CAUCHY PROBLEM WITH DENJOY-STIELTJES INTEGRAL

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Dedicated to the memory of Professor Jaroslav Kurzweil

Abstract. This work is devoted to analyzing the existence of the Cauchy fractional-type problems considering the Riemann-Liouville derivative (in the distributional Denjoy integral sense) of real order $n \geqslant 1$. These kinds of equations are a generalization of the measure differential equations. Our results extend A. A. Kilbas, H. M. Srivastava, J. J. Trujillo (2006) and H. Zhou, G. Ye, W. Liu, O. Wang (2015).

Keywords: fractional measure differential equation; Cauchy problem; Riemann-Liouville fractional integral and derivative; distributional Denjoy integral

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1. Introduction

In this paper, we study the solvability of the Cauchy-type problem for the fractional measure differential equation (FMDE)

$$\mathcal{D}_0^n x(t) = f(x(t),t) + g(x(t),t)Du, \quad \lim_{t\to 0^+} t^{m-n}x(t) = c_m, \tag{1}$$

where $t \in [0,1]$, $n \in [1,\infty)$ fixed, $m = \lceil n \rceil$ is defined as the least integer greater than or equal to n, $c_m \in \mathbb{R}$,

$$x \in C_{m-n}[0,1] = \{x : (0,1] \to \mathbb{R} \colon\ t^{m-n}x(t) \in C[0,1]\},$$

$u\colon\ [0,1] \to \mathbb{R}$ is a non-decreasing continuous function, $f\colon\ \mathbb{R}\times[0,1] \to \mathbb{R}$ is a Denjoy integrable distribution, $g\colon\ \mathbb{R}\times[0,1] \to \mathbb{R}$ is Henstock-Stieltjes (HS) integrable and

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$\mathcal{D}_0^n$ denotes the Riemann-Liouville fractional derivative of order n in the distributional sense, see below (and also [31]). Note the notation $\mathcal{D}_0^n x(t) := D^m \mathcal{J}_0^{m-n} x(t)$ means that it is a distribution that depends on the point t due to the fact that the distribution $\mathcal{J}_0^n x$ depends on the parameter t for any $n > 0$. In this sense the equation in (1) is established.

In [12], the authors generalize Carathéodory's existence theorem for the equation $x' = f(x,t)$ by using the Henstock-Kurzweil integral. If $n = 1$ and (1) is considered in the distributional sense, since Du generates a Borel measure, the problem (1) is called measure differential equation (MDE) and reads as

$$\begin{gathered} Dx(t) = f(x(t),t) + g(x(t),t)Du, \quad t \in [0,1], \\ x(0) = c_1. \end{gathered} \tag{2}$$

These kinds of equations have been studied by many authors (see [13], [14], [38], [39], [43], [47]). For example, in [47] the equivalence of MDE (2) and the integral equation

$$x(t) = c_1 + \int_0^t f(x(s),s)\,\mathrm{d}s + \int_0^t g(x(s),s)\,\mathrm{d}u(s)$$

is proved as well as the existence of at least one solution for the MDE (2) in the distributional Denjoy integral sense. In [28], the authors showed the existence of a solution of the distributional differential equation (DDE)

$$Dx(t) = g(x(t),t)Du$$

assuming that u is a regulated function. Later in 2019, the existence and uniqueness of solutions to the second-order distributional differential equations with Neumann boundary condition were analyzed via Henstock-Kurzweil-Stieltjes integrals (see [46]). Thus, as the name suggests, FMDE (1) is a generalization of a measure differential equation, see, e.g., [14].

In particular, if u is an absolutely continuous function, its distributional derivative is the usual derivative. Assuming that n is an arbitrary noninteger number bigger than one, we obtain a fractional differential equation (see [7], [20]). For the integer case, we get an ODE; see, e.g., [14].

The study of fractional integro-differential operators in the classical sense (within Lebesgue integral theory) and their applications has been developed by many authors, see, e.g., [6], [7], [15], [16], [21], [24], [26], [27], [34], [35], [37]. Here, $D_0^{n_1}$ and $J_0^{n_1}$ denote the fractional differential and integral operators in the Lebesgue sense, respectively. In [23], Kilbas et al. considered the Cauchy problem

$$\begin{gathered} D_0^{n_1} x(t) = F(x(t),t) \quad \text{a.e. on } [0,1], \\ D_0^{n_1-k} x(0^+) = b_k \quad \text{for } 1 \leqslant k \leqslant m, \end{gathered} \tag{3}$$

where n_1 is arbitrary positive, $0 \leqslant m-1 < n_1 \leqslant m$, $D_0^{n_1-m}x(t) := J_0^{m-n_1}x(t)$, $\{b_k\}_{k=1}^m \subset \mathbb{R}$ and $x(t)$, $F(x(t),t)$ belong to $L^1[0,1]$, the set of Lebesgue integrable functions. In [15], Theorem 5.1, Diethelm established the existence and uniqueness of the solution in $C(0,h]$ for (3) assuming continuity and Lipschitz conditions and proving the equivalence of (3) and the corresponding Volterra integral equation

$$x(t) = \sum_{k=1}^{m} \frac{b_k}{\Gamma(n_1-k+1)} t^{n_1-k} + J_0^{n_1} F(x(t),t) \quad \text{a.e. on } [0,1]. \tag{4}$$

In [23], the equivalence between (3) and (4) was proved assuming that x and $F(x(t),t)$ are Lebesgue integrable. Thus, the existence and uniqueness of the Cauchy problem (3) in $L^1[0,1]$ were shown under a restriction for the Lipschitz constant. Finally, in [30] the authors proved existence and uniqueness of (3) by using superposition and Lipschitz operators.

It should be noted that with the intention of solving various problems, there are new fractional derivatives combining the power law, exponential decay and Mittag-Leffler kernel; among them Liouville-Caputo, Atanga-Caputo, Atanga-Gómez and Atanga-Baleanu derivatives, see [2]–[5] and [18]. For example, in [36], a boundary value problem of fractional type (of order $1 < n \leqslant 2$) is studied, considering a pseudo fractional differential operator and vector-valued Pettis integrable functions. To the best of our knowledge, there are not enough papers concerning the fractional-order DDE, see [32].

