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NON-WEIGHT MODULES OVER THE SUPER SCHRÖDINGER ALGEBRA

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Abstract. We construct a family of non-weight modules which are free $U(\mathfrak{h})$-modules of rank 2 over the $N = 1$ super Schrödinger algebra in (1+1)-dimensional spacetime. We determine the isomorphism classes of these modules. In particular, free $U(\mathfrak{h})$-modules of rank 2 over $\mathfrak{osp}(1|2)$ are also constructed and classified. Moreover, we obtain the sufficient and necessary conditions for such modules to be simple.

Keywords: super Schrödinger algebra; simple module; U($\mathfrak{h}$)-free module, non-weight module

MSC 2020: 17B10, 17B20, 17B35, 17B66

1. Introduction

The Schrödinger group is the symmetry group of the free particle Schrödinger equation and its Lie algebra is called the *Schrödinger algebra*, which plays an important role in mathematical physics, see [3], [4], [22]. In 1994, super Schrödinger algebras as the supersymmetric extensions of the Schrödinger algebra were introduced in [16], whose representation theory plays fundamental roles in classifying non-relativistic superconformal field theories, see [21], [25]. In particular, the $N = 1$ super Schrödinger algebra in $(1 + 1)$-dimensional spacetime is the simplest super Schrödinger algebra whose even part is exactly the Schrödinger algebra in $(1 + 1)$-dimensional spacetime. The weight modules over $\mathcal{S}(1|1)$ were investigated in [1], [2], [6], [36], which naturally motivates us to study non-weight modules over $\mathcal{S}(1|1)$.

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Notoriously, as an important class of non-weight modules, free $U(\mathfrak{h})$-modules (in general, we denote by $U(\mathfrak{h})$ the universal enveloping algebra of the Cartan subalgebra $\mathfrak{h}$), which are defined by the "opposite condition" with weight modules, were introduced by Nilsson for the simple Lie algebra $\mathfrak{sl}_{n+1}$ in [23]. Almost at the same time, Tan and Zhao constructed a large class of such modules over $\mathfrak{sl}_{n+1}$ from certain modules over Witt algebras $\mathcal{W}_n$ in [27], without mentioning the "$U(\mathfrak{h})$-free" feature. Since then, this kind of modules has been widely studied over various Lie algebras, such as Kac-Moody algebras (see [7], [8]), and many infinite-dimensional Lie algebras related to Virasoro algebras, see [12], [14], [15], [17], [18], [19], [30]. Moreover, some results on $U(\mathfrak{h})$-free modules of higher rank have also been explored. For instance, the authors constructed a family of $U(\mathfrak{h})$-free modules of arbitrary rank over $\mathfrak{sl}_2$ in [24], and $U(\mathfrak{h})$-free modules of infinite rank over the Virasoro algebra were studied in [11].

Due to the complexity of Lie superalgebras themselves, the study of constructing and classifying non-weight modules over Lie superalgebras has been fairly less understood. We present here some known results on free $U(\mathfrak{h})$-modules over Lie superalgebras. It was shown that $\mathfrak{osp}(1|2n)$ is the only basic Lie superalgebra that admits such modules in [9]. Such modules were also constructed and classified for super Virasoro algebras (see [34]), (twisted) $N = 2$ superconformal algebras (see [10], [35]), Lie superalgebras of Witt type (see [31]), super Galilean conformal algebras (see [33]), topological $N = 2$ super-BMS$_3$ algebra (see [20]) and Lie superalgebras of block type, see [26], [28], [29], [32]. In this paper, we consider the free $U(\mathfrak{h})$-modules of rank 2 over the $N = 1$ super Schrödinger algebra in $(1 + 1)$-dimensional spacetime, with the aid of several results on the free $U(\mathfrak{h})$-modules of rank 1 over the Schrödinger algebra in [14].

This paper is organized as follows. In Section 2, we recall the definition of $\mathcal{S}(1|1)$, the $N = 1$ super Schrödinger algebra in $(1 + 1)$-dimensional spacetime, and some results on free $U(\mathfrak{h})$-modules of rank 1 over the Schrödinger algebra. Section 3 is devoted to studying the free $U(\mathfrak{h})$-modules of rank 2 over $\mathcal{S}(1|1)$. To be precise, we first construct such modules for $\mathcal{S}(1|1)$, and then we further determine the classification and simplicity of these modules. In particular, analogous results on the Lie superalgebra $\mathfrak{osp}(1|2)$ are also obtained.

Throughout the paper, we denote by $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{Z}$, $\mathbb{Z}_+$ the sets of all complex numbers, nonzero complex numbers, integers, and nonnegative integers, respectively. We always assume that the field is the complex number field $\mathbb{C}$. All modules over $\mathcal{S}(1|1)$ are assumed to be $\mathbb{Z}_2$-graded.

2. Preliminaries

In this section, we collect some basic notions and results that will be used for later studying.

Let us first recall the definition of the $N = 1$ super Schrödinger algebra in $(1+1)$-dimensional spacetime.

Definition 2.1 ([16]). The $N = 1$ super Schrödinger algebra in $(1+1)$-dimensional spacetime $\mathcal{S} = \mathcal{S}(1|1) = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$ is a 9-dimensional Lie superalgebra with the even part $\mathcal{S}_{\bar{0}} = \mathrm{span}_{\mathbb{C}}\{e, f, h, p, q, z\}$ and the odd part $\mathcal{S}_{\bar{1}} = \mathrm{span}_{\mathbb{C}}\{E, F, G\}$ satisfying the following non-vanishing Lie super brackets:

$$\begin{gathered}
[h,e] = 2e, \quad [h,f] = -2f, \quad [h,p] = p, \quad [h,q] = -q, \quad [e,f] = h,\\
[p,q] = z, \quad [e,q] = p, \quad [p,f] = -q, \quad [h,E] = E, \quad [h,F] = -F,\\
[e,F] = -E, \quad [f,E] = -F, \quad [p,F] = G, \quad [q,E] = G, \quad [E,E] = 2e,\\
[F,F] = -2f, \quad [G,G] = z, \quad [E,F] = h, \quad [E,G] = -p, \quad [F,G] = q.
\end{gathered}$$

Remark 2.2.

(1) The even part $\mathcal{S}_{\bar{0}} = \mathrm{span}_{\mathbb{C}}\{e, f, h, p, q, z\}$ is the Schrödinger algebra in $(1+1)$-dimensional spacetime.

(2) It is easy to see that the subspace spanned by $\{h, e, f, E, F\}$ forms a subalgebra of $\mathcal{S}$ which is isomorphic to the Lie superalgebra $\mathfrak{osp}(1|2)$. Then $\mathcal{S}$ contains two subalgebras: the super Heisenberg subalgebra $\mathcal{H} = \mathrm{span}_{\mathbb{C}}\{p, q, z, G\}$ with the even part $\mathcal{H}_{\bar{0}} = \mathrm{span}_{\mathbb{C}}\{p, q, z\}$ and the odd part $\mathcal{H}_{\bar{1}} = \mathrm{span}_{\mathbb{C}}\{G\}$, and $\mathfrak{osp}(1|2)$. In particular, the super Heisenberg subalgebra $\mathcal{H}$ is also an ideal of $\mathcal{S}$. Then $\mathcal{S}$ can be viewed as a semidirect product $\mathcal{S} = \mathcal{H} \rtimes \mathfrak{osp}(1|2)$.

(3) The super Schrödinger algebra $\mathcal{S}$ has a 1-dimensional Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$ which lies in $\mathcal{S}_{\bar{0}}$ and a 1-dimensional center $\mathbb{C}z$.

In the rest of this section, we collect some notions and results on free $U(\mathfrak{h})$-modules of rank 1 over the Schrödinger algebra $\mathcal{S}_{\bar{0}}$.

As usual, we let $U(\mathcal{S}_{\bar{0}})$-Mod be the category of all left $U(\mathcal{S}_{\bar{0}})$-modules. For convenience, we define an algebra automorphism $\sigma\colon\ U(\mathfrak{h}) \to U(\mathfrak{h})$ by $\sigma(h) = h - 1$.

Denote by τ the involution of $U(\mathcal{S}_{\bar{0}})$ defined by

$$\tau\colon\ e \mapsto -f; \quad f \mapsto -e; \quad h \mapsto -h; \quad p \mapsto q; \quad q \mapsto -p; \quad z \mapsto z.$$

It is easy to check that, when restricted to $U(\mathfrak{h})$, the automorphism τ satisfies $\tau\sigma = \sigma^{-1}\tau$.

For any $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, we also define an automorphism

$$\varphi_{\lambda,\alpha}\colon\ U(\mathcal{S}_{\bar{0}}) \to U(\mathcal{S}_{\bar{0}})$$

by

$$\varphi_{\lambda,\alpha}\colon\ e\mapsto \lambda e;\quad f\mapsto \frac{1}{\lambda}f;\quad h\mapsto h;\quad p\mapsto \alpha p;\quad q\mapsto \frac{\alpha}{\lambda}q;\quad z\mapsto \frac{\alpha^2}{\lambda}z.$$

Note that every automorphism ψ of $U(\mathcal{S}_{\bar{0}})$ induces a functor

$$\mathrm{F}_\psi\colon\ U(\mathcal{S}_{\bar{0}})\text{-Mod}\to U(\mathcal{S}_{\bar{0}})\text{-Mod},$$

which maps every module V to itself but with a new action $\bullet$ defined by

$$x\bullet v=\psi(x)\cdot v\quad \forall\, x\in\mathcal{S}_{\bar{0}},\ v\in V.$$

The functor F_ψ maps morphisms to themselves. For convenience, we denote $\mathrm{F}_{\lambda,\alpha}:=\mathrm{F}_{\varphi_{\lambda,\alpha}}$. For later use, we reformulate the following result in [14] on the classification of free $U(\mathfrak{h})$-modules of rank 1 over the Schrödinger algebra.

