Maroua Hakim; Rachid Zitouni

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AN APPROACH TO SOLVE A FUZZY BI-OBJECTIVE MULTI-INDEX FIXED CHARGE TRANSPORTATION PROBLEM

Maroua Hakim and Rachid Zitouni

In this paper, we propose a novel approach for solving a fuzzy bi-objective multi-index fixed-charge transportation problem where the aim is to minimize two objectives: the total transportation cost and transportation time. The parameters of the problem, such as fixed cost, variable cost, and transportation time are represented as fuzzy numbers. To extract crisp values from these parameters, a linear ranking function is used. The proposed approach initially separates the main problem into sub-problems. Then, it solves each sub-problem using different algorithms. After that, it determines the Pareto optimal solutions and trade-off pairs. To evaluate the performance of the proposed approach, various numerical problems of different sizes were solved. The results obtained are encouraging and show the efficiency of our approach.

Keywords: multi-index transportation problem, fixed charge transportation problem, fuzzy mathematics, multi-objective problems

Classification: 90C05, 90B06, 03B52.

1. INTRODUCTION

In recent decades, transportation (TP) problems have become one of the most attractive optimization topics. The classical transportation problem consists in delivering the required amount of products to the customers at the right time, where the total cost needs to be minimized. Several efficient methods have been proposed to solve TP. Then, these methods were extended to solve multi-index transportation problems [11, 13, 36, 37, 38].

There exist some practical situations where the transportation cost consists of two components: a variable cost that is proportional to the amount of product shipped and a fixed cost associated with each activity; this is the well-known fixed charge transportation problem (FCTP). This problem is a nonlinear programming model and may be stated as a mixed-integer problem. It was first introduced by Hirsh and Dantzig [14]. FCTP is of wide interest and appears in many real-world applications.

Several researchers have developed various methods to find optimal or near-optimal solutions for FCTP. These methods are divided into three categories: exact algorithms,

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heuristic algorithms, and metaheuristic algorithms. Exact algorithms such as the vertex ranking method [25], the branch-and-bound algorithm [28], and the cutting plane method [29]. Due to the complexity of the problem, the above exact methods are not suitable for the solution. In addition, heuristic algorithms such as [1, 2, 4, 32] have been successfully applied in solving this problem. The main disadvantage of these heuristic algorithms is that they may end up reaching a local optimum without getting close to the global optimum. Recently, researchers attempted to solve FCTP using metaheuristic algorithms; see, for instance, [5, 19, 26, 31, 34]. In 2023, Kartli et al. [15] proposed a new algorithm for determining an approximate solution for FCTP, and they performed a comparative study between this algorithm, the spanning tree genetic algorithm (st-GA), and the priority-based genetic algorithm (pb-GA). The results obtained show that their algorithm is more efficient and successful than st-GA and pb-GA.

In real-life situations, transportation problems involve multiple conflicting objectives instead of one. In 2001, Ahuja and Arora [3] investigated the multi-index bi-criterion fixed charge solid transportation problem. In 2010, Kumar et al. [17] described a new algorithm for solving the fuzzy fully bi-criterion fixed charge transportation problem using trapezoidal fuzzy numbers. Besides, Singh et al. [33] defined the multi-index bi-criterion fixed charge transportation problem with fuzzy parameters and developed an algorithm for the solution. Subsequently, Roy et al. [30] considered the multi-objective fixed charge transportation problem with product blending, in which the parameters are triangular intuitionistic fuzzy numbers. Furthermore, Haque et al. [12] discussed a budget-constrained non-linear fixed-charge solid transportation problem with closed interval parameters. Ghosh et al. [8] obtained a Pareto optimal solution for a multi-objective fully intuitionistic fuzzy fixed-charge solid transportation problem by utilizing three techniques. Recently, the multi-modal transportation problem has been rigorously studied using rough interval parameters in [23] and fuzzy stochastic parameters in [20]. Authors in [9] derived a compromise solution for the fixed-charge solid transportation problem with budget constraints based on carbon emissions in a neutrosophic environment. The time-variant multi-objective linear fractional transportation problem has been solved using interval-valued parameters in [22].

Several researchers have carried out investigations into multi-objective transportation problems. Mardanya et al. [21] developed an algorithm for obtaining the Pareto optimal solution for the multi-objective multi-item just-in-time transportation problem using some priority-based approaches. Roy et al. [24] included an algorithm to solve the fuzzy multi-objective multi-item solid transportation problem. The type-2 uncertain multi-objective fixed charge transportation problem has been handled by Ghosh et al. [10]. They proposed three methods, namely fuzzy programming, Pythagorean hesitant fuzzy programming, and the global criterion method for obtaining a Pareto optimal solution. Mondal et al. [27] investigated the multi-objective multi-item multi-choice step fixed charge solid transportation problem, assuming that the parameters of the proposed model are presented by triangular intuitionistic fuzzy numbers.

In this paper, we propose an extension in the fuzzy context of Khurana and Adlakha's work [16] for solving the fuzzy bi-objective four-index fixed charge transportation problem FBOFCTP4 using triangular fuzzy numbers. The choice of index number is not restrictive; it is just to get an idea of these types of problems while avoiding the fictitious calculation load. Our purpose is to minimize two conflicting objectives: total transportation cost and transportation time. Our proposed approach initially separates the main problem into sub-problems. Then, it solves each sub-problem using different algorithms. After that, it determines the Pareto optimal solutions and trade-offs. Various numerical instances of different sizes are solved to evaluate the performance of the proposed approach.

The paper is organized into six sections. Section 2 provides a review of fundamental concepts in fuzzy set theory. In Section 3, we outline the mathematical framework for the fuzzy bi-objective four-index fixed charge transportation problem FBOFCTP4. Section 4 details the proposed methodology. An illustrative numerical example is solved and followed by some computational results in Section 5. The last section is dedicated to conclusions and future work.

2. PRELIMINARIES

In this section, we recall some basic definitions of fuzzy set theory. See, for instance, [18, 35].

Definition 2.1. A fuzzy set $\tilde{S}$ is defined as a set of pairs $(x, \mu_{\tilde{S}}(x))$, where x is an element of the universe of discourse U and $\mu_{\tilde{S}}(x)$ is the membership function.

