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A NOTE ON LINEAR DERIVATIONS

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Abstract. At first we prove some results on a general polynomial derivation using few results of linear derivation. Then we study the ring of constants of a linear derivation for some rings. We know that any linear derivation is a nonsimple derivation. In the last section we find the smallest integer $w > 1$ such that the polynomial ring in n variables is w-differentially simple, all w derivations are nonsimple and the w derivations set contains a linear derivation.

Keywords: linear derivation; ring of constant; Fermat ring; Darboux polynomial; simple derivation

MSC 2020: 13N15

1. Introduction

Suppose R is a commutative ring with unit. An additive mapping $d\colon R \to R$ is said to be a derivation of R if it satisfies the Leibniz rule: $d(ab) = bd(a) + ad(b)$ for all a, b in R. If A is a subring of R and d is a derivation of R satisfying $d(A) = 0$, then d is called an A-derivation. The set of all derivations of R is denoted by $\mathrm{Der}(R)$, the set of all A-derivations of R by $\mathrm{Der}_A(R)$ and by R^d, the ring of constants of d, that is, $R^d = \ker(d) = \{r \in R\colon\ d(r) = 0\}$. A derivation d is locally nilpotent if for each $r \in R$ there is an integer $n \geqslant 0$ such that $d^n(r) = 0$. We denote by $\mathrm{LND}(R)$ the set of all locally nilpotent derivations of R. Let $R = k[X_1, \ldots, X_n]$ be the polynomial ring in n variables over a field k and d be a k-derivation on R. Then we say that a polynomial $f \in k[X_1, \ldots, X_n]$ is a Darboux polynomial of d if $f \neq 0$ and $d(f) = hf$ for some $h \in k[X_1, \ldots, X_n]$, and we say that h is an eigenvalue of d. If $h \in k[X_1, \ldots, X_n]$, we define $D(h) := \{f \in k[X_1, \ldots, X_n]\colon\ d(f) = hf\}$, the Darboux space corresponding to h.

Let $R = k[X_1, \ldots, X_n]$ be the polynomial ring in n variables over a field k. Suppose d is an additive map on R with $d(k) = \{0\}$ and $d(X_i) = f_i$ for $i = 1, \ldots n$.

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We say that d is linear derivation if and only if d satisfies the Leibniz rule and $f_i = \sum_{j=1}^{n} a_{ij}X_j$, where $a_{ij} \in k$ for all $i = 1, \ldots, n$. Let $Q(R) = k(X_1, \ldots, X_n)$ be the quotient field of R. Then an element φ of $k[X] \setminus k$ (or of $Q(R) \setminus k$) is said to be a polynomial (or rational) first integral of d if the following identity holds:

$$d(\varphi) = \sum_{i=1}^{n} f_i \frac{\partial \varphi}{\partial x_i} = 0.$$

Note that if $\varphi \in R$, then by using Leibniz rule it can be proved that $d(\varphi) = \sum_{i=1}^{n} (\partial\varphi/\partial X_i) d(X_i)$.

1.1. Description of the content. Suppose $R = k[X_1, \ldots, X_n]$ is a polynomial ring in n variables and $d(X_i) = f_i$, where $f_i \in R$ for $i = 1, \ldots, n$ is a derivation on R. Then it is very difficult to find the polynomial or rational first integral of d, even for $n = 2$. In [6], we study the problem for linear derivation. Using some of these results, we study in this paper the existence of polynomial or rational first integrals for general derivations.

In [8], they characterize all the linear $\mathbb{C}$-derivations of the Fermat ring over $\mathbb{C}$. They show that the Fermat ring has linear $\mathbb{C}$-derivations with trivial ring of constants. In this paper, we study all of these results on the Fermat ring over the field of characteristic p. We study the ring of constants of a linear derivation for some rings.

A ring is called w-differentially simple if it is a simple relative to a family with w derivations. If $R = k[X_1, \ldots, X_n]$, then we know that R is 1-differentially simple and $\dim(R)$-differentially simple as well. However, $n = \dim(R)$ is not necessarily the smallest w for which such a ring can be w-differentially simple, see [2]. In [1], he had asked the question: what is the smallest positive integer $w \neq 1$ such that R is w-differentially simple and all w-derivations are nonsimple and commute? In this paper, we prove what is the smallest positive integer w such that the polynomial ring in n variables is w-differentially simple, all w derivations are nonsimple and w derivations set contains a linear derivation.

2. Polynomial first integral and rational first integral

Let $R = k[X] = k[X_1, \ldots, X_n]$ be the polynomial ring in n variables over a field k of characteristic zero, and let $k(X) = k(X_1, \ldots, X_n)$ be the quotient field of R. Suppose $d(X_i) = f_i$, where $f_i \in R$ for $i = 1, \ldots, n$ is a derivation on R.