In this paper, we show the existence of at least one solution of (1) via its integral equation

$$x(t) = \sum_{k=1}^{m} c_k t^{n-k} + \mathcal{J}_0^n f(x(t),t) + \mathcal{J}_0^n g(x(t),t)Du, \tag{5}$$

where $\{c_k\}_{k=1}^m \subset \mathbb{R}$, and by using the fractional differential and integral operators introduced in [31]. Moreover, the coefficients $\{c_k\}$ can be considered as in the expression (14). This means that $\mathcal{J}_0^n$ denotes the Riemann-Liouville fractional integral operator of order n in the distributional sense. Defining

$$\mathcal{J}_0^n g(x,\cdot)Du := \mathcal{J}_0^{n-1}\mathcal{J}_0^1 g(x,\cdot)Du,$$

by Theorem 6.5.3 in [29], we have that $J_0^1 g(x,\cdot)Du \in C[0,1]$. Thus $\mathcal{J}_0^n g(x,\cdot)Du$ is a regular distribution (meaning that there exists a locally integrable function such that generates the given distribution), in fact, $\mathcal{J}_0^n g(x,\cdot)Du$ is generated by the continuous function $J_0^1 g(x,\cdot)Du$. In this case, J_0^1 denotes the Riemann-Liouville fractional integral operator of order 1 in the Lebesgue sense.

We apply theoretical implications of the Henstock-Kurzweil theory, see [29], [41], [42]. This theory is a research topic of great interest in the scientific community because it offers certain advantages. It is related to many problems not only theoretical but also practical in Statistics, Financial Mathematics, and Particle Physics, see, e.g., [10], [25], [33].

2. Preliminaries

Now we introduce the definition of the distributional Denjoy integral. Recall that the space of Lebesgue integrable functions on (a, b) is denoted by $L^1(a, b)$; the Lebesgue integral is characterized in terms of absolutely continuous functions AC. In the case of the Henstock-Kurzweil integral, there is an analogous characterization in terms of generalized absolutely continuous functions in the restricted sense ACG_*. This means, $F \in \mathrm{ACG}_*$ if and only if there exists $f \in \mathrm{HK}[a, b]$ such that $F(x) = \int_a^x f + F(a)$, hence $F' = f$ a.e., see [19]. However, if F is any continuous function, then the generalized function and the distributional derivative are needed because there exist continuous functions differentiable nowhere. Thus, Talvila (see [41]) introduced a generalized integral called Denjoy distributional integral whose theory contains Lebesgue and Henstock-Kurzweil integrals. However, there are some references that call it Henstock-Kurzweil distributional integral, see, e.g., [45].

2.1. Denjoy distributional integral A_c. Let (a, b) be a bounded open interval in $\mathbb{R}$, we define

$$\mathcal{D}(a, b) := \{\varphi\colon\ (a, b) \to \mathbb{R}\colon\ \varphi \in C^\infty \text{ and } \varphi \text{ has a compact support in } (a, b)\}.$$

Moreover, it is said that a sequence $(\varphi_n) \subset \mathcal{D}(a, b)$ converges to $\varphi \in \mathcal{D}(a, b)$ if there is a compact set $K \subset (a, b)$ such that all φ_n have support in K and, for any integer $m \geqslant 0$, the sequence of derivatives $(\varphi_n^{(m)})$ converges to $\varphi^{(m)}$ uniformly on K, see, e.g., [22].

The dual space of $\mathcal{D}(a, b)$ is denoted by $\mathcal{D}'(a, b)$ and it is the space of continuous linear functionals on (a, b). The elements of $\mathcal{D}'(a, b)$ are distributions on (a, b). It is well known that the distributional derivative of an element $T \in \mathcal{D}'(a, b)$ is the unique distribution G that satisfies

$$\langle G, \varphi\rangle = -\langle T, \varphi'\rangle \quad \forall\, \varphi \in \mathcal{D}(a, b).$$

Let

$$C^0 := \{F \in C[a, b]\colon\ F(a) = 0\}.$$

It is well known that C^0 is a Banach space with the supremum norm, $\|F\|_\infty := \sup_{t\in[a,b]} |F(t)|$. By BV we denote the set of functions $u\colon [a,b] \to \mathbb{R}$ with a bounded variation

$$\operatorname*{var}_a^b u = \sup \sum_n |u(x_n) - u(x_{n-1})| < \infty,$$

where the supremum is taken over all partitions of $[a,b]$. Recall that, when equipped with the norm $\|u\|_{\mathrm{BV}} = \mathrm{var}_a^b\, u + |u(a)|$, BV becomes a Banach space, see [29].

We follow the notation from [41] to introduce the distributional Denjoy integral.

Definition 2.1. A distribution $f \in \mathcal{D}'(a,b)$ is said to be a Denjoy integrable distribution on $[a,b]$ if there exists a continuous function $F \in C^0$ such that $\mathrm{D}F = f$ (the distributional derivative of F is f). The distributional Denjoy integral of f on $[a,b]$ is denoted by

$$\int_a^b f := F(b) - F(a).$$

Moreover, if $f \in A_c$, then f has primitives in $C[a,b]$ differing by a constant. Nevertheless, f has exactly one primitive in C^0, see [9], Theorem 6, ii).

We set $\mathrm{D}F$ to be the distributional derivative. In other words, if $\mathrm{D}F = f$, then F is the primitive of f. The space of all the Denjoy integrable distributions on $[a,b]$ is denoted by A_c. For $f \in A_c$, we define the Alexiewicz norm

$$\|f\|_A := \|F\|_\infty,$$

where $\mathrm{D}F = f$ and $F \in C^0$. In particular, if $f \in \mathrm{HK}[a,b]$, then

$$\|f\|_A := \sup_{t\in[a,b]} \left| \int_a^t f(s)\,\mathrm{d}s \right|.$$

Lemma 2.2. *Let $f \in A_c$ and $(f_k) \subset \mathrm{HK}[a,b]$ be such that*

$$\lim_{k\to\infty} \|f_k - f\|_A \to 0.$$

Then for any $\varphi \in \mathcal{D}(a,b)$,

$$\lim_{k\to\infty} \langle f_k, \varphi\rangle = \langle f, \varphi\rangle.$$

P r o o f. Let for every $k \in \mathbb{N}$, F_k be the primitive of f_k ($F_k'(t) = f_k(t)$ a.e.). Since $(f_k) \subset \mathrm{HK}[a,b]$, then for all $\varphi \in \mathcal{D}(a,b)$

$$\langle f_k, \varphi\rangle := \int_a^b f_k(t)\varphi(t)\,\mathrm{d}t = -\int_a^b F_k(t)\varphi'(t)\,\mathrm{d}t.$$

On the other hand, since $\|f - f_k\|_A \to 0$ as $k \to \infty$, (F_k) is a Cauchy sequence in $C[a,b]$. Therefore, there exists $F \in C[a,b]$ such that $F_k(t) \to F(t)$ and for every $\varphi \in \mathcal{D}(a,b)$,

$$\lim_{k\to\infty} \langle f_k, \varphi\rangle = -\lim_{k\to\infty} \int_a^b F_k(t)\varphi'(t)\,\mathrm{d}t = -\int_a^b F(t)\varphi'(t)\,\mathrm{d}t = -\langle F, \varphi'\rangle = \langle \mathrm{D}F, \varphi\rangle.$$

Thus, (f_k) converges (weakly) to $\mathrm{D}F$ in the distributional sense. Now, by the Hölder inequality (see [41], Theorem 7),

$$\left|\int_a^b (f_k - f)\varphi(t)\,\mathrm{d}t\right| \leqslant 2\|f_k - f\|_A \|\varphi\|_{\mathrm{BV}} \quad \text{for any } \varphi \in \mathcal{D}(a,b).$$

Hence, if $f_k \to f$ in the distributional sense, then $f = \mathrm{D}F$. This completes the proof. $\square$

Remark 2.3. Note that it does not depend on the Cauchy sequence because the set of continuous functions with the uniform norm is a Banach space.