Definition 2.2. A fuzzy number $\tilde{a} = (l, m, u)$ with $l \leq m \leq u$ is called a triangular fuzzy number if its membership function is given by:

$$\mu_{\tilde{a}}(x) = \begin{cases} 0 & \text{if } x < l, \\ \dfrac{x-l}{m-l} & \text{if } l \leq x < m, \\ \dfrac{u-x}{u-m} & \text{if } m \leq x < u, \\ 0 & \text{if } x \geq u. \end{cases} \tag{1}$$

2.1. Ranking Function

Let $F(\mathbb{R})$ be the set of all fuzzy numbers. The ranking function $\Re$ is a process that converts each fuzzy number into a crisp number.

$$\Re : F(\mathbb{R}) \longrightarrow \mathbb{R} \tag{2}$$

If $F(\mathbb{R})$ is the set of triangular fuzzy numbers then,

$$\Re(\tilde{a}) = \frac{l + 2m + u}{4}, \quad \tilde{a} = (l, m, u).$$

For two fuzzy numbers $\tilde{a}$ and $\tilde{b}$, we have

$$\tilde{a} <_{\Re} \tilde{b} \Longleftrightarrow \Re(\tilde{a}) < \Re(\tilde{b}). \tag{3}$$

$$\tilde{a} >_{\Re} \tilde{b} \Longleftrightarrow \Re(\tilde{a}) > \Re(\tilde{b}). \tag{4}$$

$$\tilde{a} =_{\Re} \tilde{b} \Longleftrightarrow \Re(\tilde{a}) = \Re(\tilde{b}). \tag{5}$$

2.2. Arithmetic operations on triangular fuzzy numbers

Let $\tilde{a} = (l_1, m_1, u_1)$ and $\tilde{b} = (l_2, m_2, u_2)$ be two triangular fuzzy numbers and let λ be a scalar. We define the arithmetic operations between two triangular fuzzy numbers as follows:

Addition

$$\tilde{a} \oplus \tilde{b} = (l_1 + l_2, m_1 + m_2, u_1 + u_2). \tag{6}$$

Subtraction

$$\tilde{a} \ominus \tilde{b} = (l_1 - u_2, m_1 - m_2, u_1 - l_2). \tag{7}$$

Scalar multiplication

$$\lambda\tilde{a} = \begin{cases} (\lambda l_1, \lambda m_1, \lambda u_1), \text{ if } \lambda \geq 0, \\ (\lambda u_1, \lambda m_1, \lambda l_1), \; if\, \lambda < 0. \end{cases} \tag{8}$$

3. PROBLEM POSITION

3.1. Economical interpretation

Let

- $O_1, \ldots, O_m$, m origins of availabilities $\alpha_1, \ldots, \alpha_m$, respectively.
- $D_1, \ldots, D_n$, n destinations of demands $\beta_1, \ldots, \beta_n$, respectively.
- $S_1, \ldots, S_p$, p means of transport chosen depending on reserved charges $\gamma_1, \ldots, \gamma_k$, respectively.
- $Q_1, \ldots, Q_q$, q qualities of products of quantities $\delta_1, \ldots, \delta_q$, respectively.
- $\tilde{c}_{ijkl}(i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, p, l = 1, \ldots, q)$: the variable cost for unit quantity of the product type Q_l that transported from O_i to destination D_j using the means of transport S_k.
- $\tilde{f}_{ijkl}(i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, p, l = 1, \ldots, q)$: the fixed charge for unit quantity of product type Q_l that transported from O_i to destination D_j using the means of transport S_k.
- $\tilde{t}_{ijkl}(i = 1, \ldots, m; j = 1, \ldots, n; k = 1, \ldots, p; l = 1, \ldots, q)$: the transportation time for unit quantity of product type Q_l that transported from origin O_i to destination D_j using the means of transport S_k.
- α_i: the availability at origin O_i.
- β_j: the demand at destination D_j.
- γ_k: the capacity of means of transport S_k.

- δ_l: the quantity of product type Q_l.
- $x_{ijkl}(i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, p, l = 1, \ldots, q)$: the quantity of product type Q_l shipped from origin O_i to destination D_j using the means of transport S_k.
- $y_{ijkl}(i = 1, \ldots, m, j = 1, \ldots, n, k = 1, \ldots, p, l = 1, \ldots, q)$: binary variable takes the value 1 if $x_{ijkl} > 0$ and 0 otherwise.

3.2. Problem formulation

Mathematically, a fuzzy bi-objective four-index fixed charge transportation problem FBOFCTP4 can be formulated as follows:

$$\text{Minimize}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q}(\tilde{c}_{ijkl}x_{ijkl} \oplus \tilde{f}_{ijkl}y_{ijkl}), \max[\tilde{t}_{ijkl} : x_{ijkl} > 0]\right\} \tag{9}$$

Subject to constraints

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \alpha_i, \quad \text{for all } \; i = 1, \ldots, m, \tag{10}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \beta_j, \quad \text{for all } \; j = 1, \ldots, n, \tag{11}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \gamma_k, \quad \text{for all } \; k = 1, \ldots, p, \tag{12}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \delta_l, \quad \text{for all } \; l = 1, \ldots, q, \tag{13}$$

$$y_{ijkl} = \begin{cases} 1, \text{if } \; x_{ijkl} > 0, \\ 0, \text{if } \; x_{ijkl} = 0, \end{cases} \tag{14}$$

$$x_{ijkl} \geq 0, \text{ For all } i = 1, \ldots, m, \; j = 1, \ldots, n, \; k = 1, \ldots, p \text{ and } l = 1, \ldots, q. \tag{15}$$

Where for all (i, j, k, l), we have $\alpha_i > 0, \beta_j > 0, \gamma_k, \delta_l > 0, \tilde{c}_{ijkl} \geq_{\Re} 0$, and $\tilde{f}_{ijkl} \geq_{\Re} 0$. The constraints (10), (11), (12), (13), and (14) concern the supply at origin O_i, the demand at destination D_j, the total quantity of product that the means of transport S_k can transport, the total quantity of product type Q_l at all nodes, and whether the route between origin O_i and destination D_j is opening or not, respectively.

Theorem 3.1. The fuzzy bi-objective four-index fixed charge transportation problem has a feasible solution if and only if

$$\sum_{i=1}^{m}\alpha_i = \sum_{j=1}^{n}\beta_j = \sum_{k=1}^{p}\gamma_k = \sum_{l=1}^{q}\delta_l = Q. \tag{16}$$

The problem consists of determining x_{ijkl} so that the total transportation cost and transportation time are minimized.

Definition 3.2. Let $E = \{(i, j, k, l); i = 1, .., m, j = 1, \dots, n; k = 1, \dots p, \text{and}\, l = 1, \dots q\}$. For each $(i, j, k, l) \in E$, associate a vector $P_{ijkl} \in \mathbb{R}^M$, where $M = m+n+p+q$. The P_{ijkl} vector has just four non-zero components, which are located in the lines $i, m+j, m+n+k, \text{and}\, m+n+p+l$ and share a common value of one. We define A as a matrix of vectors P_{ijkl}. Notably, matrix A has a rank of $m+n+p+q-3$.

4. SOLUTION METHOD

To solve the fuzzy bi-objective fixed charge transportation problem FBOFCTP4, we separate it into two sub-problems (P') and (P'').

$$(P') : \text{Minimize}\, \tilde{Z} =_{\Re} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q}(\tilde{c}_{ijkl}\, x_{ijkl} \oplus \tilde{f}_{ijkl}\, y_{ijkl}), \text{ s. t. c. } (10)-(15). \quad (17)$$

$$(P'') : \text{Minimize}\, \tilde{T} =_{\Re} \max[\tilde{t}_{ijkl} : x_{ijkl} > 0], \text{ s. t. c. } (10)-(15). \quad (18)$$

To solve the problem (P'), we consider the relaxed transportation problem (RP') involving variable costs only.

$$(RP') : \text{Minimize}\, \tilde{Z} =_{\Re} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q}\tilde{c}_{ijkl}\, x_{ijkl}, \text{ s. t. c. } (10)-(15). \quad (19)$$

Now, we introduce our proposed approach denoted by $\text{Al}_{FBOFCTP4}$ for solving the fuzzy bi-objective four-index fixed charge transportation problem.