Theorem 2.3 ([14])**.** *Any free $U(\mathfrak{h})$-module of rank 1 over the Schrödinger algebra $\mathcal{S}_{\bar{0}}$ is isomorphic to one of the modules:*

$$(2.1)\ \ F_{\lambda,0}(M_{b,0}),\ F_{\lambda,0}(M'_{b,0}),\ F_\tau\circ F_{\lambda,0}(M'_{b,0}),\ F_{\lambda,\alpha}(M'_{-3/4,1}),\ F_\tau\circ F_{\lambda,\alpha}(M'_{-3/4,1})$$

for some $\lambda\in\mathbb{C}^*$, $\alpha,b\in\mathbb{C}$, where $M_{b,0}$, $M'_{b,0}$, $M'_{-3/4,1}$ are the polynomial algebra $\mathbb{C}[x]$ over $\mathbb{C}$ equipped with the following $\mathcal{S}_{\bar{0}}$-action,

$$\begin{aligned}
M_{b,0}\colon\ & h\cdot g(x)=xg(x),\ e\cdot g(x)=-(\tfrac{1}{2}x-b)\sigma^2(g(x)),\\
& f\cdot g(x)=(\tfrac{1}{2}x+b)\sigma^{-2}(g(x)),\ p\cdot g(x)=q\cdot g(x)=z\cdot g(x)=0,\\
M'_{b,0}\colon\ & h\cdot g(x)=xg(x),\ e\cdot g(x)=\sigma^2(g(x)),\\
& f\cdot g(x)=-(\tfrac{1}{2}x-b)(\tfrac{1}{2}x+b+1)\sigma^{-2}(g(x)),\\
& p\cdot g(x)=q\cdot g(x)=z\cdot g(x)=0,\\
M'_{-3/4,1}\colon\ & h\cdot g(x)=xg(x),\ e\cdot g(x)=\sigma^2(g(x)),\\
& f\cdot g(x)=-(\tfrac{1}{2}x+\tfrac{3}{4})(\tfrac{1}{2}x+\tfrac{1}{4})\sigma^{-2}(g(x)),\ p\cdot g(x)=\sigma(g(x)),\\
& q\cdot g(x)=-\tfrac{1}{2}(x+\tfrac{1}{2})\sigma^{-1}(g(x)),\ z\cdot g(x)=\tfrac{1}{2}g(x),
\end{aligned}$$

respectively, where $\sigma(g(x))$ denotes $\sigma(g(h))\cdot 1=g(x-1)$.

Furthermore, $F_{\lambda,0}(M_{b,0})$ is simple if and only if $2b\notin\mathbb{Z}_+$, and $F_{\lambda,0}(M'_{b,0})$, $F_\tau\circ F_{\lambda,0}(M'_{b,0})$, $F_{\lambda,\alpha}(M'_{-3/4,1})$, $F_\tau\circ F_{\lambda,\alpha}(M'_{-3/4,1})$ are simple for all $\lambda,\alpha\in\mathbb{C}^*,b\in\mathbb{C}$.

3. Free $U(\mathfrak{h})$-modules of rank 2 over $\mathcal{S}$

Since the super Schrödinger algebra $\mathcal{S}$ is generated by odd elements $\{E, F, G\}$, $\mathcal{S}$ does not have nontrivial modules which are pure even or pure odd. Consequently, $\mathbb{Z}_2$-graded free $U(\mathfrak{h})$-modules of rank 1 over $\mathcal{S}$ do not exist. So in this section, we will discuss the $\mathbb{Z}_2$-graded free $U(\mathfrak{h})$-modules of rank 2 over $\mathcal{S}$. Moreover, we determine the classification and the simplicity of such modules.

Assume that $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is an $\mathcal{S}$-module such that it is free of rank 2 as a $U(\mathfrak{h})$-module with two homogeneous basis elements v and w. If the parities of v and w are the same, then $E \cdot v = E \cdot w = 0$ and $F \cdot v = F \cdot w = 0$. Hence,

$$h \cdot v = [E, F] \cdot v = 0, \quad h \cdot w = [E, F] \cdot w = 0,$$

a contradiction. Consequently, v and w have different parities. Set $v = 1_{\bar{0}} \in M_{\bar{0}}$ and $w = 1_{\bar{1}} \in M_{\bar{1}}$. We may assume

$$M = U(\mathfrak{h})1_{\bar{0}} \oplus U(\mathfrak{h})1_{\bar{1}} = \mathbb{C}[x_{\bar{0}}] \oplus \mathbb{C}[x_{\bar{1}}]. \tag{3.1}$$

Again, we let $\sigma(g(x_{\overline{k}}))$ denote $\sigma(g(h)) \cdot 1_{\overline{k}} = g(x_{\overline{k}} - 1)$ for all $g(x_{\overline{k}}) \in \mathbb{C}[x_{\overline{k}}]$, $\overline{k} \in \mathbb{Z}_2$.

We first present the following important observation for our further discussion.

Lemma 3.1. *Keeping notations as above, the action of $\mathcal{S}$ on M is completely determined by the action of $\mathcal{S}$ on $1_{\bar{0}} \in M_{\bar{0}}$ and $1_{\bar{1}} \in M_{\bar{1}}$. Explicitly speaking, for any $g(x_{\overline{k}}) \in \mathbb{C}[x_{\overline{k}}]$, $\overline{k} \in \mathbb{Z}_2$, we have*

$$\begin{gathered}
e \cdot g(x_{\overline{k}}) = e_{\overline{k}}(x_{\overline{k}})\sigma^2(g(x_{\overline{k}})), \quad f \cdot g(x_{\overline{k}}) = f_{\overline{k}}(x_{\overline{k}})\sigma^{-2}(g(x_{\overline{k}})), \quad h \cdot g(x_{\overline{k}}) = x_{\overline{k}}g(x_{\overline{k}}), \\
p \cdot g(x_{\overline{k}}) = p_{\overline{k}}(x_{\overline{k}})\sigma(g(x_{\overline{k}})), \quad q \cdot g(x_{\overline{k}}) = q_{\overline{k}}(x_{\overline{k}})\sigma^{-1}(g(x_{\overline{k}})), \quad z \cdot g(x_{\overline{k}}) = cg(x_{\overline{k}}), \\
E \cdot g(x_{\overline{k}}) = E_{\overline{k}}(x_{\overline{k+1}})\sigma(g(x_{\overline{k+1}})), \quad F \cdot g(x_{\overline{k}}) = F_{\overline{k}}(x_{\overline{k+1}})\sigma^{-1}(g(x_{\overline{k+1}})), \\
G \cdot g(x_{\overline{k}}) = c_{\overline{k}}g(x_{\overline{k+1}}),
\end{gathered}$$

where $e_{\overline{k}}(x)$, $f_{\overline{k}}(x)$, $p_{\overline{k}}(x)$, $q_{\overline{k}}(x)$, $E_{\overline{k}}(x)$, $F_{\overline{k}}(x)$ are polynomial algebras over $\mathbb{C}$ with indeterminate $x \in \{x_{\bar{0}}, x_{\bar{1}}\}$ such that

$$\begin{gathered}
e_{\overline{k}}(x_{\overline{k}}) = e \cdot 1_{\overline{k}}, \quad f_{\overline{k}}(x_{\overline{k}}) = f \cdot 1_{\overline{k}}, \quad p_{\overline{k}}(x_{\overline{k}}) = p \cdot 1_{\overline{k}}, \quad q_{\overline{k}}(x_{\overline{k}}) = q \cdot 1_{\overline{k}}, \\
E_{\overline{k}}(x_{\overline{k+1}}) = E \cdot 1_{\overline{k}}, \quad F_{\overline{k}}(x_{\overline{k+1}}) = F \cdot 1_{\overline{k}},
\end{gathered}$$

and $c, c_{\overline{k}} \in \mathbb{C}$ satisfying $c_{\bar{0}}c_{\bar{1}} = \frac{1}{2}c$.