In [6], Nowicki proved the necessary and sufficient conditions on the nonexistence of polynomial and rational first integrals of R for the case when $f_1, \ldots, f_n$ are homogeneous linear polynomials.

In [7], Nowicki-Zieliński gave a criterion for nonexistence of rational first integrals of d for $n = 3$ and monomial derivations, that is, when $f_1, \ldots, f_n$ are monomial.

Suppose $f_i = \sum_{j=1}^{n} a_{ij} x_j$, where $a_{ij} \in k$. Then the following two theorems were proved by Nowicki.

Theorem 2.1 ([6], Theorem 1.1)**.** *The following conditions are equivalent:*
(1) *d does not admit any polynomial first integral.*
(2) *The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $[a_{ij}]$ are $\mathbb{Z}^+$-independent.*

Theorem 2.2 ([6], Theorem 1.2)**.** *The following conditions are equivalent:*
(1) *d does not admit any rational first integral.*
(2) *The Jordan matrix of the matrix $[a_{ij}]$ has one of the following two forms:*

(a)
$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix},$$

where the eigenvalues $\lambda_1, \ldots, \lambda_n$ are $\mathbb{Z}$-independent; or

(b)
$$\begin{bmatrix} \lambda_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \lambda_{i-1} & & & & \\ & & & \lambda_i & 1 & & \\ & & & & \lambda_{i+1} & & \\ & & & & & \ddots & \\ 0 & & & & & & \lambda_n \end{bmatrix}$$

for some $i \in \{1, \ldots, n-1\}$, where $\lambda_{i+1} = \lambda_i$ and the eigenvalues $\lambda_1, \ldots \lambda_{i-1}$, $\lambda_i, \lambda_{i+2}, \ldots, \lambda_n$ are $\mathbb{Z}$-independent.

(1) $\begin{cases} \text{Suppose } d \text{ is a derivation on } R \text{ with } d(X_i) = f_i, \text{ where } f_i = \sum_{j=1}^{n} a_{ij} x_j + g_i, \\ a_{ij} \in k, \deg(g_i) \geqslant 2 \text{ for } i = 1, \ldots, n. \end{cases}$

Theorem 2.3. *Suppose d is defined as in the above, see* (1)*. If d has a polynomial first integral and $\lambda_1, \ldots, \lambda_n$ are the eigenvalves of the matrix $[a_{ij}]$, then $\lambda_1, \ldots, \lambda_n$ are $\mathbb{Z}^+$-dependent.*

P r o o f. Assume φ is a polynomial first integral of d. Then $\varphi = \varphi_r + \varphi_{r+1} + \ldots + \varphi_k$, where φ_i is the homogeneous component of degree i of φ and $\varphi_r \neq 0$ $(r > 0)$. Suppose $\delta(x_i) = \sum_{j=1}^{n} a_{ij}x_j$ is a linear homogeneous derivative of $k[X_1, \ldots, X_n]$. Then $d = \delta + d - \delta$. Now $d(\varphi) = 0$. Note that $\delta(\varphi_r)$ is the only degree r homogeneous element of $d(\varphi)$. Hence, $\delta(\varphi_r) = 0$. So φ_r is a polynomial first integral of the linear homogeneous derivation δ. Hence, the result follows by Theorem 2.1. □

Theorem 2.4. *Suppose d is defined as in the above, see* (1). *If d has a rational first integral, then the Jordan matrix of the matrix $[a_{ij}]$ cannot be any of the following two forms:*

(a)
$$\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix},$$

where the eigenvalues $\lambda_1, \ldots, \lambda_n$ are $\mathbb{Z}$-independent; or

(b)
$$\begin{bmatrix} \lambda_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \lambda_{i-1} & & & & \\ & & & \lambda_i & 1 & & \\ & & & & \lambda_{i+1} & & \\ & & & & & \ddots & \\ 0 & & & & & & \lambda_n \end{bmatrix}$$

for some $i \in \{1, \ldots, n-1\}$, where $\lambda_{i+1} = \lambda_i$ and the eigenvalues $\lambda_1, \ldots \lambda_{i-1}, \lambda_i, \lambda_{i+2}, \ldots, \lambda_n$ are $\mathbb{Z}$-independent.