4.1. Description of the proposed approach

The various steps of the proposed approach $\text{Al}_{FBOFCTP4}$ are explained as follows:

Step 1: Determine an optimal solution to the problem (RP') using the steps of Algorithm 1 shown below.

Step 2: Determine an optimal solution to the problem (P') using the steps of Algorithm 2 shown below.

Step 3: Solve the problem (P'') and determine the Pareto optimal solutions along trade-off pairs for the main problem using the steps of Algorithm 3 shown below.

4.2. Algorithm 1

This algorithm is the non-capacitated version of AL_{PT4C} introduced by Zitouni et al. [36] for solving the capacitated four-index transportation problem. It is composed of two phases.

- The first phase consists of determining an initial basic feasible solution for the problem (RP').
- The second phase involves improving a basic feasible solution.

Steps of Algorithm 1

Phase 1

We adapt the least-cost cell method to determine an initial feasible solution to the problem (RP') in a fuzzy context. The principle of this method is to determine the quantity transported with the minimum cost in each step.
Let $E = \{(i, j, k, l)$ such that $b_{ijkl} = 0\}$ where b_{ijkl} is a boolean variable taking the value 1 if x_{ijkl} has been determined and 0 in the opposite case.
Let $I = \{(i, j, k, l)$ such that x_{ijkl} is a basic variable$\}$. At the beginning, we have $I = \varnothing$.

1. Among all quadruplets $(i, j, k, l) \in E$, choose $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$, where $\tilde{c}_{\bar{i},\bar{j},\bar{k},\bar{l}} =_{\Re} \min \tilde{c}_{ijkl}$.
2. Take $x_{\bar{i}\bar{j}\bar{k}\bar{l}} = \min(\alpha_{\bar{i}}, \beta_{\bar{j}}, \gamma_{\bar{k}}, \delta_{\bar{l}})$ and $b_{\bar{i}\bar{j}\bar{k}\bar{l}} = 1$ and add $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$ to I.
3. Update $\alpha_{\bar{i}}, \beta_{\bar{j}}, \gamma_{\bar{k}}$, and $\delta_{\bar{l}}$ as follows:
   (a) $\alpha_{\bar{i}} = \alpha_{\bar{i}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$
   If $\alpha_{\bar{i}} = 0$ then let $x_{\bar{i}jkl} = 0$ and take $b_{\bar{i}jkl} = 1, \forall (j, k, l) \neq (\bar{j}, \bar{k}, \bar{l})$.
   (b) $\beta_{\bar{j}} = \beta_{\bar{j}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$
   If $b_{\bar{j}} = 0$ then let $x_{i\bar{j}kl} = 0$ and take $b_{i\bar{j}kl} = 1, \forall (i, k, l) \neq (\bar{i}, \bar{k}, \bar{l})$.
   (c) $\gamma_{\bar{k}} = \gamma_{\bar{k}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$
   If $\gamma_{\bar{k}} = 0$ then let $x_{ij\bar{k}l} = 0$ and take $b_{ij\bar{k}l} = 1, \forall (i, j, l) \neq (\bar{i}, \bar{j}, \bar{l})$.
   (d) $\delta_{\bar{l}} = \delta_{\bar{l}} - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$
   If $\delta_{\bar{l}} = 0$ then let $x_{ijk\bar{l}} = 0$ and take $b_{ijk\bar{l}} = 1, \forall (i, j, k) \neq (\bar{i}, \bar{j}, \bar{k})$.
4. Repeat from 1) to 3) until all x_{ijkl} variables are determined.

Handling degeneracy

The initial feasible solution provided by Phase 1 may be degenerate or non-degenerate. Let A_x be a matrix of column vectors P_{ijkl} such that $(i, j, k, l) \in I$.

Test of degeneracy

- If $rank(A_x) = m + n + p + q - 3$, then the solution provided by Phase 1 is non-degenerate.
- If $rank(A_x) < m+n+p+q-3$, then the solution provided by Phase 1 is degenerate. In such a case, apply the treatment of degeneracy procedure that can be found in [13, 36].

Phase 2

We adapt phase 2 of AL_{PT4C} [36] to improve the basic feasible solution.

1. Take $r = 0$ and $I^{(0)}$ had previously been defined.

2. For all $(i,j,k,l) \in I^{(r)}$, solve the linear system $\tilde{u}_i^{(r)} \oplus \tilde{v}_j^{(r)} \oplus \tilde{w}_k^{(r)} \oplus \tilde{t}_l^{(r)} =_{\Re} \tilde{c}_{ijkl}$. Where $i = 1,\ldots,m$, $j = 1,\ldots,n$, $k = 1,\ldots,p$ and $l = 1,\ldots,q$. (To solve this fuzzy system, we direct readers to papers [6, 7]).

3. For all $(i,j,k,l) \notin I^{(r)}$, determine

$$\tilde{\delta}_{ijkl}^{(r)} =_{\Re} \tilde{c}_{ijkl} \ominus (\tilde{u}_i^{(r)} \oplus \tilde{v}_j^{(r)} \oplus \tilde{w}_k^{(r)} \oplus \tilde{t}_l^{(r)}).$$

4. **If** $\forall (i,j,k,l) \notin I^{(r)}$, $\tilde{\delta}_{ijkl}^{(r)} \geq_{\Re} 0$ **then** the solution is optimal.

   Else, use
$$\tilde{\delta}_{i_0j_0k_0l_0}^{(r)} = \min\{\tilde{\delta}_{ijkl}^{(r)} : \Re(\tilde{\delta}_{ijkl}^{(r)}) < 0\}.$$

(a) For all $(i,j,k,l) \in I^{(r)}$, construct a cycle $\mu^{(r)}$ by solving the system.

$$\sum \lambda_{ijkl}^{(r)} P_{ijkl} = -P_{i_0j_0k_0l_0}.$$

(b) Determine $\theta = \min\{\dfrac{x_{ijkl}^{(r)}}{-\lambda_{ijkl}^{(r)}} \text{ with} \lambda_{ijkl}^{(r)} < 0\} = \theta_{i_sj_sk_sl_s}^{(r)}$.

(c) Determine a new set of basic solutions $x^{(r+1)}$ and basic cells $I^{(r+1)}$ as follows:

$$x^{(r+1)} = \{x_{ijkl}^{(r)} + \lambda_{ijkl}\theta : (i,j,k,l) \in \mu^{(r)}\} \cup \{x_{ijkl}^{(r)} : (i,j,k,l) \notin \mu^{(r)}\}.$$
$$I^{(r+1)} = I^{(r)} \cup \{(i_0,j_0,k_0,l_0)\} \setminus \{(i_s,j_s,k_s,l_s)\}.$$

(d) Take $r = r+1$ and repeat from 2) to 4).

