline{Q}(x,y) - \bar{a}\overline{P}(x,y)^{v_0/u}) < \deg \overline{Q}(x,y)$ for some $\bar{a} \in R/I$. By this way, there exists some $\tau(t) \in R/I[t]$ such that $Q' := Q - \tau(P)$ satisfies $u \nmid \deg \overline{Q'}(x,y)$ or $\overline{Q'}(x,y) = 0$. Since $Q \in \min(D)$, we have $Q' \in \min(D)$. So $\overline{Q'}(x,y) \neq 0$, and thus $u \nmid v$, where $v := \deg \overline{Q'}(x,y)$.

Note that $B_0 = R[P,Q] = R[P,Q']$ and $DQ' = DQ = r \in R$. Write $B_0 \ni f = \sum\limits_i \sum\limits_j r_{ij}P^iQ^j = \sum\limits_{i'} \sum\limits_{j'} {r'}_{i'j'}P^{i'}Q'^{j'}$, $r_{ij}, {r'}_{i'j'} \in R$. Since $f^m \in \operatorname{Im} D$ for all $m \geqslant 1$, we obtain by Lemma 3.4 that for all $m \geqslant 1$,

$$Q' \int_0^1 \left(\sum_{i'} \sum_{j'} {r'}_{i'j'} P^{i'} Q'^{j'} t^{j'} \right)^m \mathrm{d}t \in R[P] + rB,$$

i.e.,

$$Q' \int_0^1 \left(\sum_{i'} \sum_{j'} {r'}_{i'j'} P^{i'} Q'^{j'} t^{j'} \right)^m \mathrm{d}t + h_m(P) = rg_m(x,y)$$

for some $h_m(P) \in R[P]$ and $g_m(x,y) \in B$. Considering in $R/I[x,y]$, we have for all $m \geqslant 1$,

$$\overline{Q'} \int_0^1 \left(\sum_{i'} \sum_{j'} \overline{r'}_{i'j'} \overline{P}^{i'} \overline{Q'}^{j'} t^{j'} \right)^m \mathrm{d}t = -\bar{h}_m(\overline{P}).$$

Taking $m = um_0$, $m_0 \geqslant 1$, we have

$$\overline{Q'} \int_0^1 \left(\sum_{i'} \sum_{j'} \overline{r'}_{i'j'} \overline{P}^{i'} \overline{Q'}^{j'} t^{j'} \right)^{um_0} \mathrm{d}t = -\bar{h}_{um_0}(\overline{P}).$$

Let $\Phi = \sum_{i'}\sum_{j'} \overline{r'}_{i'j'}\overline{P}^{i'}\overline{Q'}^{j'} t^{j'} \in R/I[x,y][t]$ and consider the graded lexicographic order concerning x, y. Suppose that $\Phi \neq 0$. Let $w(t)M$ and $a\widehat{M}$ be the leading terms of Φ and $\overline{Q'}$, respectively. Then the leading term of the left side in the above equation is $a\widehat{M}M^{dm_0}\int_0^1 w(t)^{um_0}\,\mathrm{d}t$, which is of degree v (mod u). The degree of the right side in the above equation is 0 (mod u). Note that $u \nmid v$, we must have $\int_0^1 w(t)^{um_0}\,\mathrm{d}t = 0$. Since the equality holds for any $m_0 \geqslant 1$, we obtain by Lemma 2.2 that $w(t)^u = 0$, and thus $w(t) = 0$, a contradiction. Therefore, $\Phi = 0$.

It follows that $\sum_{i'}\sum_{j'} \overline{r'}_{i'j'}\overline{P}^{i'}\overline{Q'}^{j'} t^{j'} = 0$, which enforces $\overline{r'}_{i'j'}\overline{P}^{i'}\overline{Q'}^{j'} t^{j'} = 0$ for all i', j', and thus $\overline{r'}_{i'j'} = 0$ for all i', j'. Noticing that

$$f = \sum_i \sum_j r_{ij}P^iQ^j = \sum_{i'}\sum_{j'} r'_{i'j'}P^{i'}Q'^{j'},$$

where $r_{ij}, r'_{i'j'} \in R$. We have that each r_{ij} is an R/I-linear combination of some $r'_{i'j'}$. Then $\overline{r'}_{i'j'} = 0$ for all i', j', and $\overline{r}_{ij} = 0$ for all i, j. Since I is any one of I_k, $k = 1, 2, \dots, t$, we have $r_{ij} \in I_k$, $k = 1, 2, \dots, t$. Thus, Case 1 is done.