P r o o f. Obviously, by (3.1), we have

$$h \cdot g(x_{\overline{k}}) = h \cdot g(h)1_{\overline{k}} = hg(h) \cdot 1_{\overline{k}} = x_{\overline{k}} g(x_{\overline{k}}).$$

According to the defining relations $[h, e] = 2e$ and $[h, E] = E$, we have

$$\begin{aligned} e \cdot h1_{\overline{k}} &= [e, h] \cdot 1_{\overline{k}} + h \cdot e \cdot 1_{\overline{k}} = (h-2)e \cdot 1_{\overline{k}} = \sigma^2(h)e \cdot 1_{\overline{k}}, \\ E \cdot h1_{\overline{k}} &= [E, h] \cdot 1_{\overline{k}} + h \cdot E \cdot 1_{\overline{k}} = (h-1)E \cdot 1_{\overline{k}} = \sigma(h)E \cdot 1_{\overline{k}}. \end{aligned}$$

Then by induction on $n \in \mathbb{Z}_+$, we obtain that

$$e \cdot h^n 1_{\overline{k}} = (h-2)^n e \cdot 1_{\overline{k}} = \sigma^2(h^n)e \cdot 1_{\overline{k}}, \quad E \cdot h^n 1_{\overline{k}} = (h-1)^n E \cdot 1_{\overline{k}} = \sigma(h^n)E \cdot 1_{\overline{k}}.$$

As a consequence,

$$\begin{aligned} e \cdot g(x_{\overline{k}}) &= e \cdot g(h)1_{\overline{k}} = \sigma^2(g(h))e \cdot 1_{\overline{k}} = \sigma^2(g(x_{\overline{k}}))e \cdot 1_{\overline{k}}, \\ E \cdot g(x_{\overline{k}}) &= E \cdot g(h)1_{\overline{k}} = \sigma(g(h))E \cdot 1_{\overline{k}} = \sigma(g(x_{\overline{k+1}}))E \cdot 1_{\overline{k}}. \end{aligned}$$

Similar arguments yield

$$\begin{aligned} f \cdot g(x_{\overline{k}}) &= \sigma^{-2}(g(x_{\overline{k}}))f \cdot 1_{\overline{k}}, \quad p \cdot g(x_{\overline{k}}) = \sigma(g(x_{\overline{k}}))p \cdot 1_{\overline{k}}, \\ q \cdot g(x_{\overline{k}}) &= \sigma^{-1}(g(x_{\overline{k}}))q \cdot 1_{\overline{k}}, \quad z \cdot g(x_{\overline{k}}) = g(x_{\overline{k}})z \cdot 1_{\overline{k}}, \\ F \cdot g(x_{\overline{k}}) &= \sigma^{-1}(g(x_{\overline{k+1}}))F \cdot 1_{\overline{k}}, \quad G \cdot g(x_{\overline{k}}) = g(x_{\overline{k+1}})G \cdot 1_{\overline{k}}. \end{aligned}$$

Suppose $z_{\overline{k}}(x_{\overline{k}}) = z \cdot 1_{\overline{k}}$, $e_{\overline{k}}(x_{\overline{k}}) = e \cdot 1_{\overline{k}}$, $f_{\overline{k}}(x_{\overline{k}}) = f \cdot 1_{\overline{k}}$. Since

$$x_{\overline{k}} = h1_{\overline{k}} = [e, f] \cdot 1_{\overline{k}} = e_{\overline{k}}(x_{\overline{k}})\sigma^2(f_{\overline{k}}(x_{\overline{k}})) - f_{\overline{k}}(x_{\overline{k}})\sigma^{-2}(e_{\overline{k}}(x_{\overline{k}})),$$

we know that $e_{\overline{k}}(x_{\overline{k}}) \neq 0$. Therefore,

$$0 = [z, e] \cdot 1_{\overline{k}} = e_{\overline{k}}(x_{\overline{k}})(z_{\overline{k}}(x_{\overline{k}}) - \sigma^2(z_{\overline{k}}(x_{\overline{k}})))$$

implies that $\deg(z_{\overline{k}}(x_{\overline{k}})) = 0$. Hence, $z_{\overline{k}}(x_{\overline{k}}) \in \mathbb{C}$. Assume that $z_{\bar{0}}(x_{\bar{0}}) = c$, $z_{\bar{1}}(x_{\bar{1}}) = c'$ for any $c, c' \in \mathbb{C}$, set $G_{\overline{k}}(x_{\overline{k+1}}) = G \cdot 1_{\overline{k}}$, then by $[G, G] = z$ we obtain

$$\begin{aligned} z_{\bar{0}}(x_{\bar{0}}) &= z \cdot 1_{\bar{0}} = [G, G] \cdot 1_{\bar{0}} = 2G^2 \cdot 1_{\bar{0}} = 2G_{\bar{0}}(x_{\bar{0}})G_{\bar{1}}(x_{\bar{0}}), \\ z_{\bar{1}}(x_{\bar{1}}) &= z \cdot 1_{\bar{1}} = [G, G] \cdot 1_{\bar{1}} = 2G^2 \cdot 1_{\bar{1}} = 2G_{\bar{1}}(x_{\bar{1}})G_{\bar{0}}(x_{\bar{1}}), \end{aligned}$$

which implies that $\deg(G_{\overline{k}}) = 0$ and $c = c'$. Moreover, suppose $G_{\overline{k}}(x_{\overline{k+1}}) = c_{\overline{k}}$, then we have $c_{\bar{0}}c_{\bar{1}} = \frac{1}{2}c$. The proof is done. □

Suppose $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is an $\mathcal{S}$-module such that it is free of rank 2 as a $U(\mathfrak{h})$-module. Since $\mathcal{S}_{\bar{0}}$ is the Schrödinger algebra, we can naturally identify both $M_{\bar{0}}$ and $M_{\bar{1}}$ with $\mathcal{S}_{\bar{0}}$-modules which are free of rank 1 as $U(\mathfrak{h})$-modules. To understand the module structure of M, we need only consider all possible extensions of the $\mathcal{S}_{\bar{0}}$-actions presented in (2.1) to $\mathcal{S}$, that is to say, we need determine the actions of $\mathcal{S}_{\bar{1}}$ on M. By Lemma 3.1, we shall determine the actions of $\mathcal{S}_{\bar{1}}$ on $1_{\bar{0}}$ and $1_{\bar{1}}$.

Lemma 3.2. *Suppose $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is a $\mathbb{Z}_2$-graded module such that both $M_{\bar{0}}$ and $M_{\bar{1}}$ are $U(\mathcal{S}_{\bar{0}})$-modules that are free of rank 1 when restricted to $U(\mathfrak{h})$. Then M can be extended to a $U(\mathcal{S})$-module only if $M_{\bar{0}}$ and $M_{\bar{1}}$ as $U(\mathcal{S}_{\bar{0}})$-modules belong to exactly the same one of the five classes of $U(\mathcal{S}_{\bar{0}})$-modules presented in Theorem* 2.3.

P r o o f. Suppose on the contrary that $M_{\bar{0}}$ and $M_{\bar{1}}$ as $U(\mathcal{S}_{\bar{0}})$-modules belong to different classes of $U(\mathcal{S}_{\bar{0}})$-modules presented in Theorem 2.3.

Having in mind $[F, F] = -2f$ and by applying Lemma 3.1, we have

$$f_{\bar{0}}(x_{\bar{0}}) = f \cdot 1_{\bar{0}} = F \cdot (F \cdot 1_{\bar{0}}) = \sigma^{-1}(F_{\bar{0}}(x_{\bar{0}}))F_{\bar{1}}(x_{\bar{0}}),$$
$$f_{\bar{1}}(x_{\bar{1}}) = f \cdot 1_{\bar{1}} = F \cdot (F \cdot 1_{\bar{1}}) = \sigma^{-1}(F_{\bar{1}}(x_{\bar{1}}))F_{\bar{0}}(x_{\bar{1}}),$$

which imply that $\deg(f_{\bar{0}}(x_{\bar{0}})) = \deg(f_{\bar{1}}(x_{\bar{1}}))$. Therefore, as $\mathcal{S}_{\bar{0}}$-modules, we have

$$\{M_{\bar{0}}, M_{\bar{1}}\} = \{F_{\lambda,0}(M'_{b,0})F_{\mu,\alpha}(M'_{-3/4,1})\}$$

or

$$\{M_{\bar{0}}, M_{\bar{1}}\} = \{F_\tau \circ F_{\lambda,0}(M'_{b,0}), F_\tau \circ F_{\mu,\alpha}(M'_{-3/4,1})\}.$$

Here we only check that $M_{\bar{0}} = F_{\lambda,0}(M'_{b,0})$ while $M_{\bar{1}} = F_{\mu,\alpha}(M'_{-3/4,1})$ as an example. It follows from $[p, E] = [q, F] = 0$ and Lemma 3.1 that

$$0 = [p, E] \cdot 1_{\bar{0}} = p \cdot (E \cdot 1_{\bar{0}}) - E \cdot (p \cdot 1_{\bar{0}}) = \sigma(E_{\bar{0}}(x_{\bar{1}}))p_{\bar{1}}(x_{\bar{1}}),$$
$$0 = [q, F] \cdot 1_{\bar{0}} = q \cdot (F \cdot 1_{\bar{0}}) - F \cdot (q \cdot 1_{\bar{0}}) = \sigma^{-1}(F_{\bar{0}}(x_{\bar{1}}))q_{\bar{1}}(x_{\bar{1}}),$$

which is a contradiction. Hence, we have completed the proof. □

Now we introduce the category of $U(\mathcal{S})$-modules and consider the functors from the category of $U(\mathcal{S})$-modules to itself in what follows.

Let $U(\mathcal{S})$-Mod be the category of all left $U(\mathcal{S})$-modules. There is a parity change functor Π from $U(\mathcal{S})$-Mod to itself. That is, for any module $M = M_{\bar{0}} \oplus M_{\bar{1}}$, we have a new module $\Pi(M)$ with the same underlying space with the parity changed, i.e., $(\Pi(M))_{\bar{0}} = M_{\bar{1}}$ and $(\Pi(M))_{\bar{1}} = M_{\bar{0}}$.