P r o o f. Suppose φ is a rational first integral of d. Suppose $\delta(x_i) = \sum_{j=1}^{n} a_{ij}x_j$ is a linear homogeneous derivative of $k[x_1, x_2, \ldots, x_n]$. Then $d = \delta + d - \delta$. Suppose

$$\varphi = \frac{\eta_m + \eta_{m+1} + \ldots + \eta_r}{\gamma_p + \gamma_{p+1} + \ldots + \gamma_k},$$

where η_i (for $i = m, m+1, \ldots, r$) is a homogeneous component of degree i and γ_j (for $j = p, p+1, \ldots, k$) is a homogeneous component of degree j. So

$$\begin{aligned} \varphi &= \frac{\eta_m + \eta_{m+1} + \ldots \eta_r}{\gamma_p(1 + \gamma_{p+1}/\gamma_p + \ldots + \gamma_k/\gamma_p)} \\ &= \Big[\frac{\eta_m}{\gamma_p} + \frac{\eta_{m+1}}{\gamma_p} + \ldots + \frac{\eta_r}{\gamma_p}\Big]\Big[1 - \Big(\frac{\gamma_{p+1}}{\gamma_p} + \ldots + \frac{\gamma_k}{\gamma_p}\Big) + \Big(\frac{\gamma_{p+1}}{\gamma_p} + \ldots + \frac{\gamma_k}{\gamma_p}\Big)^2 - \ldots\Big] \end{aligned}$$

in $k(X)$ ($\subset k((X))$). Note that η_m/γ_p is the smallest $(m-p)$ degree element of φ. Now $d(\varphi)=0$. Note that $\delta(\eta_m/\gamma_p)$ is the only smallest $m-p$ degree element of $d(\varphi)$. Hence, $\delta(\eta_m/\gamma_p)=0$. So η_m/γ_p is a rational first integral of the linear homogeneous derivation δ. Hence, the result follows by Theorem 2.2. □

Remark 2.5. The following two examples show that the above two conditions (see Theorems 2.3 and 2.4) are necessary but not sufficient.

Example 2.6. Suppose d is a $\mathbb{C}$-derivation of $\mathbb{C}[X,Y]$ defined by $d(X)=X+X^2$ and $d(Y)=-Y$. Then $\mathbb{C}[X,Y]^d=\mathbb{C}$. Suppose $\varphi\in\mathbb{C}[X,Y]^d$. Suppose $\varphi=a_s(X)Y^s+a_{s-1}(X)Y^{s-1}+\ldots+a_0X$. Now $d(\varphi)=0$. So,

$$\begin{aligned}d(\varphi)=&[a_s'(X)Y^s+a_{s-1}'(X)Y^{s-1}+\ldots+a_0'X][X+X^2]\\&-[sa_s(X)Y^s+(s-1)a_{s-1}(X)Y^{s-1}+\ldots+a_1(X)Y]=0.\end{aligned}$$

This implies that $a_s'(X)(X+X^2)=sa_s(X)$, hence $s=0$. So, $a_0'(X)(X+X^2)=0$. This implies that $a_0(X)\in\mathbb{C}$. Hence, $\mathbb{C}[X,Y]^d=\mathbb{C}$.

Example 2.7. Suppose d is $\mathbb{R}$-derivation of $\mathbb{R}[X,Y]$ defined by $d(X)=Y$ and $d(Y)=X+X^{d-1}Y$ with $d>2$. From Theorem 1.1 of [3], we know that d has no Darboux polynomial. Hence, d has no rational first rational integral. Since f/g is the first integral of d, then by [7], Proposition 2.5, f and g are the Darboux polynomials with the same co-factor.

3. Linear derivation for some rings

Now we will discuss the rings of constants of a linear derivation on a special type of rings. In [8], they study the rings of constants of linear derivations on Fermat rings over a field of characteristic zero. For positive characteristic case the following theorem holds:

Theorem 3.1. *Suppose $\mathbb{F}_P$ is a filed of characteristic P and*

$$R=\mathbb{F}_P[x_1,\ldots,x_n]=\frac{\mathbb{F}_P[X_1,\ldots,X_n]}{(X_1^{m_1}+\ldots+X_n^{m_n})}$$

(where $n\geqslant 5$, $m_i\in\mathbb{N}$, $m_i\geqslant 3$, and $(m_i,P)=1$ for all i) is a Fermat ring. Suppose d is a linear derivation on R defined by $d(x_i)=\sum\limits_{j=1}^{n}a_{ij}x_j$ for $i=1,\ldots,n$. Then $d(x_i)=\lambda_i x_i$ for some $\lambda_i\in\mathbb{F}_P$ for all i. Also, $R^d=\mathbb{F}_P[x_1^P,\ldots,x_n^P]$ if and only if $\{\lambda_1,\ldots,\lambda_n\}$ are linearly independent over $\mathbb{F}_P$. In particular, if d is a locally nilpotent derivation, then $d=0$.