By [41], Theorems 2–3, A_c is a separable Banach space with respect to the Alexiewicz norm. On the other hand, in [8] and [9] it is shown that the completion of the Henstock-Kurzweil integrable functions space, $\widehat{\mathrm{HK}[a,b]}$, is isomorphic to A_c. Thus,

$$L^1[a,b] \subsetneq \mathrm{HK}[a,b] \subsetneq \widehat{\mathrm{HK}[a,b]} \simeq A_c,$$

where $\mathrm{HK}[a,b]$ denotes the space of Henstock-Kurzweil integrable real-valued functions on $[a,b]$. Furthermore, in [9], [41] and [45] the following result is proved.

Theorem 2.4. *A_c is isomorphic to the space C^0.*

This result inherits the partial order defined in [44], for elements $f, g \in A_c$, we say that $f \preceq g$ if and only if

$$\int_J f \leqslant \int_J g$$

for any subinterval J in $[a,b]$. In particular,

$$f \preceq g \Rightarrow \int_a^x f \leqslant \int_a^x g \quad \forall\, x \in [a,b].$$

Also, there exists a version of the Fundamental Theorem of Calculus in the distributional Denjoy integral sense.

Theorem 2.5 ([41], Theorem 4)**.**

(i) *Let $f \in A_c$ and $F(t) := \int_a^t f$. Then $F \in C^0$ and $\mathrm{D}F = f$.*

(ii) *Let $F \in C[a,b]$. Then $\int_a^t \mathrm{D}F = F(t) - F(a)$ for all $x \in [a,b]$.*

In [41], the following integration by parts result was presented.

Lemma 2.6. *Let $f \in A_c$ and $g \in \mathrm{BV}$. Put $fg = DH$, where $H(t) = F(t)g(t) - \int_a^t F(s)\,\mathrm{d}g$. Then $fg \in A_c$ and*

$$\int_a^b fg = F(b)g(b) - \int_a^b F(s)\,\mathrm{d}g.$$

In [1], a convergence theorem on A_c was proved.

Theorem 2.7. *Let (f_n) be a sequence in A_c such that $f_n \to f \in \mathcal{D}'(a,b)$. Suppose there exist $f_-, f_+ \in A_c$ satisfying $f_- \preceq f_n \preceq f_+$ for all $n \in \mathbb{N}$. Then $f \in A_c$ and $\lim\limits_{n\to\infty} \int_a^b f_n = \int_a^b f$.*

In [42], Talvila defined the convolution $f * g$ for the pair $(f,g) \in A_c \times \mathrm{BV}(\mathbb{R})$ as

$$f * g(t) = \int_{-\infty}^{\infty} (f \circ r_t)g,$$

where $r_t(s) = (t-s)$. It is clear that the convolution operator is commutative. The composition of $f \circ r_t$ for $f \in A_c$ is defined by

$$\langle f \circ r_t, \psi\rangle = \langle f, (\psi \circ r_t^{-1})/(r_t' \circ r_t^{-1})\rangle$$

for all $\psi \in \mathcal{D}(a,b)$. Clearly $r_t^{-1}(s) \neq 0$ due to $s < t$ and r_t is a bijection. The convolution $f * g$, $(f,g) \in A_c \times L^1(\mathbb{R})$ is defined as a limit, see Definition 3.2 in [42]. One can see that $f * g = g * f$. Moreover, he proved the following result.

Theorem 2.8. *If $(f,g) \in A_c \times L^1(\mathbb{R})$ then*
(i) $g * f \in A_c$,
(ii) $\|g * f\|_A \leqslant \|f\|_A \|g\|_1$, $\|g\|_1 = \int_{\mathbb{R}} |g(t)|\,\mathrm{d}t$.

2.2. Riemann-Liouville fractional integral operator on A_c. First, the set of natural numbers is denoted by $\mathbb{N} := \{1,2,\ldots\}$. In this paragraph, we recall the Riemann-Liouville fractional integral operator (fractional integral operator, for short) in the distributional Denjoy integral sense and some fundamental properties; for more details, see [31].

Definition 2.9. Let $n \in [0,\infty)$, $f \in A_c$ and

$$\varphi_n(u) := \begin{cases} u^{n-1}/\Gamma(n) & \text{if } 0 < u \leqslant b-a, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where $\Gamma\colon (0,\infty) \to \mathbb{R}$ is the Euler gamma function

$$\Gamma(s) := \int_0^{\infty} t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t.$$

Then for $n \geqslant 1$, the *Riemann-Liouville fractional integral operator of order n* is defined by

$$\mathcal{J}_a^n f(t) := \varphi_n * f(t),$$

where $f \in A_c$ and $t \in (a, b)$. For $0 < n < 1$,

$$\varphi_n * f(t) := \lim_{k \to \infty} \frac{1}{\Gamma(n)} \int_a^t (t-s)^{n-1} f_k(s) \, \mathrm{d}ts,$$

where $a \leqslant t \leqslant b$, $(f_k) \subset L^1[a, b]$ such that $\|f_k - f\|_A \to 0$ as $k \to \infty$. For $n = 0$, we set $\mathcal{J}_a^0 f := I$, the identity operator.

Remark 2.10. Note that if $n \geqslant 1$, then φ_n is increasing, non-negative and bounded on $[0, b-a]$. Thus, φ_n is a function of bounded variation on $\mathbb{R}$. When $0 < n < 1$, the function φ_n belongs to $L^1(\mathbb{R})$, but it does not have a bounded variation. Observe that if f is in $L^1[a, b]$ and $n \geqslant 0$, then $\mathcal{J}_a^n f(t) = J_a^n f(t)$ (this is, $\mathcal{J}_a^n f$ is a regular distribution induced by $J_a^n f$, the fractional integral operator of f in the Lebesgue sense), since the distributional Denjoy integral contains the Lebesgue integral. Using the Hölder inequality, [41] and [45], it is easy to see that $\mathcal{J}_a^n f$ is a temperate distribution for any $f \in A_c$ and $n \geqslant 0$ (see [17]).