Case 2: Suppose that $q := s(f) > 0$. We derive a contradiction. Let $f_1 = r^q f$. Then $s(f_1) = s(f) - q = 0$, i.e., $f_1 \in B_0 \setminus rB_0$. Write $f_1 = \sum_i \sum_j f_{ij}P^iQ^j$, and let

$$f_2 := \frac{f_1}{\gcd(\{f_{ij}\}, r^q)} = \frac{r^q}{\gcd(\{f_{ij}\}, r^q)} f = \sum_i \sum_j \frac{f_{ij}}{\gcd(\{f_{ij}\}, r^q)} P^iQ^j.$$

Noticing that $f_2 \in B_0$ and $s(f_2) \geqslant s(f_1) = 0$, we have $s(f_2) = 0$. Since $f^m \in \operatorname{Im} D$ for all $m \geqslant 1$, $f_2 = f_1/\gcd(\{f_{ij}\}, r^q)$ and $\ker D = R[P]$, we have $f_2^m \in \operatorname{Im} D$ for all $m \geqslant 1$. By the proof of Case 1, we see that

$$\frac{f_{ij}}{\gcd(\{f_{ij}\}, r^q)} \in I_k = \langle p_k \rangle, \quad k = 1, 2, \dots, t,$$

which enforces that $p_k | f_{ij}/\gcd(\{f_{ij}\}, r^q)|$ for all i, j, k, a contradiction. Hence, Case 2 cannot occur.

Therefore, Claim 3.6 is proved, i.e., if $f \in B$ is such that $f^m \in \operatorname{Im} D$ for all $m \geqslant 1$, then $s(f^l) < 0$ for some $l \geqslant 1$. Then, for any $g \in B$, we obtain by Lemma 3.3 that

$$s(gf^m) < 0 \quad \forall\, m \gg 0,$$

which implies that $gf^m \in rB_0 = \operatorname{Im} D_0 \subseteq \operatorname{Im} D$ for all $m \gg 0$. Hence, $\operatorname{Im} D$ is an MZ subspace of B. $\square$

In the rest of the paper, we consider the case, where R is a Dedekind domain. Recall that a Dedekind domain is a Noetherian integrally closed domain of dimension one. For a Dedekind domain R, every nonzero ideal is a product of prime ideals (maximal ideals) and for each prime ideal I, the localization ring R_I is a UFD, cf. [2], Chapter 9.

It is natural to ask, when R is a Dedekind domain, if the image of a locally nilpotent R-derivation of $B = R[x, y]$ is an MZ subspace. Due to Lemma 2.1, we may assume that D is irreducible, and by Lemma 3.1 there exists a local slice $Q \in B$ such that $DQ \in R$. And we may assume that $Q \in \min(D)$ as discussed in the first paragraph of the proof of Theorem 3.5. In the following theorem, we assume additionally that Q is a prime of B and $\ker D = R[P]$ for some $P \in B$, and under these conditions we show that $\operatorname{Im} D$ is an MZ subspace of B.

Theorem 3.7. *Let $B = R^{[2]} = R[x, y]$, where R is a Dedekind domain containing a field of characteristic zero, and let $D \in \mathrm{LND}_R(B)$ be irreducible. Let Q be a minimal local slice of D such that $DQ \in R$. Assume additionally that Q is a prime of B and $\ker D = R[P]$ for some $P \in B$. Then $\operatorname{Im} D$ is an MZ subspace of B.*

Proof. Let $B_0 = R[P, Q]$ and $r := DQ \in R$. Since R is a Dedekind domain, we have $rR = I_1^{s_1} I_2^{s_2} \ldots I_t^{s_t}$, where $s_1, s_2, \ldots, s_t$ are positive integers, $I_1, I_2, \ldots, I_t$ are prime ideals (maximal ideals) of R, and I_i, I_j are coprime whenever $i \neq j$.

Due to the same discussion as in Theorem 3.5, if the following claim holds then $\operatorname{Im} D$ is an MZ subspace of B.

Claim 3.8. *Let $f \in B$ be such that $f^m \in \operatorname{Im} D$ for all $m \geqslant 1$. Then we have $s(f^l) < 0$ for some $l \geqslant 1$.*

The case $s(f) < 0$ is trivial. For the case $s(f) = 0$, due to the same discussion as in Theorem 3.5, we have $s(f^l) < 0$ for some $l \geqslant 1$. To prove the statement, it suffices to show that the case $s(f) > 0$ does not occur.

Suppose in contrary that $q := s(f) > 0$. Let I be any one of I_k, $k = 1, 2, \ldots, t$. Since R is a Dedekind domain, we have that R_I is a one-dimensional UFD and IR_I is the only nonzero prime (maximal) ideal of R_I. Let D_I denote the extension of D to $R_I[x, y]$. Then $D_I \subset \mathrm{LND}_{R_I}(R_I[x, y])$. We may verify easily that $\ker D_I = R_I[P]$, $r = D_I Q = DQ \in R \subseteq R_I$ and $rR_I = (IR_I)^l$ for some l. By the same discussion as in the proof of Theorem 3.5, one may obtain that $\overline{P}, \overline{Q} \in R/I[x, y]$ are nonzero. Noticing that $R/I[x, y]$ is embedded into $R_I/IR_I[x, y]$, we have that $\overline{P}, \overline{Q} \in R_I/IR_I[x, y]$ are also nonzero.