P r o o f. We know that $x_1^{m_1} + \ldots x_n^{m_n} = 0$. Hence,

$$m_1 x_1^{m_1-1} d(x_1) + \ldots + m_n x_n^{m_n-1} d(x_n) = 0$$
$$\Rightarrow m_1 x_1^{m_1-1}\left(\sum_{j=1}^{n} a_{1j}x_j\right) + \ldots + m_n x_n^{m_n-1}\left(\sum_{j=1}^{n} a_{nj}x_j\right) = 0.$$

By using the fact $x_n^{m_n} = -x_1^{m_1} - \ldots - x_{n-1}^{m_{n-1}}$ we get that

$$(m_1a_{11} - m_na_{nn})x_1^{m_1} + \ldots + (m_{n-1}a_{n-1n-1} - m_na_{nn})x_{n-1}^{m_{n-1}}$$
$$+ m_1\sum_{j=2}^{n} a_{1j}x_1^{m_1-1}x_j + m_2\sum_{j=1,j\neq 2}^{n} a_{2j}x_2^{m_2-1}x_j$$
$$+ m_{n-1}\sum_{j=1,j\neq(n-1)}^{n} a_{n-1j}x_{n-1}^{m_{n-1}-1}x_j + m_n\sum_{j=1}^{n-1} a_{nj}x_n^{m_n-1}x_j = 0.$$

Since the set

$$\{x_1^{m_1}, \ldots, x_{n-1}^{m_{n-1}}\} \cup \{x_j x_i^{m_i-1} \colon 1 \leqslant i < j \leqslant n\} \cup \{x_j x_i^{m_i-1} \colon 1 \leqslant j < i \leqslant n\}$$

is linearly independent over $\mathbb{F}_P$ (as $m_i > 2$), we get that $a_{ij} = 0$ for all $i \neq j$ (as $(m_i, P) = 1$) and $m_1a_{11} = \ldots = m_{n-1}a_{n-1n-1} = m_na_{nn} = \alpha$ (say). Hence, $a_{ii} = \lambda_i$, where $\lambda_i = \alpha/m_i$.

Since $x_1^{m_1} + \ldots + x_n^{m_n}$ is quasi-homogeneous, R is a positively graded domain. Suppose $\lambda_1, \ldots, \lambda_n$ are linearly dependent over $\mathbb{F}_P$. Then there exist $r_1, \ldots, r_n \in \mathbb{F}_P$ such that $\sum\limits_{i=1}^{n} r_i\lambda_i = 0$, where at least one $r_i \neq 0$. Then $f = x_1^{r_1} \ldots x_n^{r_n} \notin \mathbb{F}_P[x_1^P, \ldots, x_n^P]$ and $d(f) = 0$. Hence, $R^d \neq \mathbb{F}_P[x_1^P, \ldots, x_n^P]$. Conversely, suppose $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbb{F}_P$. Clearly $\mathbb{F}_P[x_1^P, \ldots, x_n^P] \subset R^d$. If $d(f) = 0$, then d (each monomial of f) $= 0$. Suppose $d(x_1^{r_1}x_2^{r_2} \ldots x_n^{r_n}) = 0$, where $r_i \in \mathbb{N} \cap \{0\}$. Then

$$(\lambda_1 r_1 + \ldots + \lambda_n r_n)x_1^{r_1}x_2^{r_2} \ldots x_n^{r_n} = 0.$$

Hence, $\lambda_1 r_1 + \ldots + \lambda_n r_n = 0$, but λ_i's are linearly independent over $\mathbb{F}_P$. So $r_i = 0$ in $\mathbb{F}_P$ for all i. Hence, $x_1^{r_1}x_2^{r_2} \ldots x_n^{r_n} \in \mathbb{F}_P[x_1^P, \ldots, x_n^P]$. So $f \in \mathbb{F}_P[x_1^P, \ldots, x_n^P]$ and $\mathbb{F}_P[x_1^P, \ldots, x_n^P] = R^d$. If d is LND, then all $\lambda_i = 0$, hence $d = 0$. □

Theorem 3.2. *Suppose*

$$R = \frac{k[X_1, \dots, X_n]}{(X_1^{m_1} + \dots + X_n^{m_n} + X_1 X_2 \dots X_n)} = k[x_1, \dots, x_n],$$

where k is a field of characteristic zero, $m_i \neq 3$ for all i and $n \geqslant 5$. Suppose $d(x_i) = \sum_{j=1}^{n} a_{ij} x_j$ (where $i = 1, \dots, n$ and $a_{ij} \in R$) is a linear derivation. Then

$$d \neq 0 \quad \textit{if and only if} \quad \sum_{1 \leqslant i_1 \leqslant < i_2 < \dots < i_{n-1} \leqslant n} m_{i_1} \dots m_{i_{n-1}} - m_1 \dots m_n = 0.$$

If $d \neq 0$, then $d(x_i) = a_{ii} x_i$ for all $i = 1, 2, \dots, n$. In particular, if all $a_{ii} \neq 0$, then $R^d = k$.