Define an involution of $U(\mathcal{S})$ by

$$\varsigma\colon\ e \mapsto -f; \quad f \mapsto -e; \quad h \mapsto -h; \quad p \mapsto q; \quad q \mapsto -p;$$
$$z \mapsto z; \quad E \mapsto -F; \quad F \mapsto E; \quad G \mapsto G.$$

It is easy to see that the automorphism ς satisfies $\varsigma\sigma = \sigma^{-1}\varsigma$ when restricted to $\mathfrak{h}$. For any $\lambda \in \mathbb{C}^*$, $\alpha \in \mathbb{C}$, define another automorphism of $U(\mathcal{S})$ by

$$\Phi_{\lambda,\alpha}\colon\ e \mapsto \lambda^2 e;\ f \mapsto \frac{1}{\lambda^2} f;\ h \mapsto h;\ p \mapsto \alpha p;\ q \mapsto \frac{\alpha}{\lambda^2} q;\ z \mapsto \frac{\alpha^2}{\lambda^2} z;$$
$$E \mapsto \lambda E;\ F \mapsto \frac{1}{\lambda} F;\ G \mapsto \frac{\alpha}{\lambda} G.$$

Every automorphism Ψ of $U(\mathcal{S})$ also induces a functor

$$\mathcal{F}_\Psi\colon\ U(\mathcal{S})\text{-Mod} \to U(\mathcal{S})\text{-Mod},$$

which maps every module $V = V_{\bar{0}} \oplus V_{\bar{1}}$ to itself but with a new action $\star$ defined by

$$x_{\overline{k}} \star v_{\overline{k}} = \Psi(x_{\overline{k}}) \cdot v_{\overline{k}}$$

for any $x_{\overline{k}} \in \mathcal{S}_{\overline{k}}$, $v_{\overline{k}} \in V_{\overline{k}}$, $\overline{k} \in \mathbb{Z}_2$. For convenience, we put $\mathcal{F}_{\lambda,\alpha} := \mathcal{F}_{\Phi_{\lambda,\alpha}}$.

Definition 3.3. Let $M = \mathbb{C}[x_{\bar{0}}] \oplus \mathbb{C}[x_{\bar{1}}]$ be a $\mathbb{Z}_2$-graded vector space with $M_{\bar{0}} = \mathbb{C}[x_{\bar{0}}]$ and $M_{\bar{1}} = \mathbb{C}[x_{\bar{1}}]$. For any $b \in \mathbb{C}$, $g(x_{\overline{k}}) \in \mathbb{C}[x_{\overline{k}}]$, $\overline{k} \in \mathbb{Z}_2$, we define the action of $\mathcal{S}$ on M as follows.

$$\begin{aligned}
SM_{b,0}\colon\ & e \cdot g(x_{\bar{0}}) = -(\tfrac{1}{2}x_{\bar{0}} - b)\sigma^2(g(x_{\bar{0}})),\ f \cdot g(x_{\bar{0}}) = (\tfrac{1}{2}x_{\bar{0}} + b)\sigma^{-2}(g(x_{\bar{0}})),\\
& h \cdot g(x_{\bar{0}}) = x_{\bar{0}} g(x_{\bar{0}}),\ e \cdot g(x_{\bar{1}}) = -(\tfrac{1}{2}x_{\bar{1}} - b - \tfrac{1}{2})\sigma^2(g(x_{\bar{1}})),\\
& f \cdot g(x_{\bar{1}}) = (\tfrac{1}{2}x_{\bar{1}} + b + \tfrac{1}{2})\sigma^{-2}(g(x_{\bar{1}})),\ h \cdot g(x_{\bar{1}}) = x_{\bar{1}} g(x_{\bar{1}}),\\
& E \cdot g(x_{\bar{0}}) = \sigma(g(x_{\bar{1}})),\ F \cdot g(x_{\bar{0}}) = -\sigma^{-1}(g(x_{\bar{1}})),\ \mathcal{H} \cdot g(x_{\bar{0}}) = 0,\\
& E \cdot g(x_{\bar{1}}) = -(\tfrac{1}{2}x_{\bar{0}} - b)\sigma(g(x_{\bar{0}})),\ F \cdot g(x_{\bar{1}}) = (\tfrac{1}{2}x_{\bar{0}} + b)\sigma^{-1}(g(x_{\bar{0}})),\\
& \mathcal{H} \cdot g(x_{\bar{1}}) = 0;\\
SM'_{b,0}\colon\ & e \cdot g(x_{\bar{0}}) = \sigma^2(g(x_{\bar{0}})),\ f \cdot g(x_{\bar{0}}) = -(\tfrac{1}{2}x_{\bar{0}} - b)(\tfrac{1}{2}x_{\bar{0}} + b + 1)\sigma^{-2}(g(x_{\bar{0}})),\\
& h \cdot g(x_{\bar{0}}) = x_{\bar{0}} g(x_{\bar{0}}),\ e \cdot g(x_{\bar{1}}) = \sigma^2(g(x_{\bar{1}})),\\
& f \cdot g(x_{\bar{1}}) = -(\tfrac{1}{2}x_{\bar{1}} - b - \tfrac{1}{2})(\tfrac{1}{2}x_{\bar{1}} + b + \tfrac{3}{2})\sigma^{-2}(g(x_{\bar{1}})),\ h \cdot g(x_{\bar{1}}) = x_{\bar{1}} g(x_{\bar{1}}),\\
& E \cdot g(x_{\bar{0}}) = \sigma(g(x_{\bar{1}})),\ F \cdot g(x_{\bar{0}}) = (\tfrac{1}{2}x_{\bar{1}} - b - \tfrac{1}{2})\sigma^{-1}(g(x_{\bar{1}})),\ \mathcal{H} \cdot g(x_{\bar{0}}) = 0,\\
& E \cdot g(x_{\bar{1}}) = \sigma(g(x_{\bar{0}})),\ F \cdot g(x_{\bar{1}}) = (\tfrac{1}{2}x_{\bar{0}} + b + 1)\sigma^{-1}(g(x_{\bar{0}})),\ \mathcal{H} \cdot g(x_{\bar{1}}) = 0;\\
SM'_{-3/4,1}\colon\ & e \cdot g(x_{\overline{k}}) = \sigma^2(g(x_{\overline{k}})),\ f \cdot g(x_{\overline{k}}) = -(\tfrac{1}{2}x_{\overline{k}} + \tfrac{3}{4})(\tfrac{1}{2}x_{\overline{k}} + \tfrac{1}{4})\sigma^{-2}(g(x_{\overline{k}})),\\
& h \cdot g(x_{\overline{k}}) = x_{\overline{k}} g(x_{\overline{k}}),\ p \cdot g(x_{\overline{k}}) = \sigma(g(x_{\overline{k}})),\\
& q \cdot g(x_{\overline{k}}) = -\tfrac{1}{2}(x_{\overline{k}} + \tfrac{1}{2})\sigma^{-1}(g(x_{\overline{k}})),\ z \cdot g(x_{\overline{k}}) = \tfrac{1}{2} g(x_{\overline{k}}),\\
& E \cdot g(x_{\overline{k}}) = \sigma(g(x_{\overline{k+1}})),\ F \cdot g(x_{\overline{k}}) = (\tfrac{1}{2}x_{\overline{k+1}} + \tfrac{1}{4})\sigma^{-1}(g(x_{\overline{k+1}})),\\
& G \cdot g(x_{\overline{k}}) = -\tfrac{1}{2} g(x_{\overline{k+1}}),
\end{aligned}$$

where $\mathcal{H}$ is the super Heisenberg subalgebra.

Proposition 3.4. *For all $b \in \mathbb{C}$, the above relations define on $SM_{b,0}$, $SM'_{b,0}$, and $SM'_{-3/4,1}$ the structure of $\mathcal{S}$-modules.*

P r o o f. We only show that $SM_{b,0}$ is an $\mathcal{S}$-module as an example. Considering the $\mathcal{S}_{\bar{0}}$-action on $SM_{b,0}$, we note that $(SM_{b,0})_{\bar{0}} \cong M_{b,0}$, and $(SM_{b,0})_{\bar{1}} \cong M_{b+1/2,0}$, so both $(SM_{b,0})_{\bar{0}}$ and $(SM_{b,0})_{\bar{1}}$ can be regarded as $\mathcal{S}_{\bar{0}}$-modules. Hence, we only need to check the relations concerning $\mathcal{S}_{\bar{1}}$.

$$\begin{aligned}
2E^2 \cdot g(x_{\bar{0}}) &= 2E \cdot (-\sigma(g_{\bar{1}})) = -2(\tfrac{1}{2}x_{\bar{0}} - b)\sigma^2(g(x_{\bar{0}})) = 2e \cdot g(x_{\bar{0}}) = [E,E] \cdot g(x_{\bar{0}}),\\
2E^2 \cdot g(x_{\bar{1}}) &= 2E \cdot (\tfrac{1}{2}x_{\bar{0}} - b)\sigma(g_{\bar{0}}) = -2(\tfrac{1}{2}x_{\bar{0}} - b - \tfrac{1}{2})\sigma^2(g(x_{\bar{1}}))\\
&= 2e \cdot g(x_{\bar{1}}) = [E,E] \cdot g(x_{\bar{1}}),
\end{aligned}$$

which imply that $2E^2 \cdot g(x_{\overline{k}}) = [E,E] \cdot g(x_{\overline{k}})$.