Now, we recall some fundamental properties of Riemann-Liouville fractional integrals, see [31].

Theorem 2.11. *Let $n \in [0, \infty)$, $f \in A_c$ and $\mathcal{J}_a^n f(t)$ be as in Definition 2.9. Then:*

(i) $\mathcal{J}_a^n \colon A_c \to A_c$.

(ii) *$\mathcal{J}_a^n$ is a bounded linear operator with respect to the Alexiewicz norm. In other words, for $(f_k) \subset A_c$ which convergs in the Alexiewicz norm to f, we have that $(\mathcal{J}_a^n f_k)$ convergences in the Alexiewicz norm to $\mathcal{J}_a^n f$.*

(iii) *Moreover, if $n \geqslant 1$ and $(f_k) \subset L^1[a, b]$ such that $\|f_k - f\|_A \to 0$ as $k \to \infty$, then*

$$\mathcal{J}_a^n f(t) = \lim_{k \to \infty} \varphi_n * f_k(t)$$

in A_c and in $C[a, b]$.

Theorem 2.12. *Let $m, n \in [0, \infty)$ and $f \in A_c$. Then $\mathcal{J}_a^m \mathcal{J}_a^n f = \mathcal{J}_a^{m+n} f$ in A_c. Moreover, if $m \geqslant 1$ or $n \geqslant 1$, then the identity holds everywhere in $C[a, b]$.*

2.3. Riemann-Liouville fractional differential operator on A_c. Now, we consider an extension of the Riemann-Liouville differential operator (see [15], [37]) in a more general sense, this means in the distributional Denjoy integral sense; this study was developed in [31].

Definition 2.13. Let $n \in [0, \infty)$, $m := \lceil n \rceil$ and $f \in A_c$. The *Riemann-Liouville fractional differential operator of order* n is

$$\mathcal{D}_a^n f := D^m \mathcal{J}_a^{m-n} f,$$

where D^m denotes the m-fold iterates of the distributional derivative. For $n = 0$, we set $\mathcal{D}_a^0 := I$, the identity operator. Also, $\mathcal{D}_0^{-n} f := \mathcal{J}_0^n f$.

Remark 2.14. Observe that the operator $\mathcal{D}_a^n$ is well defined, since $\mathcal{J}_a^{m-n} f \in A_c$, and the distributional derivative of a distribution is a distribution, see [22]. Therefore, for any $n \in \mathbb{R}^+$,

$$\mathcal{D}_a^n \colon A_c \to \mathcal{D}'(a, b).$$

The following two results can be consulted [31].

Theorem 2.15. *Let* $n \in [0, \infty)$. *Then, for every* $f \in A_c$,

$$\mathcal{D}_a^n \mathcal{J}_a^n f = f.$$

Theorem 2.16. *Let* $n \in [0, \infty)$. *Then, for* $f \in A_c$,

$$\mathcal{J}_a^n f = D(\mathcal{J}_a^n F),$$

where $F \in C_0$ *is the primitive of* f. *In consequence, for* $j \in \mathbb{N}$ *and* $\varphi \in \mathcal{D}(a, b)$, *then*

$$\langle D^j(\mathcal{J}_a^n f), \varphi \rangle = (-1)^{j+1} \langle \mathcal{J}_a^n F, \varphi^{(j+1)} \rangle. \tag{7}$$

Proposition 2.17. *The fractional integral operator of arbitrary order* $n > 0$ *keeps the partial order in* A_c.

Proof. Let $f, g \in A_c$ such that $f \preceq g$ (meaning that $F \leqslant G$, where $\mathrm{D}F = f$ and $\mathrm{D}G = g$). Clearly,

$$\mathcal{J}_a^n F(t) \leqslant \mathcal{J}_a^n G(t) \quad \forall\, t \in [a, b].$$

Applying Theorems 2.12 and Theorem 2.16, we have $\int_a^t \mathcal{J}_0^n f \leqslant \int_a^t \mathcal{J}_0^n g$ for all $t \in [a, b]$. It implies that

$$\mathcal{J}_0^n f \preceq \mathcal{J}_0^n g.$$

□

3. Main results

Let $\gamma > 0$ and

$$C_\gamma[a,b] = \{f\colon\ (a,b] \to \mathbb{R}\colon\ (t-a)^\gamma f(t) \in C[a,b]\},$$

where the norm is given by

$$\|f\|_\gamma = \sup_{t\in[a,b]} |(t-a)^\gamma f(t)|.$$

Clearly, the space $C_\gamma[a,b]$ endowed with the norm $\|\cdot\|_\gamma$ is a Banach space. For $\gamma = 0$, $C_0[a,b] := C[a,b]$. Without loss of generality, we will consider $[a,b] = [0,1]$. Let $\{c_k\}_{k=1}^m \subset \mathbb{R}$ and

$$x_0(t) = \sum_{k=1}^m c_k t^{n-k} \quad \text{for } t \in [0,1].$$

Then $x_0 \in C_{m-n}[0,1]$. Further, for $r > 0$, set

$$B_r(x_0) = \{x \in C_{m-n}[0,1] : \|x - x_0\|_{m-n} \leqslant r\}.$$

Now we impose some assumptions on f, u and g from (1). For the following conditions, there is $r > 0$ such that:

(C1) $f(x(\cdot),\cdot)$ is in A_c for any $x \in B_r$.
(C2) $f(\cdot,t)$ is continuous for all $t \in [0,1]$.
(C3) There exist f_-, $f_+ \in A_c$ such that $f_-(\cdot) \preceq f(x,\cdot) \preceq f_+(\cdot)$.
(C4) $g(x(\cdot),\cdot)$ is Henstock-Stieljes integrable for any $x \in B_r$ with respect to u.
(C5) $g(\cdot,t)$ is continuous for all $t \in [0,1]$.
(C6) There exist Henstock-Stieljes integrable functions g_-, g_+ such that $g_-(\cdot) \leqslant g(x,\cdot) \leqslant g_+(\cdot)$.
(C7) u is a non-decreasing continuous function on $[0,1]$.

Note that these conditions are analogous to those used in [47].

Definition 3.1. A solution of the Cauchy problem (1) is a real-valued function $x \in C(0,1]$ such that there exists the finite limit

$$\lim_{t\to 0^+} t^{m-n} x(t), \quad \text{i.e., } x \in C_{m-n}[0,1],$$

and x satisfies (1).