4.3. Algorithm 2

The algorithm consists of determining the optimal solution to the problem (P'). It starts with the solution provided by Algorithm 1. This solution is an extreme point of the convex set of feasible solutions. The algorithm then searches all nearby extreme points and finds the one with the lowest cost. If this cost is less than the cost of the current solution, the algorithm moves to this new extreme point and repeats the entire process. The process of replacing solution variables is continued until there are no more adjacent extreme points that yield a better solution. The different steps are as follows:

Steps of Algorithm 2

Input: Define the data of problem: m, n, p, q, $\tilde{c}_{ijkl}$ variable costs, $\tilde{f}_{ijkl}$ fixed costs, and $x_{(RP')}^{(opt)}$ the optimal solution of the problem (RP').

Output: Return the best solution of the problem (P').

1. Initialization:
   Let $I^{(h)}$ be the set of interesting quadruplets (i,j,k,l) in iteration h. At the beginning of this algorithm, we know an optimal solution for the problem (RP'). First, take $h=1$; $x^{(1)} = x^{(opt)}_{(RP')}$; and $I^{(1)}$ was previously defined.

2. Determine the total fixed cost of the current basic feasible solution and denote this by $\tilde{F}^{(h)}(current)$.

$$\tilde{F}^{(h)}(current) =_{\Re} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} \tilde{f}_{ijkl}\, y_{ijkl}.$$

3. For all $(i,j,k,l) \in I^{(h)}$, solve the linear system.

$$\tilde{u}_i^{(h)} \oplus \tilde{v}_j^{(h)} \oplus \tilde{w}_k^{(h)} \oplus \tilde{t}_l^{(h)} =_{\Re} \tilde{c}_{ijkl}. \text{ Where}$$
$$i=1,\ldots,m,\ j=1,\ldots,n,\ k=1,\ldots,p \,\text{and}\, l=1,\ldots,q.$$

4. For all $(i,j,k,l) \notin I^{(h)}$, determine

$$\tilde{\delta}_{ijkl}^{(h)} =_{\Re} \tilde{c}_{ijkl} \ominus (\tilde{u}_i^{(h)} \oplus v_j^{(h)} \oplus \tilde{w}_k^{(h)} \oplus \tilde{t}_l^{(h)}).$$

5. For all $(i',j',k',l') \notin I^{(h)}$, determine all cycles $\mu^{(h)}$ by solving the system.

$$\sum_{(i,j,k,l)\in I^{(h)}} \lambda_{ijkl}^{(h)} P_{ijkl} = -P_{i'j'k'l'}.$$

6. For all $(i',j',k',l') \notin I^{(h)}$, determine

$$\theta_{i'j'k'l'}^{(h)} = \min\{\frac{x_{ijkl}^{(h)}}{\lambda_{ijkl}^{(h)}}, \lambda_{ijkl}^{(h)} < 0\}.$$

7. For all $(i,j,k,l) \notin I^{(h)}$, determine

$$\tilde{A}_{ijkl}^{(h)} =_{\Re} \theta_{ijkl}^{(h)}\, \tilde{\delta}_{ijkl}^{(h)},$$

   where $\tilde{A}_{ijkl}^{(h)}$ is the change in fuzzy costs when a non-basic variable enters the base with value $\theta_{ijkl}^{(h)}$.

8. For all $(i,j,k,l) \notin I^{(h)}$, determine

$$\tilde{F}_{ijkl}^{(h)}(\text{diff}) =_{\Re} \tilde{F}_{ijkl}^{(h)}(NB) - \tilde{F}^{(h)}(current),$$

   where $\tilde{F}_{ijkl}^{(h)}(NB)$ is the total fixed cost obtained by introducing variable $x_{ijkl}^{(h)}$ with value $\theta_{ijkl}^{(h)}$, for all $(i,j,k,l) \notin I^{(h)}$.

9. For all $(i,j,k,l) \notin I^{(h)}$ determine

$$\tilde{\Delta}_{ijkl}^{(h)} =_{\Re} \tilde{A}_{ijkl}^{(h)} \oplus \tilde{F}_{ijkl}^{(h)}(\text{diff}).$$

10. **If** $\forall(i,j,k,l) \notin I^{(h)}$, $\Re(\tilde{\Delta}_{ijkl}^{(h)}) \geq 0$ **then** the current solution is optimal.

    Else use

$$\tilde{\Delta}_{i_0 j_0 k_0 l_0}^{(h)} = \min\{\tilde{\Delta}_{ijkl}^{(h)} : \Re(\tilde{\Delta}_{ijkl}^{(h)}) < 0\}.$$

11. For all $(i,j,k,l) \in I^{(h)}$, construct a cycle $\mu^{(h)}$ by solving the system.

$$\sum \lambda_{ijkl}^{(h)} P_{ijkl} = -P_{i_0 j_0 k_0 l_0}.$$

12. Determine $\theta = \min\{\dfrac{x_{ijkl}^{(h)}}{-\lambda_{ijkl}^{(h)}} \text{ with} \lambda_{ijkl}^{(h)} < 0\} = \theta_{i_s j_s k_s l_s}^{(h)}$.

    $x_{i_0 j_0 k_0 l_0}$ is the variable entering the base with value θ, it corresponds to $\tilde{\Delta}_{i_0 j_0 k_0 l_0}^{(h)}$.

13. Determine a new set of basic solutions $x^{(h+1)}$ and basic cells $I^{(h+1)}$ as follows:

$$x^{(h+1)} = \{x_{ijkl}^{(r)} + \lambda_{ijkl}\theta : (i,j,k,l) \in \mu^{(h)}\} \cup \{x_{ijkl}^{(h)} : (i,j,k,l) \notin \mu^{(h)}\}.$$
$$I^{(h+1)} = I^{(h)} \cup \{(i_0, j_0, k_0, l_0)\} \setminus \{(i_s, j_s, k_s, l_s)\}.$$

14. Take $h = h + 1$ and repeat from 2) to 10).

4.4. Algorithm 3

The algorithm consists of determining Pareto optimal solutions and trade-off pairs for the main problem FBOFCTP4. The algorithm begins with the optimal solution provided by Algorithm 2, denoting this by $X^{(1)}$ and $\tilde{Z}_1$ is the value of the objective associated with the soluton $X^{(1)}$. It then determines the value of $\tilde{T}_1$ corresponding to $X^{(1)}$ to get the first trade-off pair $(\tilde{Z}_1, \tilde{T}_1)$. Thereafter, the algorithm modifies the cost matrix $\tilde{c}^{(t)}$ and solves (P_t') using both Algorithms 1 and 2 for obtaining the solution $X^{(2)}$ and trade-off pair $(\tilde{Z}_2, \tilde{T}_2)$. The algorithm repeats the whole process until no further feasible solution is obtained. The different steps of Algorithm 3 are as follows:

Steps of Algorithm 3

Input:
Define the data of the problem: m, n, p, q, $\tilde{c}_{ijkl}$ variable costs, $\tilde{f}_{ijkl}$ fixed costs. $x_{(P')}^{(opt)}$ the optimal solution of the problem (P'), and $\tilde{Z}$ the value of the objective associated with the solution $x_{(P')}^{(opt)}$.

Output:
Return the Pareto optimal solutions and trade-off pairs.