$$\begin{aligned}
E \cdot F \cdot g(x_{\bar{0}}) + F \cdot E \cdot g(x_{\bar{0}}) &= (\tfrac{1}{2}x_{\bar{0}} - b)g(x_{\bar{0}}) + (\tfrac{1}{2}x_{\bar{0}} + b)g(x_{\bar{0}})\\
&= x_{\bar{0}}g(x_{\bar{0}}) = h \cdot g(x_{\bar{0}}) = [E,F] \cdot g(x_{\bar{0}}),\\
E \cdot F \cdot g(x_{\bar{1}}) + F \cdot E \cdot g(x_{\bar{1}}) &= (\tfrac{1}{2}x_{\bar{1}} + b - \tfrac{1}{2})g(x_{\bar{1}}) + (\tfrac{1}{2}x_{\bar{1}} - b + \tfrac{1}{2})g(x_{\bar{1}})\\
&= x_{\bar{1}}g(x_{\bar{1}}) = h \cdot g(x_{\bar{1}}) = [E,F] \cdot g(x_{\bar{1}}),
\end{aligned}$$

which imply that $E \cdot F \cdot g(x_{\overline{k}}) + F \cdot E \cdot g(x_{\overline{k}}) = [E,F] \cdot g(x_{\overline{k}})$.

$$\begin{aligned}
&e \cdot E \cdot g(x_{\bar{0}}) - E \cdot e \cdot g(x_{\bar{0}}) = (\tfrac{1}{2}x_{\bar{1}} - b - \tfrac{1}{2})\sigma^3(g(x_{\bar{1}})) - (\tfrac{1}{2}x_{\bar{1}} - b - \tfrac{1}{2})\sigma^3(g(x_{\bar{1}})) = 0,\\
&e \cdot E \cdot g(x_{\bar{1}}) - E \cdot e \cdot g(x_{\bar{1}})\\
&\quad = -(\tfrac{1}{2}x_{\bar{0}} - b - 1)(\tfrac{1}{2}x_{\bar{0}} - b)\sigma^3(g(x_{\bar{0}})) + (\tfrac{1}{2}x_{\bar{0}} - b - 1)(\tfrac{1}{2}x_{\bar{0}} - b)\sigma^3(g(x_{\bar{0}})) = 0,
\end{aligned}$$

which imply that $e \cdot E \cdot g(x_{\overline{k}}) - E \cdot e \cdot g(x_{\overline{k}}) = [e,E] \cdot g(x_{\overline{k}})$.

$$\begin{aligned}
&e \cdot F \cdot g(x_{\bar{0}}) - F \cdot e \cdot g(x_{\bar{0}})\\
&\quad = -(\tfrac{1}{2}x_1 - b - \tfrac{1}{2})\sigma(g(x_{\bar{1}})) + (\tfrac{1}{2}x_1 - b + \tfrac{1}{2})\sigma(g(x_{\bar{1}}))\\
&\quad = -\sigma(g(x_{\bar{1}})) = -E \cdot g(x_{\bar{0}}) = [e,F] \cdot g(x_{\bar{0}}), e \cdot F \cdot g(x_{\bar{1}}) - F \cdot e \cdot g(x_{\bar{1}})\\
&\quad = (\tfrac{1}{2}x_0 + b - 1)(\tfrac{1}{2}x_0 - b)\sigma(g(x_{\bar{0}})) - (\tfrac{1}{2}x_0 + b)(\tfrac{1}{2}x_0 - b)\sigma(g(x_{\bar{0}}))\\
&\quad = -(\tfrac{1}{2}x_0 - b)\sigma(g(x_{\bar{0}})) = -E \cdot g(x_{\bar{1}}) = [e,F] \cdot g(x_{\bar{1}}),
\end{aligned}$$

which imply that $e \cdot F \cdot g(x_{\overline{k}}) - F \cdot e \cdot g(x_{\overline{k}}) = [e,F] \cdot g(x_{\overline{k}})$. Similar computations yield that the remaining relations hold, i.e.,

$$\begin{gathered}
2F^2 \cdot g(x_{\overline{k}}) = [F,F] \cdot g(x_{\overline{k}}), \quad f \cdot E \cdot g(x_{\overline{k}}) - E \cdot f \cdot g(x_{\overline{k}}) = [f,E] \cdot g(x_{\overline{k}}),\\
f \cdot F \cdot g(x_{\overline{k}}) - F \cdot f \cdot g(x_{\overline{k}}) = [f,F] \cdot g(x_{\overline{k}}).
\end{gathered}$$

In the same way, one can check that both $SM'_{b,0}$ and $SM'_{-3/4,1}$ are $\mathcal{S}$-modules. Here we omit the specific calculations. □

Remark 3.5. Suppose that M is one of the $\mathcal{S}$-modules determined in Definition 3.3. Applying the functor $\mathcal{F}_{\lambda,\alpha}$ to M, we obtain that $\mathcal{F}_{\lambda,\alpha}(M)$ is also an $\mathcal{S}$-module with the action $\star$. Then we consider the following action of $\mathcal{S}$ on $\mathcal{F}_{\lambda,\alpha}(M)$:

$$x_{\bar{0}} * v_{\overline{k}} = x_{\bar{0}} \star v_{\overline{k}}, \quad x_{\bar{1}} * v_{\bar{0}} = \frac{a}{\lambda} x_{\bar{1}} \star v_{\bar{0}}, \quad x_{\bar{1}} * v_{\bar{1}} = \frac{\lambda}{a} x_{\bar{1}} \star v_{\bar{1}}$$

for any $x_{\overline{k}} \in \mathcal{S}_{\overline{k}}$, $v_{\overline{k}} \in M_{\overline{k}}$, $\overline{k} \in \mathbb{Z}_2$, $a \in \mathbb{C}^*$. Evidently, $\mathcal{F}_{\lambda,\alpha}(M)$ still forms an $\mathcal{S}$-module under the action $*$, denoted $\mathcal{F}^a_{\lambda,\alpha}(M)$.

Now we present the main result of this section.

Theorem 3.6. *Any free $U(\mathfrak{h})$-module of rank 2 over $\mathcal{S}$ is isomorphic to one of the following modules:*

$$\mathcal{F}^a_{\lambda,\alpha}(SM_{b,0}),\ \mathcal{F}^a_{\lambda,\alpha}(\Pi(SM_{b,0})),\ \mathcal{F}^a_{\lambda,\alpha}(SM'_{b,0}),\ \mathcal{F}^a_{\lambda,\alpha}(\Pi(SM'_{b,0})),$$
$$\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(SM'_{b,0}),\ \mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(\Pi(SM'_{b,0})),\ \mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,1}),\ \mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,1})$$

for some $\lambda, a \in \mathbb{C}^*$ and $\alpha, b \in \mathbb{C}$.

P r o o f. Let $M = M_{\bar{0}} \oplus M_{\bar{1}}$ be any free $U(\mathfrak{h})$-module of rank 2 over $\mathcal{S}$. By Lemma 3.2, we can divide the proof into five cases.

Case 1: As $\mathcal{S}_{\bar{0}}$-modules, $M_{\bar{0}} \cong F_{\lambda,0}(M_{b,0})$, $M_{\bar{1}} \cong F_{\mu,0}(M_{b',0})$. Suppose that $E \cdot 1_{\overline{k}} = E_{\overline{k}}(x_{\overline{k+1}})$, by $[E,E] \cdot 1_{\bar{0}} = 2e \cdot 1_{\bar{0}}$ and Lemma 3.1, we have

$$E \cdot (E \cdot 1_{\bar{0}}) = \sigma(E_{\bar{0}}(x_{\bar{0}}))E_{\bar{1}}(x_{\bar{0}}) \quad \text{and} \quad E \cdot (E \cdot 1_{\bar{0}}) = e \cdot 1_{\bar{0}} = -\lambda(\tfrac{1}{2}x_{\bar{0}} - b),$$

which imply that $\sigma(E_{\bar{0}}(x_{\bar{0}}))E_{\bar{1}}(x_{\bar{0}}) = -\lambda(\frac{1}{2}x_{\bar{0}} - b)$. Hence, $E_{\bar{0}}(x_{\bar{1}}) = a$, $E_{\bar{1}}(x_{\bar{0}}) = -(\lambda/a)(\frac{1}{2}x_{\bar{0}} - b)$, or $E_{\bar{0}}(x_{\bar{1}}) = -a(\frac{1}{2}x_{\bar{1}} - b + \frac{1}{2})$, $E_{\bar{1}}(x_{\bar{0}}) = \lambda/a$ for some $a \in \mathbb{C}^*$.

Similarly, from

$$E \cdot (E \cdot 1_{\bar{1}}) = \sigma(E_{\bar{1}}(x_{\bar{1}}))E_{\bar{0}}(x_{\bar{1}}) \quad \text{and} \quad E \cdot (E \cdot 1_{\bar{1}}) = e \cdot 1_{\bar{1}} = -\mu(\tfrac{1}{2}x_{\bar{1}} - b'),$$

we obtain that $\sigma(E_{\bar{1}}(x_{\bar{1}}))E_{\bar{0}}(x_{\bar{1}}) = -\mu(\frac{1}{2}x_{\bar{1}} - b')$. On the one hand, if $E_{\bar{0}}(x_{\bar{1}}) = a$, $E_{\bar{1}}(x_{\bar{0}}) = -(\lambda/a)(\frac{1}{2}x_{\bar{0}} - b)$, then $\sigma(E_{\bar{1}}(x_{\bar{1}}))E_{\bar{0}}(x_{\bar{1}}) = -\lambda(\frac{1}{2}x_{\bar{1}} - b - \frac{1}{2})$. This implies that $\lambda = \mu$, $b' = b + \frac{1}{2}$. On the other hand, if $E_{\bar{0}}(x_{\bar{1}}) = -a(\frac{1}{2}x_{\bar{1}} - b + \frac{1}{2})$, $E_{\bar{1}}(x_{\bar{0}}) = \lambda/a$, then $\sigma(E_{\bar{1}}(x_{\bar{1}}))E_{\bar{0}}(x_{\bar{1}}) = -\lambda(\frac{1}{2}x_{\bar{1}} - b + \frac{1}{2})$. This implies that $\lambda = \mu$, $b' = b - \frac{1}{2}$.