Remark 3.2. Clearly $x \in A_c$ and $\mathcal{D}_0^n x$ is a distribution depending on t due to the fact that the distribution $\mathcal{J}_0^n x$ depends on the parameter t for any $n > 0$.

Lemma 3.3. *Let $n \in [1,\infty)$ be fixed. Assume that* (C1)–(C7) hold. Then x satisfies (5) if and only if x is a solution to the Cauchy problem (1).

P r o o f. Let $1 \leqslant n < \infty$ be fixed. Note that $\mathcal{J}_0^n f(x(t),t)$ exists by (C1), (C3) and Theorem 2.11 (iii). Since $C[0,1] \subset A_c$, $\mathcal{J}_0^n g(x(t),t)$ makes sense due to (C4), (C6), (C7) and Theorem 6.5.3 of [29].

Suppose that there exists x satisfying (5). Clearly x satisfies that

$$\lim_{t\to 0^+} t^{m-n} x(t) = c_m.$$

Applying $\mathcal{D}_0^n$ to (5), by the linearity of $\mathcal{D}_0^n$ (see [31], Theorem 4.7), Theorem 2.15 and Example 2.4 in [15] we have that

$$\mathcal{D}_0^n \mathcal{J}_0^n f(x(t),t) = f(x(t),t), \quad \mathcal{D}_0^n x_0(t) = 0.$$

Clearly if $n = 1$, the solvability of (5) implies that (1) holds. Now assume that $n > 1$. By the relation (7) proved in Theorem 2.16, for any $\varphi \in \mathcal{D}(0,1)$ we obtain

$$\begin{aligned}&\langle D^m \mathcal{J}_0^{m-n} \mathcal{J}_0^{n-1} \mathcal{J}_0^1 g(x(t),t)Du, \varphi\rangle \\ &\qquad = (-1)^{m+1} \langle \mathcal{J}_0^{m-n} \mathcal{J}_0^1 \mathcal{J}_0^{n-1} \mathcal{J}_0^1 g(x(t),t)Du, \varphi^{(m+1)}\rangle.\end{aligned}$$

By the semigroup property (Theorem 2.12), we have

$$\begin{aligned}(8)\quad &(-1)^{m+1}\langle \mathcal{J}_0^{m-n} \mathcal{J}_0^1 \mathcal{J}_0^{n-1} \mathcal{J}_0^1 g(x(t),t)Du, \varphi^{(m+1)}\rangle \\ &\qquad = (-1)^{m+1}\langle \mathcal{J}_0^m \mathcal{J}_0^1 g(x(t),t)Du, \varphi^{(m+1)}\rangle \\ &\qquad = (-1)^{m+1} \int_0^1 \mathcal{J}_0^m \mathcal{J}_0^1 g(x(t),t)Du(t)\varphi^{(m+1)}(t)\,\mathrm{d}t.\end{aligned}$$

Applying the integration by parts, i.e., Lemma 2.6, to (8), and Theorem 6.6.1 from [29] we obtain

$$\begin{aligned}(9)\quad &(-1)^{m+1} \int_0^1 \mathcal{J}_0^m \mathcal{J}_0^1 g(x(t),t)Du(t)\varphi^{(m+1)}(t)\,\mathrm{d}t \\ &\qquad = (-1)^{2m+1} \int_0^1 \mathcal{J}_0^1 g(x(t),t)Du(t)\varphi^{(1)}(t)\,\mathrm{d}t \\ &\qquad = \int_0^1 \varphi(t)\,\mathrm{d}\Big(\int_0^t g(x(s),s)\,\mathrm{d}u(s)\Big)(t) = \int_0^1 g(x(t),t)\varphi(t)\,\mathrm{d}u(t).\end{aligned}$$

Thus, by (8) and (9), we get that $\mathcal{D}_0^n \mathcal{J}_0^n g(x(t),t)Du$ is identified with $g(x(t),t)Du$. Therefore,

$$\mathcal{D}_0^n x(t) = f(x(t),t) + g(x(t),t)Du.$$

Now, assume that x is a solution to the Cauchy problem (1). By definition,

$$D^m \mathcal{J}_0^{m-n} x(t) = f(x(t),t) + g(x(t),t)Du.$$

Note that $x \in L^1(0,1)$, then by Theorem 2.16

$$D^m \mathcal{J}_0^{m-n} x = D^{m+1} \mathcal{J}_0^{m-n} X,$$

where $\mathrm{D}X = x$. Applying $\mathcal{J}_0^1$ to both sides of the equality, we have

$$D^{m-1} \mathcal{J}_0^{m-n} x(t) = D^m \mathcal{J}^{m-n} X(t) = \mathcal{J}_0^1 f(x(t),t) + \mathcal{J}_0^1 g(x(t,t)) Du. \tag{10}$$

According to (C4) and (C7), and by Theorem 6.5.3 from [29],

$$\mathcal{J}_0^1 g(x(t),t) Du(t) = \int_0^t g(x(s),s)\,\mathrm{d}u(s)$$

is continuous. Since (C1) holds, we have that $\mathcal{J}_0^1 f(x(t),t)$ is continuous, too. Therefore, $D^{k-1} \mathcal{J}_0^{m-n} x$ is induced by a continuous function for $k = 1, \dots, m$. Apply the operator to $\mathcal{J}_0^{m-1}$ to (10). Then by Theorem 2.5 (ii) and Theorem 2.12,

$$\mathcal{J}_0^{m-2} D^{m-2} (\mathcal{J}_0^{m-n} x(t) - \mathcal{J}_0^{m-n} x(0)) = \mathcal{J}_0^m (f(x(t),t) + g(x(t),t) Du).$$

One can apply Theorem 2.5 (ii) $m+2$ times to the left hand side of the above expression,

$$\mathcal{J}_0^{m-n} x(t) - \sum_{i=0}^{m-1} \mathcal{J}_0^{m-n} x(0) t^i = \mathcal{J}_0^m (f(x(t),t) + g(x(t),t) Du). \tag{11}$$

Then applying $\mathcal{D}_0^{m-n}$ to the expression (11), by Theorem 2.15, Example 2.4 in [15] and Theorem 2.12, we obtain that the integral equation (5) holds, where

$$c_k = \mathcal{D}_0^{n-k} x(0) / \Gamma(n-k+1)$$

for $k = 1, \dots, m$. □

Theorem 3.4. *Let $n \in [1, \infty)$ be a fixed number. Assume f, u, g satisfy the conditions* (C1)–(C7). Then there exists at least one solution of the Cauchy problem (1) given by (5).