1. Initialization:
   At the beginning of this algorithm, we know an optimal solution $x^{(opt)}_{(P')}$ for the problem (P'). Let $\tilde{Z}$ be the value of the objective associated with the solution $x^{(opt)}_{(P')}$.
   Take $t = 1$, $(P'_t) = (P')$, $X^{(t)} = x^{(opt)}_{(P')}$, and $\tilde{Z}_t = \tilde{Z}$.
   Let $\tilde{M} = (M_1, M_2, M_3)$ be a sufficiently large triangular fuzzy number with $M_1 \leq M_2 \leq M_3$.

   While $(\tilde{Z}_t <_{\Re} \tilde{M})$ **do**

2. Determine $\tilde{T}_t = \max\{\tilde{t}_{ijkl} : x_{ijkl} > 0 \text{ according to } X^{(t)}\}$.

3. Take $t = t + 1$.

4. Define $\tilde{c}^{(t)}_{ijkl} = \begin{cases} \tilde{c}_{ijkl} & \text{if}\, \tilde{t}_{ijkl} <_{\Re} \tilde{T}_{t-1}, \\ \tilde{M} & \text{if}\, \tilde{t}_{ijkl} \geq_{\Re} \tilde{T}_{t-1}. \end{cases}$

5. Determine an optimal solution $X^{(t)}$ to the problem (P'_t) with variable cost $\tilde{c}^{(t)}_{ijkl}$ using both algorithms 1 and 2. Let $\tilde{Z}_t$ be the value of the objective associated with the optimal solution $X^{(t)}$.

$$\tilde{Z}_t =_{\Re} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q} (\tilde{c}_{ijkl} x_{ijkl} \oplus \tilde{f}_{ijkl}\, y_{ijkl}).$$

   End While.

6. Take $t = \bar{q} + 1$ and $\tilde{Z}_{\bar{q}+1} >_{\Re} \tilde{M}$.

7. Let L be the complete list of Pareto optimal solutions $\{X^{(1)}, \ldots, X^{(\bar{q})}\}$ with trade-off pairs: $(\tilde{Z}_1, \tilde{T}_1), (\tilde{Z}_2, \tilde{T}_2), (\tilde{Z}_3, \tilde{T}_3), \ldots, (\tilde{Z}_{\bar{q}}, \tilde{T}_{\bar{q}})$ where $\tilde{Z}_1 <_{\Re} \tilde{Z}_2 <_{\Re} \ldots <_{\Re} \tilde{Z}_{\bar{q}}$ and $\tilde{T}_1 >_{\Re} \tilde{T}_2 >_{\Re} \ldots >_{\Re} \tilde{T}_{\bar{q}}$.

8. Determine the distance d_r between all pairs $(\tilde{Z}_r, \tilde{T}_r)$ and the ideal trade-off pair $(\tilde{Z}_1, \tilde{T}_{\bar{q}})$.
$$d_r = |\Re(\tilde{Z}_r) - \Re(\tilde{Z}_1)| + |\Re(\tilde{T}_r) - \Re(\tilde{T}_{\bar{q}})|.$$

9. Determine s where

$$d_s = \min\{d_r, r = 1, \ldots, \bar{q}\},$$
$$d_s = |\Re(\tilde{Z}_s) - \Re(\tilde{Z}_1)| + |\Re(\tilde{T}_s) - \Re(\tilde{T}_{\bar{q}})|.$$

Remark 4.1. A feasible solution x is said to be an efficient (non-dominated) solution if and only if there exists no other solution y such that $\tilde{Z}(y) <_{\Re} \tilde{Z}(x)$ and $\tilde{T}(y) \leq_{\Re} \tilde{T}(x)$ or $\tilde{Z}(y) \leq_{\Re} \tilde{Z}(x)$ and $\tilde{T}(y) <_{\Re} \tilde{T}(x)$.

Convergence of the Algorithm 3 Based on the variable cost $\tilde{c}_{ijkl}^{(t)}$ stated in step 4 of Algorithm 3, the algorithm will provide an infeasible solution after a certain number of iterations. This assumption guarantees the convergence of the algorithm.

5. NUMERICAL IMPLEMENTATION

This section presents a detailed resolution of a numerical example and some results obtained by carrying out different numerical tests.

Example 5.1. Let us consider a fuzzy bi-objective four-index fixed charge transportation problem with $(m = n = p = q = 2)$, whose quantities are $\alpha_i, \beta_j, \gamma_k, \delta_l, \tilde{c}_{ijkl}, \tilde{f}_{ijkl}$, and $\tilde{t}_{ijkl}$ are given by the following tables.
In this example, $x_B^{(r)}$ and $x_H^{(r)}$ are the set of basic variables and non-basic variables at iteration r, respectively.

α_1	α_2	β_1	β_2	γ_1	γ_2	δ_1	δ_2
32	9	13	28	27	14	15	26

Tab. 1. Table of $\alpha_i, \beta_j, \gamma_k$, and δ_l quantities.

$\tilde{c}_{1111}$	$\tilde{c}_{1112}$	$\tilde{c}_{1121}$	$\tilde{c}_{1122}$
$(2,6,11)$ $\Re = 6.25$	$(1,4,13)$ $\Re = 5.5$	$(6,15,16)$ $\Re = 13$	$(1,12,18)$ $\Re = 10.75$
$\tilde{c}_{1211}$	$\tilde{c}_{1212}$	$\tilde{c}_{1221}$	$\tilde{c}_{1222}$
$(7,11,12)$ $\Re = 10.25$	$(3,9,16)$ $\Re = 9.25$	$(7,16,19)$ $\Re = 14.5$	$(2,6,12)$ $\Re = 6.5$
$\tilde{c}_{2111}$	$\tilde{c}_{2112}$	$\tilde{c}_{2121}$	$\tilde{c}_{2122}$
$(10,15,16)$ $\Re = 14$	$(7,8,19)$ $\Re = 10.5$	$(3,7,11)$ $\Re = 7$	$(3,4,11)$ $\Re = 5.5$
$\tilde{c}_{2211}$	$\tilde{c}_{2212}$	$\tilde{c}_{2221}$	$\tilde{c}_{2222}$
$(4,8,16)$ $\Re = 9$	$(3,10,17)$ $\Re = 10$	$(8,11,15)$ $\Re = 11.25$	$(10,17,18)$ $\Re = 15.5$

Tab. 2. Matrix of variable costs.