Then if $b' = b + \frac{1}{2}$, by $F = [E,f]$, we obtain

$$\begin{aligned}
F \cdot 1_{\bar{0}} &= E \cdot f \cdot 1_{\bar{0}} - f \cdot E \cdot 1_{\bar{0}} = \frac{a}{\lambda}\Big(\frac{1}{2}x_{\bar{1}} + b - \frac{1}{2}\Big) - \frac{a}{\lambda}\Big(\frac{1}{2}x_{\bar{1}} + b + \frac{1}{2}\Big) = -\frac{a}{\lambda}, \\
F \cdot 1_{\bar{1}} &= E \cdot f \cdot 1_{\bar{1}} - f \cdot E \cdot 1_{\bar{1}} \\
&= -\frac{1}{a}\Big(\frac{1}{2}x_{\bar{0}} + b\Big)\Big(\frac{1}{2}x_{\bar{0}} - b\Big) + \frac{1}{a}\Big(\frac{1}{2}x_{\bar{0}} + b\Big)\Big(\frac{1}{2}x_{\bar{0}} - b + 1\Big) = \frac{1}{a}\Big(\frac{1}{2}x_{\bar{0}} + b\Big).
\end{aligned}$$

Similarly, if $b' = b - \frac{1}{2}$, then we have $F \cdot 1_{\bar{0}} = (a/\lambda)(\frac{1}{2}x_{\bar{1}} + b - \frac{1}{2})$ and $F \cdot 1_{\bar{1}} = -1/a$. Since the action of $\mathcal{H}_{\bar{0}}$ on $M_{\bar{0}}$ and $M_{\bar{1}}$ are zero and $[p, F] = G$, we know that $G \cdot 1_{\bar{0}} = G \cdot 1_{\bar{1}} = 0$.

Hence, by Lemma 3.1, we obtain that

$$M \cong \begin{cases} \mathcal{F}^a_{\sqrt{\lambda},\alpha}(SM_{b,0}), & \text{if } b' = b + \frac{1}{2}; \\ \mathcal{F}^a_{\sqrt{\lambda},\alpha}(\Pi(SM_{b-1/2,0})), & \text{if } b' = b - \frac{1}{2}. \end{cases}$$

Case 2: As $\mathcal{S}_{\bar{0}}$-modules, $M_{\bar{0}} \cong F_{\lambda,0}(M'_{b,0})$, $M_{\bar{1}} \cong F_{\mu,0}(M'_{b',0})$. Note that $\deg(e \cdot 1_{\overline{k}}) = 0$ and $[E, E] = 2e$, then a computation similar to the proof of Case 1 yields that $\lambda = \mu$ and $E \cdot 1_{\bar{0}} = a$, $E \cdot 1_{\bar{1}} = \lambda/a$ for some $a \in \mathbb{C}^*$. By $F = [E, f]$, we have

$$\begin{aligned} F \cdot 1_{\bar{0}} &= E \cdot f \cdot 1_{\bar{0}} - f \cdot E \cdot 1_{\bar{0}} \\ &= -\frac{a}{\lambda}\Big(\frac{1}{2}x_{\bar{1}} - b - \frac{1}{2}\Big)\Big(\frac{1}{2}x_{\bar{1}} + b + \frac{1}{2}\Big) + \frac{a}{\lambda}\Big(\frac{1}{2}x_{\bar{1}} - b'\Big)\Big(\frac{1}{2}x_{\bar{1}} + b' + 1\Big) \\ &= \frac{a}{\lambda}\Big(\frac{1}{2}x_{\bar{1}} + \Big(b + \frac{1}{2}\Big)^2 - b'(b'+1)\Big), \\ F \cdot 1_{\bar{1}} &= E \cdot f \cdot 1_{\bar{1}} - f \cdot E \cdot 1_{\bar{1}} \\ &= -\frac{1}{a}\Big(\frac{1}{2}x_{\bar{0}} - b' - \frac{1}{2}\Big)\Big(\frac{1}{2}x_{\bar{0}} + b' + \frac{1}{2}\Big) + \frac{1}{a}\Big(\frac{1}{2}x_{\bar{0}} - b\Big)\Big(\frac{1}{2}x_{\bar{0}} + b + 1\Big) \\ &= \frac{1}{a}\Big(\frac{1}{2}x_{\bar{0}} + (b' + \frac{1}{2})^2 - b(b+1)\Big). \end{aligned}$$

Then by $[F, F] = -2f$, we obtain

$$-f \cdot 1_{\bar{0}} = F \cdot F \cdot 1_{\bar{0}} = \frac{1}{\lambda}\Big(\frac{1}{2}x_{\bar{0}} + \Big(b + \frac{1}{2}\Big)^2 - b'(b'+1) + \frac{1}{2}\Big)\Big(\frac{1}{2}x_{\bar{0}} + \Big(b' + \frac{1}{2}\Big)^2 - b(b+1)\Big),$$

which implies that $b' = b + \frac{1}{2}$ or $b' = b - \frac{1}{2}$.

Therefore, by Lemma 3.1, we have that

$$M \cong \begin{cases} \mathcal{F}^a_{\sqrt{\lambda},\alpha}(SM'_{b,0}) & \text{if } b' = b + \frac{1}{2}; \\ \mathcal{F}^a_{\sqrt{\lambda},\alpha}(\Pi(SM'_{b-1/2,0})) & \text{if } b' = b - \frac{1}{2}. \end{cases}$$

Case 3: As $\mathcal{S}_{\bar{0}}$-modules, $M_{\bar{0}} \cong F_\tau \circ F_{\lambda,0}(M'_{b,0})$, $M_{\bar{1}} \cong F_\tau \circ F_{\mu,0}(M'_{b',0})$. Since τ is an involution of $U(\mathcal{S}_{\bar{0}})$, it follows that $F_\tau(M_{\bar{0}}) \cong F_{\lambda,0}(M'_{b,0})$ and $F_\tau(M_{\bar{1}}) \cong F_{\mu,0}(M'_{b,0})$ as $U(\mathcal{S}_{\bar{0}})$-modules. Note that $\mathcal{F}_\varsigma(M) = \mathcal{F}_\varsigma(M_{\bar{0}}) \oplus \mathcal{F}_\varsigma(M_{\bar{1}})$ and $\varsigma|_{\mathcal{S}_{\bar{0}}} = \tau$, so $\mathcal{F}_\varsigma(M_{\bar{0}}) = F_\tau(M_{\bar{0}})$ and $\mathcal{F}_\varsigma(M_{\bar{1}}) = F_\tau(M_{\bar{1}})$ as $\mathcal{S}_{\bar{0}}$-modules. Then by Case 2, we have that $\lambda = \mu$ and if $b' = b + \frac{1}{2}$, $\mathcal{F}_\varsigma(M) \cong \mathcal{F}^a_{\sqrt{\lambda},\alpha}(SM'_{b,0})$, or if $b' = b - \frac{1}{2}$, $\mathcal{F}_\varsigma(M) \cong \mathcal{F}^a_{\sqrt{\lambda},\alpha}(\Pi(SM'_{b-1/2,0}))$. Since ς is also an involution of $U(\mathcal{S})$, we obtain that

$$M \cong \begin{cases} \mathcal{F}_\varsigma \circ \mathcal{F}^a_{\sqrt{\lambda},\alpha}(SM'_{b,0}) & \text{if } b' = b + \frac{1}{2}; \\ \mathcal{F}_\varsigma \circ \mathcal{F}^a_{\sqrt{\lambda},\alpha}(\Pi(SM'_{b-1/2,0})) & \text{if } b' = b - \frac{1}{2}. \end{cases}$$

Case 4: As $\mathcal{S}_{\bar{0}}$-modules, $M_{\bar{0}} \cong F_{\lambda,\alpha}(M'_{-3/4,1})$, $M_{\bar{1}} \cong F_{\mu,\beta}(M'_{-3/4,1})$. In this case, we note that the actions of e, f on M are the same as in Case 2 by setting $b = -\frac{3}{4}$. Then by a similar computation, we obtain that $\lambda = \mu$ and

$$E \cdot 1_{\bar{0}} = a, \quad E \cdot 1_{\bar{1}} = \frac{\lambda}{a}, \quad F \cdot 1_{\bar{0}} = \frac{a}{\lambda}\Big(\frac{1}{2}x_{\bar{1}} + \frac{1}{4}\Big), \quad F \cdot 1_{\bar{1}} = \frac{1}{a}\Big(\frac{1}{2}x_{\bar{0}} + \frac{1}{4}\Big)$$

for some $a \in \mathbb{C}^*$. From $G = [q, E] = [p, F]$, one sees that

$$G\cdot 1_{\bar{0}} = q\cdot E\cdot 1_{\bar{0}} - E\cdot q\cdot 1_{\bar{0}} = -\frac{a\beta}{2\lambda}\Big(x_{\bar{1}}+\frac{1}{2}\Big)+\frac{a\alpha}{2\lambda}\Big(x_{\bar{1}}-\frac{1}{2}\Big) = \frac{a}{2\lambda}\Big((\alpha-\beta)x_{\bar{1}}-\frac{1}{2}(\alpha+\beta)\Big),$$

and

$$G\cdot 1_{\bar{0}} = p\cdot F\cdot 1_{\bar{0}} - F\cdot p\cdot 1_{\bar{0}} = \frac{a\beta}{\lambda}\Big(\frac{1}{2}x_{\bar{1}}-\frac{1}{4}\Big)-\frac{a\alpha}{\lambda}\Big(\frac{1}{2}x_{\bar{1}}+\frac{1}{4}\Big) = \frac{a}{2\lambda}\Big((\beta-\alpha)x_{\bar{1}}-\frac{1}{2}(\alpha+\beta)\Big).$$

This implies $\alpha - \beta = 0$. Hence,

$$G \cdot 1_{\bar{0}} = -\frac{a\alpha}{2\lambda}, \quad G \cdot 1_{\bar{1}} = -\frac{\alpha}{2a}.$$

Thus, by Lemma 3.1, we have $M \cong \mathcal{F}^a_{\sqrt{\lambda},\alpha}(SM'_{-3/4,1})$.