P r o o f. Let $1 \leqslant n < \infty$ be fixed. Let us further fix h_1 and h_2 in $\mathbb{R}$ by

$$h_1 = \max_{t \in [0,1]} \{ |\mathcal{J}_0^n f_-(t)|, |\mathcal{J}_0^n f_+(t)| \}, \quad h_2 = \max_{t \in [0,1]} \{ |\mathcal{J}_0^n g_-(t)\,\mathrm{d}u(t)|, |\mathcal{J}_0^n g_+(t)\,\mathrm{d}u(t)| \}.$$

By the conditions (C3), (C6) and Proposition 2.17 we have

$$|\mathcal{J}_0^n f(x(t),t)| \leqslant h_1 \quad \text{and} \quad |\mathcal{J}_0^n g(x(t),t) Du(t)| \leqslant h_2.$$

Let $r = h_1 + h_2$,

$$B_r = \{ x \in C_{m-n}[0,1] \colon \|x - x_0\|_{m-n} \leqslant r \}.$$

Now we define an operator T on B_r to $C_{m-n}[0,1]$ by

$$Tx(t) = \mathcal{J}_0^n f(x(t),t) + \mathcal{J}_0^n g(x(t),t)Du + x_0(t).$$

First, we prove that $T\colon\ B_r \to B_r$.

$$\begin{aligned}\|Tx - x_0\|_{m-n} &= \sup_{t\in[0,1]} t^{m-n}|\mathcal{J}_0^n f(x(t),t) + \mathcal{J}_0^n g(x(t),t)Du| \\ &\leqslant \sup_{t\in[0,1]} t^{m-n}[|\mathcal{J}_0^n f(x(t),t)| + |\mathcal{J}_0^n g(x(t),t)Du|] \leqslant h_1 + h_2 = r.\end{aligned}$$

This implies that $T(B_r) \subset B_r$.

As a second step, we show that T is a continuous operator with respect to the norm in $C_{m-n}[0,1]$. Let $x \in B_r$ and (x_j) be a sequence in B_r such that x_j converges to x in the norm of $C_{m-n}[0,1]$. By the condition (C2), for all $t \in [0,1]$,

$$f(x_j(t),t) \to f(x(t),t) \quad \text{as } j \to \infty.$$

By (C3) and Proposition 2.17, we have for all $j \in \mathbb{N}$,

$$\mathcal{J}_0^{n-1} f_-(\cdot) \preceq \mathcal{J}_0^{n-1} f(x_j(\cdot),\cdot) \preceq \mathcal{J}_0^{n-1} f_+(\cdot).$$

Moreover, by (C1) we have $\mathcal{J}_0^{n-1} f(x_j(\cdot),\cdot), \mathcal{J}_0^{n-1} f(x(\cdot),\cdot) \in A_c$ for $j \in \mathbb{N}$. By the continuity of the fractional integral operator, Theorem 2.11,

$$\mathcal{J}_0^{n-1}(f(x_j(\cdot),\cdot) - f(x(\cdot),\cdot)) \to 0 \quad \text{as } j \to \infty.$$

Thus, applying Theorem 2.7,

$$\lim_{j\to\infty} \mathcal{J}_0^n f(x_j(\cdot),\cdot) = \mathcal{J}_0^n f(x(\cdot),\cdot).$$

Further, by Theorem 2.11 (iii), $\mathcal{J}_0^n f(x_j(\cdot),\cdot), \mathcal{J}_0^n f(x(\cdot),\cdot) \in C[0,1]$ for $j \in \mathbb{N}$.

On the other hand, by (C5),

$$g(x_j(t),t) \to g(x(t),t) \quad \text{as } j \to \infty.$$

Due to the fact that (C4), (C6), and (C7) hold, applying Theorem 6.8.11 of [29] we have

$$\lim_{j\to\infty} \int_0^t g(x_i(s),s)\,\mathrm{d}u(s) = \int_0^t g(x(s),s)\,\mathrm{d}u(s).$$

Clearly, for any $j \in \mathbb{N}$, $\mathcal{J}_0^1 g(x(\cdot),\cdot)\,\mathrm{d}u, \mathcal{J}_0^1 g(x_j(\cdot),\cdot)\,\mathrm{d}u \in C[0,1]$, by Theorem 6.5.3 of [29]. Since $L^1[1,0] \subsetneq A_c$ for any $j \in \mathbb{N}$,

$$\mathcal{J}_0^n g(x(\cdot),\cdot)\,\mathrm{d}u, \mathcal{J}_0^n g(x_j(\cdot),\cdot)\,\mathrm{d}u \in C[0,1],$$

see [15]. Again, by (C5)

$$\mathcal{J}_0^n g(x_j(\cdot),\cdot)\,\mathrm{d}u \to \mathcal{J}_0^n g(x(\cdot),\cdot)\,\mathrm{d}u$$

uniformly as $j\to\infty$. Thus, $(Tx_j(t))$ converges to $Tx(t)$ in the norm of $C_{m-n}[0,1]$.

Finally, we prove that the operator T is compact. It means, we show that there exists a compact set $K'\subset B_r$ such that $T(B_r)\subset K'$. It is clear by (C3) and (C6) that $T(B_r)$ is uniformly bounded with respect to the norm $\|\cdot\|_{m-n}$. Thus, $T'x(t) := t^{m-n}Tx(t)$ is uniformly bounded in $C[0,1]$. We show that T' is equicontinuos in $C[0,1]$. Let $\varepsilon>0$ and $0<t_1<t_2<1$,

$$\begin{aligned}
&|t_1^{m-n}\mathcal{J}_0^n f(x,\cdot)(t_1) - t_2^{m-n}\mathcal{J}_0^n f(x,\cdot)(t_2)|\\
&\qquad = \frac{1}{\Gamma(n)}\left|\int_0^{t_1} t_1^{m-n}(t_1-s)^{n-1}f(x,s)\,\mathrm{d}s - \int_0^{t_2} t_2^{m-n}(t_2-s)^{n-1}f(x,s)\,\mathrm{d}s\right|\\
&\qquad \leqslant \frac{1}{\Gamma(n)}\left|\int_0^{t_1}(t_1^{m-n}(t_1-s)^{n-1} - t_2^{m-n}(t_2-s)^{n-1})f(x,s)\,\mathrm{d}s\right|\\
&\qquad\quad + \frac{1}{\Gamma(n)}\left|\int_{t_1}^{t_2} t_2^{m-n}(t_2-s)^{n-1}f(x,s)\,\mathrm{d}s\right|.
\end{aligned}$$

Note that by the condition (C3), we have that

$$\|f(x,\cdot)\|_A \leqslant \max_{t\in[0,1]}\left\{\left|\int_0^t f_-\right|,\left|\int_0^t f_+\right|\right\}.$$

By Theorem 2.8,

$$\left|\int_{t_1}^{t_2} t_2^{m-n}(t_2-s)^{n-1}f(x,s)\,\mathrm{d}s\right| \leqslant \|f(x,\cdot)\|_A\int_{t_1}^{t_2}\tau_{t_2}^{n-1}(s)\,\mathrm{d}s.$$

Clearly, the map $t_1\to\int_{t_1}^{t_2}\tau_{t_2}^{n-1}(s)\,\mathrm{d}s$ defines a function in AC, where $\tau_{t_2}(s)=(t_2-s)$ when $1>t_2\geqslant s>0$ and zero otherwise. Then given $\varepsilon>0$, there exists δ such that if $|t_1-t_2|<\delta$, then $\left|\int_{t_1}^{t_2}\tau_{t_2}^{n-1}(s)\,\mathrm{d}s\right|<\varepsilon$.