P r o o f. We know that $x_1^{m_1} + \dots + x_n^{m_n} + x_1 \dots x_n = 0$. So,

$$\begin{aligned}
& m_1 x_1^{m_1 - 1} d(x_1) + \dots + m_n x_n^{m_n - 1} d(x_n) + \sum_{i=1}^{n} x_1 \dots d(x_i) \dots x_n = 0 \\
& \quad \Rightarrow m_1 x_1^{m_1 - 1}(a_{11} x_1 + \dots + a_{1n} x_n) + m_2 x_2^{m_2 - 1}(a_{21} x_1 + \dots + a_{2n} x_n) \\
& \qquad + \dots + m_n x_n^{m_n - 1}(a_{n1} x_1 + \dots + a_{nn} x_n) \\
& \qquad + \sum_{i=1}^{n} x_1 \dots (a_{i1} x_1 + \dots + a_{in} x_n) \dots x_n = 0 \\
& \quad \Rightarrow m_1 a_{11} x_1^{m_1} + m_2 a_{22} x_2^{m_2} + \dots + m_n a_{nn} x^{m_n} \\
& \qquad + m_1 a_{12} x_1^{m_1 - 1} x_2 + \dots + m_1 a_{1n} x_1^{m_1 - 1} x_n \\
& \qquad \vdots \\
& \qquad + m_n a_{n1} x_1 x_n^{m_n - 1} + \dots + m_n a_{nn-1} x_{n-1} x_n^{m_1 - 1} \\
& \qquad + \sum_{i=1}^{n} (a_{ii}) x_1 \dots x_n \\
& \qquad + a_{12} x_2^2 x_3 \dots x_n + \dots + a_{1n} x_2 \dots x_{n-1} x_n^2 \\
& \qquad \vdots \\
& \qquad + a_{n1} x_1^2 \dots x_{n-1} + \dots + a_{nn-1} x_1 \dots x_{n-1}^2 = 0.
\end{aligned}$$

Now we will use the following three conditions:

(1) $m_i \neq 0$ for $i = 1, \dots, n$.

(2) $x_1^{m_1} + \dots + x_n^{m_n} + x_1 \dots x_n = 0$.

(3) $\{x_1^2 x_2 \dots \widehat{x_i} \dots x_n \colon i = 2, \dots, n\} \cup \dots \cup \{x_1 \dots \widehat{x_i} \dots x_{n-1} x_n^2 \colon i = 1, \dots, n-1\}$ (here $\widehat{x_i}$ means omit this variable) are linearly independent over k.

Hence, we get: $a_{ij} = 0$ if $i \neq j$ and

$$m_2a_{22} - m_1a_{11} = 0, \quad m_3a_{33} - m_1a_{11} = 0, \quad \dots \quad m_na_{nn} - m_1a_{11} = 0,$$
$$\sum_{i=1}^{n} a_{ii} - m_1a_{11} = 0.$$

Hence,

$$m_1a_{11} - m_2a_{22} = 0, \quad m_1a_{11} - m_3a_{33} = 0, \quad \dots \quad m_1a_{11} - m_na_{nn} = 0,$$
$$(1 - m_1)a_{11} + \sum_{i=2}^{n} a_{ii} = 0.$$

This system of equations has a nonzero solution if and only if

$$\begin{vmatrix} (1-m_1) & 1 & 1 & \dots & 1 \\ m_1 & -m_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_1 & 0 & \dots & 0 & -m_n \end{vmatrix} = 0.$$

So, $(a_{11}, \dots, a_{nn}) \neq (0, \dots, 0)$ if and only if $\sum_{1 \leqslant i_1 \leqslant < i_2 < \dots < i_{n-1} \leqslant n} m_{i_1} \dots m_{i_{n-1}} - m_1 \dots m_n = 0$. Hence, $d \neq 0$ implies that $d(x_i) = a_{ii}x_i$ for all $i = 1, \dots, n$. Therefore, if $a_{ii} \neq 0$ for all $i = 1, \dots, n$, then $R^d = k$. □

4. Simple derivations

Suppose k is a field of characteristic zero and $R = k[X_1, \dots, X_n]$ is the polynomial ring in n variables over k. Suppose $D \subset \mathrm{Der}_k(R)$ is a nonempty family of k-derivations. An ideal I of R is called D-stable if $d(I) \subset I$ for all $d \in D$. For example, the ideals 0 and R are always D-stable. If R has no other D-stable ideal, it is called D-simple. When $D = \{d\}$, d is often called a simple derivation. Many useful properties on simple derivations are given by Nowicki in [5].

Definition 4.1. A ring is called w-differentially simple if it is a simple relative to a family with w nonsimple derivations.