Case 5: As $\mathcal{S}_{\bar{0}}$-modules, $M_{\bar{0}} \cong F_\tau \circ F_{\lambda,\alpha}(M'_{-3/4,1})$, $M_{\bar{1}} \cong F_\tau \circ F_{\mu,\beta}(M'_{-3/4,1})$. Similar to the discussion in Case 3, we have that $M \cong \mathcal{F}_\varsigma \circ \mathcal{F}^a_{\sqrt{\lambda},\alpha}(SM'_{-3/4,1})$.

Therefore, the proof is completed. □

Theorem 3.7. *There exist no isomorphisms between any two modules belonging to different classes listed in Theorem* 3.6 *except for the following two pairs:*

$$(\mathcal{F}^a_{\lambda,\alpha}(SM'_{b,0}), \mathcal{F}^a_{\lambda,\alpha}(\Pi(SM'_{b,0}))), \quad (\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(SM'_{b,0}), \mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(\Pi(SM'_{b,0}))).$$

P r o o f. Suppose that M and M' are any two $\mathcal{S}$-modules which are free of rank 2 as $U(\mathfrak{h})$-modules. Let $\phi\colon M \to M'$ be an isomorphism of $\mathcal{S}$-modules. Since $\phi(g(x_{\overline{k}})) = g(x_{\overline{k}})\phi(1_{\overline{k}})$ for all $\overline{k} \in \mathbb{Z}_2$, ϕ is determined by $\phi(1_{\overline{k}})$ and the same is true for ϕ^{-1}. Since

$$1_{\overline{k}} = \phi(\phi^{-1}(1_{\overline{k}})) = \phi^{-1}(1_{\overline{k}})\phi(1_{\overline{k}}),$$

we obtain that $\phi(1_{\overline{k}}) = c_{\overline{k}} \in \mathbb{C}^*$. Thus, $\phi(g(x_{\overline{k}})) = c_{\overline{k}}g(x_{\overline{k}})$.

Then, let $E_{\overline{k}}(x_{\overline{k+1}}) = E \cdot 1_{\overline{k}} \in M$, $E'_{\overline{k}}(x_{\overline{k+1}}) = E \circ 1_{\overline{k}} \in M'$ and we have

$$\begin{aligned} c_{\overline{k}}E'_{\overline{k}}(x_{\overline{k+1}}) &= E'_{\overline{k}}(x_{\overline{k+1}})\sigma(\phi(1_{\overline{k}})) = E \circ \phi(1_{\overline{k}}) = \phi(E \cdot 1_{\overline{k}}) \\ &= \phi(E_{\overline{k}}(x_{\overline{k+1}})) = E_{\overline{k}}(x_{\overline{k+1}})\phi(1_{\overline{k}}) = c_{\overline{k}}E_{\overline{k}}(x_{\overline{k+1}}), \end{aligned}$$

which implies $E_{\overline{k}}(x_{\overline{k+1}}) = E'_{\overline{k}}(x_{\overline{k+1}})$.

Also, let

$$f_{\overline{k}}(x_{\overline{k}}) = f \cdot 1_{\overline{k}} \in M, \quad f'_{\overline{k}}(x_{\overline{k}}) = f \circ 1_{\overline{k}} \in M',$$
$$p_{\overline{k}}(x_{\overline{k}}) = p \cdot 1_{\overline{k}} \in M, \quad p'_{\overline{k}}(x_{\overline{k}}) = p \circ 1_{\overline{k}} \in M',$$

then similar computations yield that $f_{\overline{k}}(x_{\overline{k}}) = f'_{\overline{k}}(x_{\overline{k}})$ and $p_{\overline{k}}(x_{\overline{k}}) = p'_{\overline{k}}(x_{\overline{k}})$.

So two modules can be isomorphic only if

$$E \cdot 1_{\overline{k}} = E \circ 1_{\overline{k}}, \quad f \cdot 1_{\overline{k}} = f \circ 1_{\overline{k}} \quad \text{and} \quad p \cdot 1_{\overline{k}} = p \circ 1_{\overline{k}}. \tag{3.2}$$

In fact, $E \cdot 1_{\overline{k}}$, $f \cdot 1_{\overline{k}}$, $p \cdot 1_{\overline{k}}$ are determined by their leading coefficient and the sets of zeros. It is easy to see that all these conditions coincide nontrivially only in pairs

$$\begin{gathered}(\mathcal{F}^a_{\lambda,\alpha}(SM'_{b,0}), \mathcal{F}^a_{\lambda,\alpha}(\Pi(SM'_{-b-3/2,0}))),\\ (\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(SM'_{b,0}), \mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(\Pi(SM'_{-b-3/2,0})))\end{gathered} \tag{3.3}$$

for some $a \in \mathbb{C}^*$. Hence, the identities in (3.2) are impossible by Definition 3.3, if they belong to different classes in Theorem 3.6 except for the pairs given in (3.3). Whence, the proof is completed. □

Remark 3.8. It is worth mentioning that if we take $b \in \mathbb{C}_{\geqslant -3/4}$ in the definition of the module $SM'_{b,0}$, where

$$\mathbb{C}_{\geqslant -3/4} = \{c \in \mathbb{C}\colon \operatorname{Re}(c) > -\tfrac{3}{4} \text{ or } c \in -\tfrac{3}{4} + \mathbb{Z}_+\sqrt{-1}\},$$

then unless both of the pairs in (3.3) coincide trivially, precisely one of b and $-b-\frac{3}{2}$ lies in the set $\mathbb{C}_{\geqslant -3/4}$. In this case, we obtain that any two modules belonging to different classes in Theorem 3.6 are pairwise non-isomorphic.

As a direct consequence of Theorems 3.6, 3.7, and Remark 3.8, we have the following theorem.

Theorem 3.9. *All pairwise non-isomorphic free $U(\mathfrak{h})$-modules of rank 2 over $\mathcal{S}$ consist of the modules*

$$\begin{aligned}&\{\mathcal{F}^a_{\lambda,\alpha}(SM_{b,0})\colon\ \lambda, a \in \mathbb{C}^*, \alpha, b \in \mathbb{C}\}\\ &\quad\cup \{\mathcal{F}^a_{\lambda,\alpha}(\Pi(SM_{b,0}))\colon\ \lambda, a \in \mathbb{C}^*, \alpha, b \in \mathbb{C}\}\\ &\quad\cup \{\mathcal{F}^a_{\lambda,\alpha}(SM'_{b,0})\colon\ \lambda, a \in \mathbb{C}^*, \alpha \in \mathbb{C}, b \in \mathbb{C}_{\geqslant -3/4}\}\\ &\quad\cup \{\mathcal{F}^a_{\lambda,\alpha}(\Pi(SM'_{b,0}))\colon\ \lambda, a \in \mathbb{C}^*, \alpha \in \mathbb{C}, b \in \mathbb{C}_{\geqslant -3/4}\}\\ &\quad\cup \{\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(SM'_{b,0})\colon\ \lambda, a \in \mathbb{C}^*, \alpha \in \mathbb{C}, b \in \mathbb{C}_{\geqslant -3/4}\}\\ &\quad\cup \{\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(\Pi(SM'_{b,0}))\colon\ \lambda, a \in \mathbb{C}^*, \alpha \in \mathbb{C}, b \in \mathbb{C}_{\geqslant -3/4}\}\\ &\quad\cup \{\mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,1})\colon\ \lambda, a \in \mathbb{C}^*, \alpha, b \in \mathbb{C}\}\\ &\quad\cup \{\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,1})\colon\ \lambda, a \in \mathbb{C}^*, \alpha, b \in \mathbb{C}\}.\end{aligned}$$

In the following theorems, we provide sufficient and necessary conditions for the simplicity of $U(\mathcal{S})$-modules given in Theorem 3.6.

Theorem 3.10. *For all $\lambda, \alpha, a \in \mathbb{C}^*$, $b \in \mathbb{C}$,*

$$\mathcal{F}^a_{\lambda,0}(SM'_{b,0}), \quad \mathcal{F}^a_{\lambda,0}(\Pi(SM'_{b,0})), \quad \mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,0}(SM'_{b,0}),$$
$$\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,0}(\Pi(SM'_{b,0})), \quad \mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,1}), \quad \mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,1})$$

are all simple.