On the other hand, applying the Hölder inequality and Theorem 7 from [42],

$$\begin{aligned}
&\left|\int_0^{t_1}(t_1^{m-n}(t_1-s)^{n-1} - t_2^{m-n}(t_2-s)^{n-1})f(x,s)\,\mathrm{d}s\right|\\
&\qquad\qquad \leqslant 2\|f(x,\cdot)\|_A\|\tau_{t_1}^{n-1}-\tau_{t_2}^{n-1}\|_{\mathrm{BV}},
\end{aligned}$$

where $\tau_{t_1}(s)$ is defined analogously to $\tau_{t_2}(s)$. Thus, there exists $\delta>0$ such that $\|\tau_{t_1}^{n-1}-\tau_{t_2}^{n-1}\|_{\mathrm{BV}}<\varepsilon$ when $|t_1-t_2|<\delta$. Applying linearity of the integral, and by

(C4), (C5), (C6), and (C7), we have

$$
\begin{aligned}
&|t_1^{m-n}\mathcal{J}_0^n g(x,\cdot)\,\mathrm{d}u(t_1) - t_2^{m-n}\mathcal{J}_0^n g(x,\cdot)\,\mathrm{d}u(t_2)| \\
&\quad\leqslant \frac{1}{\Gamma(n-1)}\biggl|\int_0^{t_1}(t_1^{m-n}(t_1-s)^{n-2} - t_2^{m-n}(t_2-s)^{n-2})\mathcal{J}_0^1 g(x,\tau)\,\mathrm{d}u(\tau)(s)\,\mathrm{d}s\biggr| \\
&\qquad + \frac{1}{\Gamma(n-1)}\biggl|\int_{t_1}^{t_2} t_2^{m-n}(\tau_2-s)^{n-2}\mathcal{J}_0^1 g(x,\tau)\,\mathrm{d}u(\tau)(s)\,\mathrm{d}s\biggr| \\
&\quad\leqslant \frac{h}{\Gamma(n-1)}\biggl(\|\tau_{t_1}^{n-2} - \tau_{t_2}^{n-2}\|_{1,[0,1]} + \int_{t_1}^{t_2}\tau_{t_2}^{n-2}(s)\,\mathrm{d}s\biggr),
\end{aligned}
$$

where

$$
h = \max_t\{|\mathcal{J}_0^1 g_-(t)\,\mathrm{d}u(t)|, |\mathcal{J}_0^1 g_+(t)\,\mathrm{d}u(t)|\}.
$$

In general, $\tau_{t_i}^{n-2} \in L^1[0,1]$ for $i = 1,2$. The map $t_1 \to \int_{t_1}^{t_2}\tau_{t_2}^{n-2}(s)\,\mathrm{d}s \in \mathrm{AC}$. And

$$
\lim_{t_2\to t_1}\tau_{t_2}^{n-2}(s) = \tau_{t_1}^{n-2}(s), \quad s \in (0,t_1].
$$

Thus, by Lebesgue's dominated convergence theorem, $\|\tau_{t_1}^{n-2} - \tau_{t_2}^{n-2}\|_{1,[0,1]} < \varepsilon$ as $|t_2 - t_1| < \delta$. Therefore, we prove that the set

$$
\{t^{m-n}\mathcal{J}_0^n g(x,\cdot)Du(t)\colon\ x \in B_r\}
$$

is equicontinuous in $C[0,1]$. Thus, we obtain that $T'(B_r)$ is equicontinuous. Applying the Arzelà-Ascoli theorem, $T'(B_r)$ is relatively compact. Hence, there exists a compact set $K \subset C[0,1]$ such that

$$
T(B_r) \subset \{t^{n-m}f\colon\ f \in K\}.
$$

We denote the set $\{t^{n-m}f\colon\ f \in K\} \cap B_r$ as K'. It is easy to show that K' is a compact set in B_r and $T(B_r) \subset K'$. Consequently, T is a compact continuous operator. By Theorem 4.1.1 from [40], there exists a fixed point x of the operator T. Applying Lemma 3.3, the Cauchy problem (1) has at least one solution given by (5). $\square$

Lemma 3.3 and Theorem 3.4 can be extended for the case $f\colon \mathbb{R}^r \times [0,1] \to \mathbb{R}^r$ and $g\colon \mathbb{R}^r \times [0,1] \to \mathbb{R}^r$, assuming that any component of $f(x(\cdot),\cdot)$ is in A_c, any component of $g(x,(\cdot),\cdot)$ is Henstock-Kurzweil integrable for any $x \in B_r$ and the corresponding suitable changes for the conditions (C1)–(C7).

Corollary 3.5. *Under the assumptions of Theorem* 3.4, *then Cauchy problem*

$$
\begin{aligned}
\mathcal{D}_0^n x(t) &= f(x(t),t) + g(x(t),t)Du, \\
\lim_{t\to 0^+}\mathcal{D}_0^{n-k}x(t) &= b_k \quad \text{for } 1 \leqslant k \leqslant m-1, \\
\lim_{t\to 0^+}\mathcal{J}_0^{m-n}x(t) &= b_m
\end{aligned}
\tag{12}
$$

has at least one solution given by the integral equation

$$x(t)=\sum_{k=1}^{m}\frac{b_k}{\Gamma(n-k+1)}t^{n-k}+\mathcal{J}_0^n f(x(t),t)+\mathcal{J}_0^n g(x(t),t)Du. \tag{13}$$

R e m a r k 3.6. Note that if the equation (13) holds, then $x\in C_{m-n}[0,1]$ under the assumptions of Theorem 3.4.

P r o o f. By arguments similar to those of the first part in the proof of Lemma 3.3, one can easily verify that (13) is a solution to the Cauchy problem (12). Thus, it is enough to show that (13) holds.