The problem has a feasible solution because:

$$\sum_{i=1}^{2} \alpha_i = \sum_{j=1}^{2} \beta_j = \sum_{1}^{2} \gamma_k = \sum_{1}^{2} \delta_l = 41.$$

To solve this problem, we initially separate the main problem FBOFCTP4 into two

$\tilde{f}_{1111}$	$\tilde{f}_{1112}$	$\tilde{f}_{1121}$	$\tilde{f}_{1122}$
$(20, 22, 37)$ $\Re = 25.25$ $\tilde{f}_{1211}$	$(24, 27, 32)$ $\Re = 27.5$ $\tilde{f}_{1212}$	$(9, 19, 25)$ $\Re = 18$ $\tilde{f}_{1221}$	$(8, 8, 35)$ $\Re = 14.75$ $\tilde{f}_{1222}$
$(6, 18, 26)$ $\Re = 17$	$(13, 25, 30)$ $\Re = 23.25$	$(2, 15, 34)$ $\Re = 16.5$	$(11, 27, 36)$ $\Re = 25.25$
$\tilde{f}_{2111}$	$\tilde{f}_{2112}$	$\tilde{f}_{2121}$	$\tilde{f}_{2122}$
$(32, 34, 40)$ $\Re = 35$	$(16, 22, 27)$ $\Re = 21.75$	$(24, 32, 33)$ $\Re = 30.25$	$(32, 35, 40)$ $\Re = 35.5$
$\tilde{f}_{2211}$	$\tilde{f}_{2212}$	$\tilde{f}_{2221}$	$\tilde{f}_{2222}$
$(14, 20, 35)$ $\Re = 22.25$	$(3, 9, 20)$ $\Re = 10$	$(11, 28, 34)$ $\Re = 25.25$	$(27, 28, 32)$ $\Re = 28.75$

Tab. 3. Matrix of fixed costs.

sub-problems (P') and (P'').

$$(P') : \text{Minimize } \tilde{Z} =_{\Re} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}(\tilde{c}_{ijkl}x_{ijkl} \oplus \tilde{f}_{ijkl}y_{ijkl}). \tag{20}$$

$$(P'') : \text{Minimize } \tilde{T} =_{\Re} \max[\tilde{t}_{ijkl} : x_{ijkl} > 0]. \tag{21}$$

Which are subjected to constraints

$$\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} x_{ijkl} = \alpha_i, \text{ for } i = 1, 2, \tag{22}$$

$$\sum_{i=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} x_{ijkl} = \beta_j, \text{ for } j = 1, 2, \tag{23}$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{l=1}^{2} x_{ijkl} = \gamma_k, \text{ for } k = 1, 2, \tag{24}$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} x_{ijkl} = \delta_l, \text{ for } l = 1, 2, \tag{25}$$

$$y_{ijkl} = \begin{cases} 1, \text{if } x_{ijkl} > 0, \\ 0, \text{if } x_{ijkl} = 0, \end{cases} \tag{26}$$

$$x_{ijkl} \geq 0, \; i = 1, 2, \; j = 1, 2, \; k = 1, 2 \text{ and } l = 1, 2. \tag{27}$$

Then, we consider the relaxed transportation problem (RP') involving variable costs only.

$$(RP') : \text{Minimize } \tilde{Z} =_{\Re} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{p}\sum_{l=1}^{2}(\tilde{c}_{ijkl}\, x_{ijkl}), \text{ s. t. c. } (22)-(27).$$

$\tilde{t}_{1111}$	$\tilde{t}_{1112}$	$\tilde{t}_{1121}$	$\tilde{t}_{1122}$
$(4, 8, 15)$ $\Re = 8.75$	$(4, 4, 5)$ $\Re = 4.25$	$(5, 10, 12)$ $\Re = 9.25$	$(2, 3, 5)$ $\Re = 3.25$
$\tilde{t}_{1211}$	$\tilde{t}_{1212}$	$\tilde{t}_{1221}$	$\tilde{t}_{1222}$
$(4, 14, 15)$ $\Re = 11.75$	$(1, 1, 9)$ $\Re = 3$	$(7, 8, 13)$ $\Re = 9$	$(4, 10, 14)$ $\Re = 9.5$
$\tilde{t}_{2111}$	$\tilde{t}_{2112}$	$\tilde{t}_{2121}$	$\tilde{t}_{2122}$
$(9, 12, 14)$ $\Re = 11.75$	$(6, 6, 10)$ $\Re = 7$	$(2, 3, 6)$ $\Re = 3.5$	$(11, 11, 13)$ $\Re = 11.5$
$\tilde{t}_{2211}$	$\tilde{t}_{2212}$	$\tilde{t}_{2221}$	$\tilde{t}_{2222}$
$(9, 13, 14)$ $\Re = 12.25$	$(1, 2, 8)$ $\Re = 3.25$	$(13, 15, 15)$ $\Re = 14.5$	$(5, 6, 15)$ $\Re = 8$

Tab. 4. Matrix of transportation time.

Now, we apply the steps of the proposed approach $\mathrm{Al}_{FBOFCTP4}$ mentioned in Section 4.

Application of Algorithm 1

We determine an optimal solution for the relaxed transportation problem (RP') using the steps of Algorithm 1.

Phase 1

- Take $I = \varnothing$.
- We have $\min \tilde{c}_{ijkl} =_{\Re} \tilde{c}_{1112}$.
- Determine x_{1112} as follows:

$$x_{1112} = \min(\alpha_1, \beta_1, \gamma_1, \delta_2) = \min(32, 13, 27, 26) = 13 \text{ and } b_{1112} = 1.$$

- Add $(1, 1, 1, 2)$ to I.
- Update $\alpha_1, \beta_1, \gamma_1$, and δ_2 as follows:

$$\alpha_1 = 19,\ \beta_1 = 0,\ \gamma_1 = 14,\ \delta_2 = 13.$$

- For all $(i, k, l) \neq (1, 1, 2), x_{i1kl} = 0$ and $b_{i1kl} = 1$.

Repeat the steps of Phase 1 of Algorithm 1 until all x_{ijkl} are determined.
The initial basic feasible solution given by the least cost cell method is $x^{(0)} = x_B^{(0)} \cup x_H^{(0)}$, where:

$$x_B^{(0)} = \{x_{1112} = 13, x_{1211} = 5, x_{1221} = 1, x_{1222} = 13, x_{2211} = 9\}.$$

Test of degeneracy

The number of non-zero elements of $x_B^{(0)}$ equal to $5 = M - 3$. Therefore, the obtained solution is non-degenerate.

Phase 2

The optimality test of phase 2 shows that $x^{(0)}$ is not optimal. After 4 other iterations we get the following optimal solution: $x^{opt} = x_B^{opt} \cup x_H^{opt}$ where:

$$x_B^{(opt)} = \{x_{1111} = 6, x_{2211} = 9, x_{1112} = 7, x_{1212} = 5, x_{1222} = 14\}.$$

Application of Algorithm 2

We determine an optimal solution for the problem (P') using the steps of Algorithm 2.

- Take $x^{(1)} = x^{opt}_{(RP')}$.

$$x_B^{(1)} = \{x_{1111} = 6, x_{2211} = 9, x_{1112} = 7, x_{1212} = 5, x_{1222} = 14\}.$$

- The set of basic cells is as follows:

$$I^{(1)} = \{(1,1,1,1), (1,1,1,2), (1,2,1,2), (1,2,2,2), (2,2,1,1)\}.$$

- Determine $\tilde{F}^1(current)$ as follows:

$$\begin{aligned}
&\tilde{F}^1(current) =_{\Re} \tilde{f}_{1111} \oplus \tilde{f}_{1112} \oplus \tilde{f}_{1212} \oplus \tilde{f}_{1222} \oplus \tilde{f}_{2211},\\
&\tilde{F}^1(current) =_{\Re} (82, 121, 170),\\
&\Re(\tilde{F}^1(current)) = 123.5.
\end{aligned}$$

- We determine the values of $\tilde{\delta}_{ijkl}^{(h)}, \theta_{ijkl}^{(h)}, \tilde{A}_{ijkl}^{(h)}, \tilde{F}_{ijkl}^{(h)}(\text{diff})$ and $\tilde{\varDelta}_{ijkl}^{(h)}$.