Note that if $R = k[X_1, \dots, X_n]$, then we know that R is 1-differentially simple and $\dim(R)$-differentially simple as well. However, $n = \dim(R)$ is not necessarily the smallest w for which such a ring can be w-differentially simple, see [2].

Question 4.2. See [1]. What is the smallest positive integer $w \neq 1$ such that $R = k[X_1, \dots, X_n]$ is w-differentially simple and all w derivations are nonsimple and commute?

Suppose $R = k[X_1, \ldots, X_n]$ is a polynomial ring of n variables and d is a linear derivation of R defined by $d(X_i) = \sum_{j=1}^{n} a_{ij}X_j$ for $i = 1, \ldots, n$. Then $d(m) \subset m$, where $m = (X_1, \ldots, X_n)$. So, d is a nonsimple derivation.

Question 4.3. What is the smallest positive integer w such that R is w-differentially simple, all w derivations are nonsimple and w derivations set contains d?

The following lemma is easy to prove.

Lemma 4.4. *Let d be a k-derivation of $k[X_1, \ldots, X_n]$. Then:*
(1) *The product of two Darboux polynomial is again a Darboux polynomial.*
(2) *The sum of two Darboux polynomials with the same eigenvalue is again a Darboux polynomial.*
(3) *$D(\lambda_1) \cap D(\lambda_2) = \{0\}$ for $\lambda_1 \neq \lambda_2$.*

Theorem 4.5. *Suppose $R = k[X_1, \ldots, X_n]$ (k is a field of characteristic 0) and $d(X_i) = \sum_{j=1}^{n} a_{ij}X_j$ for $i = 1, \ldots, n$ is a linear derivation of R. Then $R = k[Y_1, \ldots, Y_n]$, where Y_i is a new variable and Y_i is a Darboux polynomial of d for $i = 1, \ldots, n$ if and only if $A := (a_{ij})_{n\times n}$ is a diagonalizable matrix.*

P r o o f. Suppose A is a digonalizable matrix. Then there exists $P \in GL(n, k)$ such that $P^{-1}AP = D$, where D is a diagonal matrix (says $D = (d_{11}, \ldots, d_{nn})$). Now suppose

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = P^{-1} \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}.$$

As d is a linear map, we get

$$\begin{bmatrix} d(Y_1) \\ \vdots \\ d(Y_n) \end{bmatrix} = d\left(\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}\right) = d\left(P^{-1}\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}\right) = P^{-1}d\left(\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}\right)$$
$$= P^{-1}A\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = DP^{-1}\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} = D\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}.$$

So, $d(Y_i) = d_{ii}Y_i$ for $i = 1, \ldots, n$. Hence, Y_i is a Darboux polynomial for all $i = 1, \ldots, n$. Since P is an invertible matrix, Y_i is a new variable for $i = 1, \ldots, n$. So, $R = k[Y_1, \ldots, Y_n]$.

Conversely, suppose $R = k[Y_1, \ldots, Y_n]$ and Y_i is a Darboux polynomial of d for $i = 1, \ldots, n$. So, there exists $\lambda_i \in R$ such that $d(Y_i) = \lambda_i Y_i$ for $i = 1, \ldots, n$ and there exists $P \in GL(k, n)$ such that

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = P \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}.$$

Since d is a linear derivation, $\lambda_i \in k$.

$$\begin{bmatrix} d(Y_1) \\ \vdots \\ d(Y_n) \end{bmatrix} = d\left(\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}\right) = d\left(P \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}\right) = Pd\left(\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}\right) = PA \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$

and

$$\begin{bmatrix} d(Y_1) \\ \vdots \\ d(Y_n) \end{bmatrix} = \begin{bmatrix} \lambda_1 Y_1 \\ \vdots \\ \lambda_n Y_n \end{bmatrix} = D \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = DP \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix},$$

where D is the diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_n$. Hence, we get $PA = DP$. So, $PAP^{-1} = D$. □

Theorem 4.6. *Suppose d is a linear derivation on $R := k[Y_1, \ldots, Y_n]$, where Y_i is a Darboux polynomial of d for all $i = 1, \ldots, n$ and k is a field of characteristic 0. Then $f \in R$ is a Darboux polynomial iff each monomial of f is a Darboux polynomial of the same eigenvalue λ.*

P r o o f. As d is a linear derivation, the eigenvalue of all Darboux polynomials is a scalar element. Suppose $d(Y_i) = \lambda_i Y_i$, where $\lambda_i \in k$ for $i = 1, \ldots, n$. Note that

$$d(Y_1^{i_1} \ldots Y_n^{i_n}) = \left(\sum_{j=1}^{n} i_j \lambda_j\right) Y_1^{i_1} \ldots Y_n^{i_n}.$$

So all monomials are Darboux polynomials of d.