Since R_I is a one-dimensional UFD, by the proof of Theorem 3.5 for $f \in B \subseteq R_I[x,y]$ such that $f^m \in \operatorname{Im} D \subseteq \operatorname{Im} D_I$ for all $m \geqslant 1$, we have $f \in R_I[P,Q]$ and all coefficients of f belong to the radical of $rR_I = (IR_I)^l$, namely, belong to IR_I. Since I is any one of I_k, $k = 1, 2, \ldots, t$, we have

$$f \in \bigcap_{k=1}^{t} R_{I_k}[P,Q] \subseteq K[P,Q]$$

and all coefficients of $f \in K[P,Q]$ belong to $\bigcap\limits_{k=1}^{t} I_k R_{I_k}$. Write

$$f = \sum_i \sum_j f_{ij} P^i Q^j,$$

where $f_{i,j} \in \bigcap\limits_{k=1}^{t} I_k R_{I_k}$. Then for each k there exists some $a_k \in R \setminus I_k$ such that $a_k f_{ij} \in I_k$.

Since $q := s(f) > 0$, we have $r^q f \in B_0 \setminus rB_0$ and then

$$r^q f = \sum_i \sum_j h_{ij} P^i Q^j = \sum_i \sum_j r^q f_{ij} P^i Q^j,$$

where $h_{ij} \in R$ for any i, j. So $r^q f_{ij} = h_{ij} \in R$ for any i, j. It follows that

$$a_k h_{ij} = r^q a_k f_{ij} \in rR \subseteq I_k^{s_k}.$$

Since each I_k is a maximal ideal, we have that $I_k^{s_k}$ is a primary ideal of B. And noticing that a_k does not belong to I_k, the radical of $I_k^{s_k}$, we get $h_{ij} \in I_k^{s_k}$ for $k = 1, 2, \ldots, t$. Then

$$h_{ij} \in \bigcap_{k=1}^{t} I_k^{s_k} = I_1^{s_1} I_2^{s_2} \ldots I_t^{s_t} = rR,$$

and thus $r^q f \in rB_0$, a contradiction. Hence, the theorem is proved. □

Theorem 3.9. *Let $B = R^{[2]} = R[x,y]$, where R is a Dedekind domain containing a field of characteristic zero and let $D \in \mathrm{LND}_R(B)$ be irreducible. Let Q be a minimal local slice of D such that $DQ \in R$. Assume additionally that $(DQ)R$ is a power of a maximal ideal of R. Then $\operatorname{Im} D$ is an MZ subspace of B.*

P r o o f. Let $r := DQ$. By assumption, $rR = I^k$, where I is a maximal ideal of R and k is a positive interger.

Let $A = \ker D$. Then $B_r = A_r[Q] = (A[Q])_r$. Let D_r be the extension of D to B_r. Then $D_r \in \mathrm{LND}_{A_r}(B_r)$ and $\operatorname{Im} D \subseteq \operatorname{Im} D_r$. Since $1 = D_r(Q/r) \in \operatorname{Im} D_r$, we have $\operatorname{Im} D_r = B_r$.

Since R is a Dedekind domain, we have that R_I is a UFD of dimension one. Let D_I denote the extension of D to $R_I[x,y]$. Then $D_I \in \mathrm{LND}_{R_I}(R_I[x,y])$, and $\operatorname{Im} D_I$ is an MZ subspace of $R_I[x,y]$ due to Theorem 3.5. One may see that $\operatorname{Im} D \subseteq \operatorname{Im} D_I$. For $f, g \in B$ with $f^m \in \operatorname{Im} D \subseteq \operatorname{Im} D_I$ for all $m \geqslant 1$, we then have $gf^m \in \operatorname{Im} D_I$ for all $m \gg 0$, and thus for each sufficiently large m there exists some $s_m \in R \setminus I$ such that $s_m g f^m \in \operatorname{Im} D$.

For each $m \geqslant 1$, since $\operatorname{Im} D_r = B_r$, there is some $p_m \geqslant 1$ such that $r^{p_m} g f^m \in \operatorname{Im} D$. And for each sufficiently large m, since I is a maximal ideal of R, we have $I + s_m R = R$. Then $I^k + s_m R = R$, i.e., $rR + s_m R = R$ and thus $r^{p_m} R + s_m R = R$. Thus, when m is sufficiently large,

$$gf^m = (u_m r^{p_m} + v_m s_m) g f^m = u_m r^{p_m} g f^m + v_m s_m g f^m \in \operatorname{Im} D$$

for some $u_m, v_m \in R$. Therefore, $\operatorname{Im} D$ is an MZ subspace of B. □

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Authors' address: Xiaosong Sun (corresponding author), Beini Wang, School of Mathematics, Jilin University, No. 2699 Qianjin Street, Changchun 130012, Jilin Province, P. R. China, e-mail: `sunxs@jlu.edu.cn`, `wangbn21@jlu.edu.cn`.