The optimality test of the algorithm 2 shows that the solution $x^{(1)}$ is not optimal. The optimal solution is $x^{(opt)} = x_B^{(opt)} \cup x_H^{(opt)}$ where:

$$x_B^{(opt)} = \{x_{1111} = 13, x_{1211} = 2, x_{1212} = 3, x_{1222} = 14, x_{2212} = 9\}.$$

The optimal cost corresponding to the optimal solution $x^{(opt)}$ is:

$$\begin{aligned}
&\tilde{Z} =_{\Re} \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2}(\tilde{c}_{ijkl}x_{ijkl} \oplus \tilde{f}_{ijkl}y_{ijkl}),\\
&\tilde{Z} =_{\Re} (156, 402, 685).\Re(\tilde{Z}) = 411.25.
\end{aligned}$$

(i,j,k,l)	$\tilde{\delta}^{(h)}_{ijkl}$	$\theta^{(h)}_{ijkl}$	$\tilde{A}^{(h)}_{ijkl}$	$\tilde{F}^{(h)}_{ijkl}$(diff)	$\tilde{\Delta}^{(h)}_{ijkl}$
(1,1,2,1)	(-1,12,15) $\Re=9.5$	6	(-6,72,90) $\Re=57$	(-99,-3,76) $\Re=-7.25$	(-105,69,166) $\Re=49.75$
(1,1,2,2)	(-8,11,18) $\Re=8$	7	(-56,77,126) $\Re=56$	(-104,-19,91) $\Re=-12.75$	(-160,58,217) $\Re=43.25$
(1,2,1,1)	(-7,0,8) $\Re=0.25$	5	(-35,0,40) $\Re=1.25$	(-95,-7,84) $\Re=-6.25$	(-130,-7,124) $\Re=-5$
(1,2,2,1)	(-3,8,16) $\Re=7.25$	6	(-18,48,96) $\Re=43.5$	(-106,-7,85) $\Re=-8.75$	(-124,41,181) $\Re=34.75$
(2,1,1,1)	(-3,12,14) $\Re=8.75$	7	(-21,84,98) $\Re=61.25$	(-80,7,96) $\Re=7.5$	(-101,91,194) $\Re=68.75$
(2,1,1,2)	(-8,7,18) $\Re=6$	3.5	(-28,24.5,63) $\Re=21$	(-96,-5,83) $\Re=-5.75$	(-124,19.5,146) $\Re=15.25$
(2,1,2,1)	(-6,7,10) $\Re=4.5$	7	(-42,49,70) $\Re=31.5$	(-88,5,89) $\Re=2.75$	(-130,54,159) $\Re=34.25$
(2,1,2,2)	(-8,6,11) $\Re=3.75$	3.5	(-28,21,38.5) $\Re=13.125$	(-80,8,96) $\Re=8$	(-108,29,134.5) $\Re=21.125$
(2,2,1,2)	(-15,4,14) $\Re=1.75$	7	(-105,28,98) $\Re=12.25$	(-110,-18,76) $\Re=-17.5$	(-215,10,174) $\Re=-5.25$
(2,2,2,1)	(-4,6,12) $\Re=5$	9	(-36,54,108) $\Re=45$	(-91,8,87) $\Re=3$	(-127,62,195) $\Re=48$
(2,2,2,2)	(-4,14,16) $\Re=10$	7	(-28,98,112) $\Re=70$	(-85,1,88) $\Re=1.25$	(-113,99,200) $\Re=71.25$

Tab. 5. Table of $\tilde{\delta}^{(h)}_{ijkl}, \theta^{(h)}_{ijkl}, \tilde{A}^{(h)}_{ijkl}, \tilde{F}^{(h)}_{ijkl}$(diff), and $\tilde{\Delta}^{(h)}_{ijkl}$ quantities.

Application of Algorithm 3

We solve the problem (P'') and determine the Pareto optimal solutions along with trade-offs for the main problem using the steps of Algorithm 3.

- Take $t = 1$.
- Consider $(P'_1) = (P')$.
- Take $X^{(1)} = x^{(opt)}_{(P')}$ and $\tilde{c}^{(1)}_{ijkl} = \tilde{c}_{ijkl}$ for all (i,j,k,l).
- The optimal solution of the problem (P'_1) is $X^{(1)} = X^{(1)}_B \cup X^{(1)}_H$ where:

$$X^{(1)}_B = \{x_{1111} = 13, x_{1211} = 2, x_{1212} = 3, x_{1222} = 14, x_{2212} = 9\}.$$

- The optimal cost corresponding to $X^{(1)}$ is:

$$\tilde{Z}_1 = (156, 402, 685).$$

- We have $\Re(\tilde{Z}_1) = 411.25$ and $\tilde{Z}_1 <_{\Re} \tilde{M}$.

- Determine $\tilde{T}_1$ as follows:

$$\tilde{T}_1 =_{\Re} \max\{\tilde{t}_{ijkl} : x_{ijkl} > 0 \text{ according to } X^{(1)}\}.$$
$$\tilde{T}_1 =_{\Re} (4, 14, 15), \Re(\tilde{T}_1) = 11.75$$

The first fuzzy cost-time trade off pair is:

$$(\tilde{Z}_1, \tilde{T}_1) = ((156, 402, 685), (4, 14, 15)).$$

- Define $\tilde{c}^{(2)}_{ijkl} = \begin{cases} \tilde{c}_{ijkl} & \text{if } \tilde{t}_{ijkl} <_{\Re} \tilde{T}_1, \\ \tilde{M} & \text{if } \tilde{t}_{ijkl} \geq_{\Re} \tilde{T}_1. \end{cases}$

- Determine an optimal solution to the problem (P_2') with variable costs $\tilde{c}^{(2)}_{ijkl}$ using both algorithms 1 and 2.

- The optimal solution of the problem (P_2') is $X^{(2)} = X_B^{(2)} \cup X_H^{(2)}$ where:

$$X_B^{(2)} = \{x_{1111} = 13, x_{1221} = 2, x_{1212} = 5, x_{1222} = 12, x_{2212} = 9\}.$$

- The optimal cost corresponding to the optimal solution $X^{(2)}$ is:

$$\tilde{Z}_2 = (154, 415, 715).$$

We have $\Re(\tilde{Z}_2) = 424.75$ then $\tilde{Z}_2 <_{\Re} \tilde{M}$.

The second fuzzy cost-time trade-off pair is:

$$(\tilde{Z}_2, \tilde{T}_2) = ((154, 415, 715), (4, 10, 14)).$$

- Define $\tilde{c}^{(3)}_{ijkl} = \begin{cases} \tilde{c}_{ijkl} & \text{if } \tilde{t}_{ijkl} <_{\Re} \tilde{T}_2, \\ \tilde{M} & \text{if } \tilde{t}_{ijkl} \geq_{\Re} \tilde{T}_2. \end{cases}$

We determine an optimal solution to the problem (P_3') with variable costs $\tilde{c}^{(3)}_{ijkl}$ using both algorithms 1 and 2.