Suppose $d(f) = \lambda f$, where $\lambda \in k$ and $f \in R$. We can write $f = f_r + \ldots + f_m$, where f_j is the homogeneous component of f of degree j for $j = r, \ldots, m$. By comparing degrees on both sides of $d(f) = \lambda f$, we get $d(f_j) = \lambda f_j$ for $j = r, \ldots, m$. The results follow easily. □

An immediate consequence of the above results is as follows.

Corollary 4.7. *Suppose d is a linear derivation on $R := k[Y_1, \ldots, Y_n]$, where Y_i is a Darboux polynomial of d for all $i = 1, \ldots, n$ and k is a field of characteristic 0. Suppose $d(Y_i) = \lambda_i Y_i$ for $i = 1, \ldots, n$. Then $D(\lambda)$ is the linear span of $Y_1^{i_1} \ldots Y_n^{i_n}$ such that $\sum_{j=1}^{n} i_j \lambda_j = \lambda$, where $D(\lambda)$ is the Darboux space corresponding to the eigenvalue λ.*

Clearly for $n = 2$ case w is 2. Now suppose $d(X_1) = a_{11}X_1 + a_{12}X_2$, $d(X_2) = a_{21}X_1 + a_{22}X_2$, where $\left|\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right| \neq 0$, is a linear derivation of $k[X_1, X_2]$. Note that if $\left|\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right| = 0$, then $d(X_1) = rd(X_2)$ for some $r \in k$ and which is not an intersecting case. Suppose $\delta = f\partial/\partial X_1 + g\partial/\partial X_2$, where $f, g \in k[X_1, X_2]$, is a nonsimple derivation of $k[X_1, X_2]$ with either $f(0,0) \neq 0$ or $g(0,0) \neq 0$. Then we will give the condition on f and g for which $k[X_1, X_2]$ is a $\{d, \delta\}$-simple.

Note that if $f(0,0) = 0$, $g(0,0) = 0$, then (X_1, X_2) is a $\{d, \delta\}$-stable.

Lemma 4.8. *Let $k[X_1, X_2]$ be a $\{d, \delta\}$-simple if and only if d and δ do not have any common Darboux polynomial.*

P r o o f. ($\Rightarrow$) Obvious.

($\Leftarrow$) From [4], we know that if $I \subset k[X_1, X_2]$ is d-stable, then all minimal prime ideals of I are d-stable. So if an ideal $I \subset k[X_1, X_2]$ is $\{d, \delta\}$-stable, then all minimal prime ideals of I are $\{d, \delta\}$-stable. Hence, it is enough to prove that no prime ideal of $k[X_1, X_2]$ is $\{d, \delta\}$-stable. The only maximal ideal of $k[X_1, X_2]$ that is d-stable is (X_1, X_2), because any maximal ideal of $k[X_1, X_2]$ looks like $(X_1 - a_1, X_2 - a_2)$ for some $a_1, a_2 \in k$. If $(X_1 - a_1, X_2 - a_2)$ is d-stable, then $d(X_1 - a_1) \in (X_1 - a_1, X_2 - a_2)$ and $d(X_2 - a_2) \in (X_1 - a_1, X_2 - a_2)$. Hence, $d(X_1) = a_{11}X_1 + a_{12}X_2 \in (X_1 - a_1, X_2 - a_2)$ and $d(X_2) = a_{21}X_1 + a_{22}X_2 \in (X_1 - a_1, X_2 - a_2)$. This implies that $a_1 = a_2 = 0$ since $\left|\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right| \neq 0$. But $\delta((X_1, X_2)) \not\subseteq (X_1, X_2)$, as otherwise $f, g \in (X_1, X_2)$ and hence $f(0,0), g(0,0) \in (X_1, X_2)$. So we need to check only for height one prime ideal. All height one prime ideals of $k[X_1, X_2]$ are principal. Suppose (p) is a height one prime ideal which is $\{d, \delta\}$-stable. Hence, $d(p) \in (p)$ and $\delta(p) \in (p)$. So p is a common Darboux polynomial of d and δ, which is a contradiction. □

Theorem 4.9. *Suppose d and δ are as defined above with the matrix $A := \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$ being diagonalizable. Then $R = k[X_1, X_2]$ is a $\{d, \delta\}$-simple.*