P r o o f. Since ς is an involution of $\mathcal{S}$, we have $\mathcal{F}_\varsigma \circ \mathcal{F}_\varsigma = \mathrm{Id}$, where Id is the identity functor of $U(\mathcal{S})$-Mod. Further, it is straightforward to check that

$$\mathcal{F}^{a^{-1}}_{\lambda^{-1},0}(\mathcal{F}^a_{\lambda,0}(SM'_{b,0})) \cong SM'_{b,0}, \quad \mathcal{F}^{a^{-1}}_{\lambda^{-1},\alpha^{-1}}(\mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,1})) \cong SM'_{-3/4,1}.$$

So it suffices to prove that $SM'_{b,0}$ and $SM'_{-3/4,1}$ are simple for all $b \in \mathbb{C}$.

Suppose $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a nonzero submodule of $SM'_{-3/4,1}$. It follows from Definition 3.3 that the module action of $\mathcal{S}_{\bar{1}}$ maps the nonzero elements of $V_{\bar{0}}$ into the nonzero elements of $V_{\bar{1}}$ and the nonzero elements of $V_{\bar{1}}$ into the nonzero elements of $V_{\bar{0}}$. Hence, $V_{\bar{0}} \neq 0$ and $V_{\bar{1}} \neq 0$. Moreover, if $V_{\bar{0}} = (SM'_{-3/4,1})_{\bar{0}}$ according to the action of E on $1_{\bar{0}}$, we have $V_{\bar{1}} = (SM'_{-3/4,1})_{\bar{1}}$, and consequently, $V = SM'_{-3/4,1}$. This implies that $SM'_{-3/4,1}$ is a simple $\mathcal{S}$-module provided that $(SM'_{-3/4,1})_{\bar{0}}$ is a simple $\mathcal{S}_{\bar{0}}$-module. Then, by Theorem 2.3, we obtain that $SM'_{-3/4,1}$ is simple.

A similar argument yields that $SM'_{b,0}$ is simple. We complete the proof. □

Theorem 3.11. *The following statements hold.*

(1) *$\mathcal{F}^a_{\lambda,\alpha}(SM_{b,0})$ and $\mathcal{F}^a_{\lambda,\alpha}(\Pi(SM_{b,0}))$ are simple if and only if $b \neq 0$.*
(2) *$\mathcal{F}^a_{\lambda,\alpha}(SM_{0,0})$ has a unique proper submodule $\mathcal{F}^a_{\lambda,\alpha}(\Gamma)$, where $\Gamma = x_{\bar{0}}(SM_{0,0})_{\bar{0}} \oplus (SM_{0,0})_{\bar{1}}$, and the corresponding simple quotient is a 1-dimensional trivial $U(\mathcal{S})$-module.*

P r o o f. First note that it is sufficient to prove the two statements for $SM_{b,0}$ using arguments as in the proof of Theorem 3.10.

Then, by Theorem 2.3, we see that $(SM_{b,0})_{\bar{0}}$ is a simple $\mathcal{S}_{\bar{0}}$-module if and only if $2b \notin \mathbb{Z}_+$. Again, by the same arguments as in the proof of Theorem 3.10, we obtain that $SM_{b,0}$ is a simple $\mathcal{S}$-module if $(SM_{b,0})_{\bar{0}}$ is a simple $\mathcal{S}_{\bar{0}}$-module, i.e., $2b \notin \mathbb{Z}_+$.

For $2b \in \mathbb{Z}_+$, we claim that $SM_{b,0}$ contains a nonzero proper submodule only if $b = 0$. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be an arbitrary nonzero proper submodule of $SM_{b,0}$. Then $V_{\bar{0}} \neq (SM_{b,0})_{\bar{0}}$, otherwise, it follows from the action of E or F on $(SM_{b,0})_{\bar{0}}$ that $V = SM_{b,0}$, which is a contraction. So $V_{\bar{0}}$ is a nonzero proper submodule of $(SM_{b,0})_{\bar{0}}$ as $\mathcal{S}_{\bar{0}}$-module.

Note that $(SM_{b,0})_{\bar{0}}$ as an $\mathcal{S}_{\bar{0}}$-module can be naturally regarded as an $\mathfrak{sl}_2$-module which is free of rank 1 as $U(\mathfrak{h})$-module. Then by [23], Theorem 12, we see that

$$V_{\bar{0}} = \mathbb{C}[x_{\bar{0}}] \prod_{j=0}^{2b} (x_{\bar{0}} + 2b - 2j).$$

Furthermore, it follows from the action of E or F on $V_{\bar{0}}$ that $V_{\bar{1}} = (SM_{b,0})_{\bar{1}}$. Since $1_{\bar{1}} \in V_{\bar{1}}$ and $E \cdot 1_{\bar{1}} = -(\frac{1}{2}x_{\bar{0}} - b)1_{\bar{0}} \in V_{\bar{0}}$, we know that $b = 0$. In conclusion, the claim follows and we have shown that $SM_{b,0}$ is simple if and only if $b \neq 0$.

Furthermore, if $b = 0$, we see that $V \cong \Gamma$ and $SM_{0,0}/\Gamma$ is a 1-dimensional trivial $U(\mathcal{S})$-module. The proof is completed. □

We complete this section with a description of free $U(\mathfrak{h})$-modules over the Lie superalgebra $\mathfrak{osp}(1|2)$.

First, we see that there do not exist $\mathbb{Z}_2$-graded $\mathfrak{osp}(1|2)$-modules which are free over $U(\mathfrak{h})$ of rank 1, since $\mathfrak{osp}(1|2) = \mathrm{span}_{\mathbb{C}}\{h, e, f, E, F\}$ is generated by odd elements $\{E, F\}$. Hence, we also consider the $\mathbb{Z}_2$-graded free $U(\mathfrak{h})$-modules of rank 2 over $\mathfrak{osp}(1|2)$.

Note that $\mathfrak{osp}(1|2)$ is a subalgebra of $\mathcal{S}$. Then any $\mathcal{S}$-module M presented in Theorem 3.6 can be naturally viewed as an $\mathfrak{osp}(1|2)$-module, denoted as $\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}M$. In this way, we obtain a lot $\mathbb{Z}_2$-graded free $U(\mathfrak{h})$-modules of rank 2 over $\mathfrak{osp}(1|2)$.

Furthermore, using $\mathfrak{osp}(1|2)_{\bar{0}} = \mathfrak{sl}_2$ and the fact that $F_{\lambda,0}(M_{b,0})$, $F_{\lambda,0}(M'_{b,0})$, and $F_\tau \circ F_{\lambda,0}(M'_{b,0})$ are all free $U(\mathfrak{h})$-modules of rank 1 over $\mathfrak{sl}_2$, similar arguments as above show that

$$\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}^a_{\lambda,\alpha}(SM_{b,0}), \quad \mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}^a_{\lambda,\alpha}(\Pi(SM_{b,0})), \quad \mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}^a_{\lambda,\alpha}(SM'_{b,0}),$$
$$\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}^a_{\lambda,\alpha}(\Pi(SM'_{b,0})), \quad \mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(SM'_{b,0}),$$
$$\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(\Pi(SM'_{b,0})),$$

are all $\mathbb{Z}_2$-graded free $U(\mathfrak{h})$-modules of rank 2 over $\mathfrak{osp}(1|2)$, where

$$\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,1}) \quad \text{and} \quad \mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,1})$$

disappear since

$$\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,1}) = \mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,0}),$$
$$\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,1}) = \mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}\mathcal{F}_\varsigma \circ \mathcal{F}^a_{\lambda,\alpha}(SM'_{-3/4,0}).$$

In the end, for any $\mathbb{Z}_2$-graded free $U(\mathfrak{h})$-modules of rank 2 over $\mathcal{S}$, say M, we illustrate a criterion for $\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}M$ to be simple.

Proposition 3.12. *Suppose that M is any $\mathcal{S}$-module which is free over $U(\mathfrak{h})$ of rank 2. Then $\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)}M$ is a simple $\mathfrak{osp}(1|2)$-module if and only if M is a simple $\mathcal{S}$-module.*

P r o o f. The necessity is obvious. For the sufficiency, since $\mathcal{F}_\varsigma$ is an auto-equivalence and $\mathcal{F}^a_{\lambda,\alpha}$ is invertible, it suffices to prove that the statement holds for the $\mathcal{S}$-modules $SM_{b,0}$ $(2b \notin \mathbb{Z}_+)$, $SM'_{b,0}$ $(b \in \mathbb{C})$, and $SM'_{-3/4,1}$.

By Definition 3.3, we note that the super Heisenberg subalgebra $\mathcal{H}$ acts trivially on $SM_{b,0}$ and $SM'_{b,0}$, so the $\mathcal{S}$-modules $SM_{b,0}$ and $SM'_{b,0}$ are actually $\mathfrak{osp}(1|2)$-modules. Then $\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)} SM_{b,0}$ and $\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)} SM'_{b,0}$ are simple $\mathfrak{osp}(1|2)$-modules. We also note that

$$\mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)} SM'_{-3/4,1} = \mathrm{Res}^{\mathcal{S}}_{\mathfrak{osp}(1|2)} SM'_{-3/4,0},$$

so it is a simple $\mathfrak{osp}(1|2)$-module. Hence, the proof is completed. □

Remark 3.13. It is worthwhile to mention that the simple modules of $\mathfrak{osp}(1|2)$ are classified in [5]. However, all the $\mathfrak{osp}(1|2)$-modules considered in [5] are not $\mathbb{Z}_2$-graded, which are thus different from the above $\mathfrak{osp}(1|2)$-modules we have concerned. Moreover, the results about $U(\mathfrak{h})$-free modules of rank 1 over $\mathfrak{osp}(1|2)$ that are not $\mathbb{Z}_2$-graded are also obtained in [9], [13].

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