By Theorem 3.4, the integral equation (5) holds. Applying $\mathcal{D}_0^{n-k}$ to (5), by Example 2.4 from [15], Theorem 2.12, Theorem 2.15, and Theorem 2.16, we have that for $k\in\{1,2,3,\dots,m-1\}$,

$$\mathcal{D}_0^{n-k}x(t)=\sum_{i=1}^{k}c_i\frac{\Gamma(n-k+1)}{\Gamma(k-i+1)}t^{k-i}+\mathcal{J}_0^k f(x(t),t)+\mathcal{J}_0^k g(x(t),t)Du.$$

By Theorem 2.11 (iii) and Theorem 6.5.3 in [29], we have that $\mathcal{J}_0^k f(x(t),t)$ and $\mathcal{J}_0^k g(x(t),t)Du$ are continuous for any $k\in\{1,2,3,\dots,m-1\}$, respectively. Thus,

$$\lim_{t\to0^+}\mathcal{D}_0^{n-k}x(t)=c_k\Gamma(n-k+1). \tag{14}$$

Then $b_k=c_k\Gamma(n-k+1)$ for $k\in\{1,2,3,\dots,m-1\}$.

Similarly, applying the fractional operator $\mathcal{J}_0^{m-n}$ to (5), by Example 2.4 in [15], Theorem 2.12 and Theorem 2.16, we have

$$\mathcal{J}_0^{m-n}x(t)=\sum_{i=1}^{m}c_i\frac{\Gamma(n-i+1)}{\Gamma(m-i+1)}t^{m-i}+\mathcal{J}_0^m f(x(t),(t))+\mathcal{J}_0^m g(x(t),t)Du.$$

Thus,

$$\lim_{t\to0^+}\mathcal{J}_0^{m-n}x(t)=c_m\Gamma(n-m+1).$$

Since $c_m=\lim\limits_{t\to0^+}t^{m-n}x(t)$ exists, the integral equation (13) holds and is a solution of (12). □

R e m a r k 3.7. For the case $n=1$, our results (Lemma 3.3 and Theorem 3.4) are particular cases of Lemma 3.1 and Theorem 3.3 in [47], respectively. However, Lemma 3.3 and Theorem 3.4 proved here generalize the order of the Cauchy problem (1) to any arbitrary real number $n\geqslant1$. In particular, for the integer case $n\in\mathbb{N}$, a solution x of (1) is given by (5) belonging to $C[0,1]$ and $x(0)=c_n$. Finally, Corollary 3.5 is a generalization of [24], Theorem 3.3 where the existence of the Cauchy problem (3) has been proved in the Lebesgue integral sense, without considering the extra term $g(x(t),t)Du$.

Finally, we present an example to illustrate to scope of our results.

Example 3.8. Let $n \in [1, \infty)$ be a fixed number. We consider the Cauchy problem

$$\mathcal{D}_0^n x(t) = w + t^{m-n} x(t) + \frac{t^{m-n}}{t+1} x(t) Du, \quad \lim_{t \to 0^+} t^{m-n} x(t) = c_m, \tag{15}$$

and $t \in [0,1]$, where $u(t) = C(t)$, C denotes the Cantor ternary function on $[0,1]$, see [11]. Clearly,

$$u \in C[0,1] \setminus \mathrm{AC}[0,1]$$

and u is non-decreasing, thus (C7) holds. Observe that $\mathcal{D}_0^n x$ is a distribution that depends on x and the point t.

The function w is the distributional derivative of the Weierstrass function

$$W(t) = \sum_{n=1}^{\infty} \sin(n^2 \pi t)/n^2.$$

Since $W \in C[0,1]$ but is differentiable nowhere on $[0,1]$, $w \in A_c$. For $n = 1$ the Cauchy problem (15) is a particular case of the equation (1.1) in [47]. If $n > 1$, then (15) can be considered as a particular case of FMDEs. In fact, $\|t^{m-n} x(t)\|_\infty \leqslant M$ and $w \in A_c$, then $w \pm M \in A_c$ as well. We set

$$f(x(t), t) = w + t^{m-n} x(t) \quad \text{and} \quad g(x(t), t) = \frac{t^{m-n}}{t+1} x(t).$$

Thus,

$$w - M \preceq f(x,t) \preceq w + M, \quad t \in [0,1],$$

and (C1)–(C3) hold.

Clearly, $g(x(t), t)$ is continuous with respect to x and t, hence, (C4)–(C6) hold. Then applying Theorem 3.4, the Cauchy problem (15) has at least one solution given by

$$x(t) = x_0(t) + \mathcal{J}_0^n [w + (\cdot)^{m-n} x(\cdot)](t) + \mathcal{J}_0^n \Big[\frac{(\cdot)^{m-n}}{(\cdot)+1} x(\cdot) Du \Big](t),$$

where $x \in C_{m-n}[0,1]$. For the case $n \in \mathbb{N}$, clearly $x \in C[0,1]$.

Perspectives.

(1) An essential issue to analyse is the uniqueness of the Cauchy problems (1) and (12).

(2) It is an open problem to extend Lemma 3.3 and Theorem 3.4 for the Cauchy problem (1) of order $0 < n < 1$.

More precisely, for the point (2) a mathematical meaning to the fractional integral of $g(x(\cdot),\cdot)Du$ of order n, $\mathcal{J}_0^n g(x(\cdot),\cdot)Du(t)$ is needed. For this purpose, for example, we require that for any $t\in[0,1]$ and $x\in B_r$, the function

$$((t-s)^+)^{n-1}g(x(s),s)$$

depending on s be Henstock-Stieltjes integrable with respect to $u\in\mathrm{BV}$ on $[0,1]$. To the best of our knowledge, one of the more general results to prove the existence of the Henstock-Stieltjes integral is Theorem 6.3.11 of [29]. According to this result, since $u\in\mathrm{BV}$, the function $((t-s)^+)^{n-1}g(x(s),s)$ must be regulated (it means, its left and right limits exist for all $s\in[0,1]$). Nevertheless, it might have a singularity.

Another approach to solve (2) is the following. Let us assume that the Lebesgue-Stieltjes integral

$$\int_{(0,1)}((t-s)^+)^{n-1}g(x(s),s)\,\mathrm{d}_{\mu_u}$$

exists, where d_{μ_u} is the generated measure given by u. By Theorem 6.12.3 of [29],

$$\mathcal{J}_0^n g(x,\cdot)Du(t)=\frac{1}{\Gamma(n)}\int_{(0,1)}((t-s)^+)^{n-1}g(x(s),s)\,\mathrm{d}_{\mu_u}.$$

However, the conditions (C4) and (C6) for g have to change according to Lebesgue integral theory.

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