P r o o f. By Lemma 4.8, it is enough to prove that d and δ do not have any common Darboux polynomial. As A is a diagonalizable matrix by Theorem 4.5, we know that there exist $Y_1, Y_2 \in R$ such that $R = k[Y_1, Y_2]$, $d(Y_1) = \lambda_1 Y_1$ and $d(Y_2) = \lambda_2 Y_2$. Note that here $\left[\begin{smallmatrix} Y_1 \\ Y2 \end{smallmatrix}\right] = P \left[\begin{smallmatrix} X_1 \\ X_2 \end{smallmatrix}\right]$, where $P^{-1}AP = \left[\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right]$. Therefore, $\left[\begin{smallmatrix} \delta(Y_1) \\ \delta(Y2) \end{smallmatrix}\right] = P \left[\begin{smallmatrix} \delta(X_1) \\ \delta(X_2) \end{smallmatrix}\right] = P \left[\begin{smallmatrix} f(X_1, X_2) \\ g(X_1, X_2) \end{smallmatrix}\right]$. Corollary 4.7 implies that it is enough to prove that no monomial of $k[Y_1, Y_2]$ is a Darboux polynomial of δ. Suppose $Y_1^{i_1} Y_2^{i_2}$ is a Darboux polynomial of δ with eigenvalue $h \in R$. So,

$$\delta(Y_1^{i_1} Y_2^{i_2}) = hY_1^{i_1} Y_2^{i_2} \Rightarrow i_1 Y_1^{i_1 - 1} Y_2^{i_2} \delta(Y_1) + i_2 Y_1^{i_1} Y_2^{i_2 - 1} \delta(Y_2) = hY_1^{i_1} Y_2^{i_2}.$$

If $h \neq 0$, then the minimal degree of $i_1 Y_1^{i_1-1} Y_2^{i_2} \delta(Y_1) + i_2 Y_1^{i_1} Y_2^{i_2-1} \delta(Y_2)$ is i_1+i_2-1 and the minimal degree of $hY_1^{i_1} Y_2^{i_2}$ is at least $i_1 + i_2$, which is a contradiction. If $h = 0$, then $i_1 Y_2 \delta(Y_1) + i_2 Y_1 \delta(Y_2) = 0$, which is not possible. Hence, no monomial of $k[Y_1, Y_2]$ is a Darboux polynomial. □

Example 4.10. Suppose $R = k[X_1, X_2]$ and $d = (X_1 + X_2)\partial/\partial X_1 + 2X_2\partial/\partial X_2$, $\delta = (X_1 - 1)\partial/\partial X_1 + (X_2 - 2)\partial/\partial X_2$ are two derivations on R. Clearly, d and δ are two nonsimple derivations of R. But R is $\{d, \delta\}$-simple, as $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right]$ is a diagonalizable matrix.

Theorem 4.11. *Suppose $R = k[X_1, \ldots, X_n]$ d is a linear derivation on R defined by $d(X_i) = X_i$ for $i = 1, 2, 3$ and $\delta := (X_1 - 1)\partial/\partial X_1 + \ldots + (X_n - 1)\partial/\partial X_n$ is the derivation on R. Then R is $\{d, \delta\}$-simple.*

P r o o f. Note that

$$d(X_1^{i_1} \ldots X_n^{i_n}) = \sum_{j=1}^{n} (i_j) X_1^{i_1} X_2^{i_2} X_3^{i_3} = (\text{degree of } X_1^{i_1} \ldots X_n^{i_n}) X_1^{i_1} \ldots X_n^{i_n}. \tag{2}$$

So if f is a homogeneous polynomial of degree γ, then $d(f) = \gamma \cdot f$. Hence, f is the Darboux polynomial of d. From 2.1, we know that d has no Darboux polynomial of eigenvalue 0. Now suppose $d(f) = \lambda f$ for some $f, \lambda(\neq 0) \in R$. Then comparing the degrees on both sides, we can say that $\lambda \in k$. Equation (2) implies that f is a homogeneous polynomial of degree λ. Therefore, f is Darboux polynomial of d with eigenvalue λ if and only if f is a homogeneous polynomial of degree λ. Suppose $I \subseteq R$ with $d(I) \subseteq I$ and $I = \langle f_1, \ldots, f_r \rangle$. It is easy to prove, using equation (2), that $d(f_j) = \lambda_j f_j$ for $j = 1, \ldots, r$ and $\lambda_j \in k \setminus \{0\}$. Hence, any d-stable ideal I of R is generated by a homogeneous Darboux polynomial.

We claim that δ has no homogeneous Darboux polynomial. Suppose f is a homogeneous Darboux polynomial of δ. So $\delta(f) = \lambda f$ for some $\lambda \in R$. It is easy to see that $\lambda \neq 0$. Without loss generality we can assume that the variable X_1 appears in some of the monomials of f. Suppose minimal degree of X_1 in f is i_1. The minimal degree of X_1 in $\delta(f)$ is $i_1 - 1$ but the minimal degree of X_1 in λf is greater than or equal i_1. Hence, $\delta(f) \neq \lambda f$. Therefore, R is a $\{d, \delta\}$-simple. □

Corollary 4.12. *The answer to Question* 4.3 *is* 2.

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