- The optimal solution of problem (P_3') is $X^{(3)} = x_B^{(3)} \cup x_H^{(3)}$ where:

$$X_B^{(3)} = \{x_{1111} = 2.5, x_{1221} = 3.5, x_{2121} = 9, x_{1212} = 24.5, x_{1122} = 1.5\}.$$

- The optimal cost corresponding to $X^{(3)}$ is:

$$\tilde{Z}_3 = (198.5, 474.5, 781).$$

We have $\Re(\tilde{Z}_3) = 482.125$ then $\tilde{Z}_3 <_{\Re} \tilde{M}$.

The third fuzzy cost-time trade-off pair is:

$$(\tilde{Z}_3, \tilde{T}_3) = ((198.5, 474.5, 781), (7, 8, 13)).$$

- Define $\tilde{c}^{(4)}_{ijkl} = \begin{cases} \tilde{c}_{ijkl} & \text{if } \tilde{t}_{ijkl} <_{\Re} \tilde{T}_3, \\ \tilde{M} & \text{if } \tilde{t}_{ijkl} \geq_{\Re} \tilde{T}_3. \end{cases}$.

We determine an optimal solution to the problem (P'_4) with variable costs $\tilde{c}^{(4)}_{ijkl}$ using both algorithms 1 and 2.

- The optimal solution to the problem (P'_4) is $X^{(4)} = X^{(4)}_B \cup X^{(4)}_H$ where:

$$X^{(4)} = \{x_{1111} = 2.5, x_{1221} = 3.5, x_{2121} = 9, x_{1212} = 24.5, x_{1122} = 1.5\}.$$

- The optimal cost corresponding to the optimal solution $X^{(4)}$ is:

$$\tilde{Z}_4 = (3.5M_1 + 174, 3.5M_2 + 418.75, 3.5M_3 + 714.5).$$

- We have $\Re(\tilde{Z}_4) = \dfrac{3.5M_1 + 7M_2 + 3.5M_3}{4} + 431.5$ then $\Re(\tilde{Z}_4) > \Re(\tilde{M})$ so $\tilde{Z}_4 >_{\Re} \tilde{M}$. Therefore the algorithm ends here.

- We have obtained three fuzzy cost-time trade-off pairs:

$$(\tilde{Z}_1, \tilde{T}_1) = ((156, 402, 685), (4, 14, 15)),$$
$$(\tilde{Z}_2, \tilde{T}_2) = ((154, 415, 715), (4, 10, 14)),$$
$$(\tilde{Z}_3, \tilde{T}_3) = ((198.5, 474.5, 781), (7, 8, 13)).$$

- The Ideal fuzzy cost-time trade-off pair is:

$$(\tilde{Z}_1, \tilde{T}_3) = ((156, 402, 685), (7, 8, 13)).$$

- Now, we determine the optimum fuzzy cost-time trade-pair:

$$d_1 = (\Re(\tilde{Z}_1) - \Re(\tilde{Z}_1)) + (\Re(\tilde{T}_1) - \Re(\tilde{T}_3)) = 2.75.$$
$$d_2 = (\Re(\tilde{Z}_2) - \Re(\tilde{Z}_1)) + (\Re(\tilde{T}_2) - \Re(\tilde{T}_3)) = 14.$$
$$d_3 = (\Re(\tilde{Z}_3) - \Re(\tilde{Z}_1)) + (\Re(\tilde{T}_3) - \Re(\tilde{T}_3)) = 70.875.$$
$$d^* = min\{d_1, d_2, d_3\} = d_1 = 2.75.$$

- Therefore, the optimum fuzzy cost-time trade-off pair is:

$$(\tilde{Z}_1, \tilde{T}_1) = ((156, 402, 685), (4, 14, 15))$$

5.1. Computational results

To test the performance of the proposed approach, we solve various numerical examples of different sizes. For each problem, a set of data, variable cost, fixed cost, and transportation time matrices are randomly generated. Denote by

- $M \times N$: the size of the problem, where $M = m + n + p + q$ and $N = mnpq$.
- Iter: the number of efficient cost-time trade-off pairs.

The table below summarizes the obtained results.

Size ($M \times N$)	Iter	Total time(s)
8×16	5	0.1548
10×36	4	0.2117
12×81	5	0.5339
14×144	6	1.4447
16×256	9	6.1573
18×400	10	9.5902
20×625	8	32.1369
24×1296	6	81.1712
26×1764	9	304.7870
32×1096	9	1.3921e+03
34×5184	4	1.4313e+03
40×10000	4	3.885e+03

Tab. 6. Computational results of the proposed approach for FBOFCTP4.

Comments

- From our tests, we observe that our approach is stable, it can be used to solve different problems with different sizes (from 8×16 until more than 40×10000).
- Note that in each problem solved, the ranges of fixed costs and variable costs are different from the others.
- Based on our experiments, we note that the proposed approach is efficient and provides an optimum solution in less time, especially for relatively large instances.
- The obtained results are indepedent of the number of indices. Therefore, the proposed approach can be extended to solve bi-objective fixed-charge transportation problems with an index number greater than four under crisp or fuzzy envirnments.

6. CONCLUSION AND FUTURE RESEARCH SCOPES

In this paper, we have considered a fuzzy bi-objective four-index fixed charge transportation problem FBOFCTP4. For realistic situations, the parameters of the problem, such as fixed cost, variable cost, and transportation time are represented by triangular fuzzy numbers. To solve the aforementioned problem, we have introduced a novel approach $\mathrm{Al}_{FBOFCTP4}$ consisting of three major steps. In the first step, we have presented an algorithm (Algorithm 1) composed of two phases. The first phase involves determining an initial feasible solution for the problem (RP'). After treating the degeneracy case, we initiate the second phase to determine an optimal solution for (RP'). In the second step, we have provided an algorithm (Algorithm 2) to find an optimal solution to the problem (P'). In the third step, we presented an algorithm (Algorithm 3) to solve the problem (P'') and identify the Pareto optimal solutions along trade-off pairs for the fuzzy bi-objective four-index fixed charge transportation problem. To assess the performance of the proposed approach, we have established an experiment with various test problems of FBOFCTP4 with different sizes.

The results obtained are encouraging and show the efficiency of the approach for yielding an optimal solution in a short period, especially for larger instances. Our approach is independent of the number of indices, as shown in the computational results. Therefore, it can be extended to solve bi-objective fixed-charge transportation problems with more than four indices in a fuzzy environment.

In future studies, interested researchers may extend the proposed approach to incorporate additional objectives beyond cost and time in an uncertain environment.

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Maroua Hakim, Laboratory of Fundamental and Numerical Mathematics, Faculty of Sciences, University Setif 1 Ferhat Abbas, Setif, 19000. Algeria.
e-mail: maroua.hakim@univ-setif.dz

Rachid Zitouni, Laboratory of Fundamental and Numerical Mathematics, Faculty of Sciences, University Setif 1 Ferhat Abbas, Setif, 19000. Algeria.
e-mail: rzitouni@univ-setif.dz