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GEOMETRIC APPROACHES TO ESTABLISH THE FUNDAMENTALS OF LORENTZ SPACES $\mathbb{R}_2^3$ AND $\mathbb{R}_1^2$

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Abstract. The aim of this paper is to investigate the orthogonality of vectors to each other and the Gram-Schmidt method in the Minkowski space $\mathbb{R}_2^3$. Hyperbolic cosine formulas are given for all triangle types in the Minkowski plane $\mathbb{R}_1^2$. Moreover, the Pedoe inequality is explained for each type of triangle with the help of hyperbolic cosine formulas. Thus, the Pedoe inequality allowed us to establish a connection between two similar triangles in the Minkowski plane. In the continuation of the study, the rotation matrix that provides both point and axis rotation in the Minkowski plane is obtained by using the Lorentz matrix multiplication. Also, it is stated to be an orthogonal matrix. Moreover, the orthogonal projection formulas on the spacelike and timelike lines are given in the Minkowski plane. In addition, the distances of any point from the spacelike or timelike line are formulated.

Keywords: Gram-Schmidt method; Lorentz triangle; hyperbolic cosine formulas; Pedoe inequality; Lorentz matrix multiplication; orthogonal projection

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1. Introduction

The angle definition between any two timelike vectors, timelike-pure triangle, area information of timelike-pure triangle, and hyperbolic cosine rules are given in [4]. In addition, some properties related to hyperbolic angles are examined in [1], [3]. This properties for all vector types are detailed and regulated in [15]. The information of the area spacelike-pure triangle and the non-pure triangle is given in [10]. This resource also includes hyperbolic cosine rules for spacelike pure triangles and some non-pure triangle types. The angle measure and some properties of the angle are

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expressed in [9]. This study is given for all possible non-pure triangle types, and hyperbolic cosine formulas are found for these triangles.

Moreover, in the Minkowski plane, only spacelike and timelike vector can be perpendicular to each other (see [4]). The following conditions apply for orthogonality between vectors in the Minkowski space $\mathbb{R}^3_1$:

- ▷ A spacelike vector can be perpendicular to a null vector, a spacelike vector, and a timelike vector.
- ▷ A timelike vector can be orthogonal only to a spacelike vector.
- ▷ A null vector can be orthogonal only to a spacelike vector. Also, two null vectors can be orthogonal to each other only if they are linearly dependent (see [7], [13]).

The Gram-Schmidt method is expressed with orthogonality information between vectors in the Minkowski space $\mathbb{R}^3_1$ (see [13]). In this article, it is examined which vectors can be perpendicular to each other in the Minkowski space $\mathbb{R}^3_2$. After that, the Gram-Schmidt method is applied in the Minkowski space $\mathbb{R}^3_2$.

Another title of the study is the Pedoe inequality. The Pedoe inequality in Euclidean space is given with the similarity condition of the two triangles in [8], [12], [14]. In this article, the Pedoe inequality is found for spacelike-pure, timelike-pure and all non-pure triangle types.

In addition, the rotation matrix is given in the Minkowski plane in [10]. This rotation matrix that provides both axis and point rotation is obtained by the Euclidean inner product. An orthogonal matrix must satisfy the condition $\mathbf{A}^{-1} = \varepsilon \mathbf{A}^t \varepsilon$ in the Minkowski plane (see [11]). So, this rotation matrix is the Lorentz orthogonal matrix. In this article, the rotation matrix is obtained using the Lorentz inner product instead of Euclidean inner product. The orthogonal matrix obtained by the Lorentz matrix multiplication satisfies the condition $\mathbf{A}_L^{-1} = \mathbf{A}_L^t$ (see [5]). Therefore, it is stated that the rotation matrix obtained by using the Lorentz inner product is orthogonal in this study.

Moreover, timelike and spacelike line equations are given with Hesse coordinates in [2] such that

$$L_{\mathrm{t}} \equiv x \cosh A_{\mathrm{SS}} - y \sinh A_{\mathrm{SS}} - c = 0, \quad L_{\mathrm{s}} \equiv x \sinh A_{\mathrm{ST}} - y \cosh A_{\mathrm{ST}} - c = 0$$

for $A_{\mathrm{SS}} = \angle(\overrightarrow{AB}, \overrightarrow{AC}) = A_{\mathrm{ST}}$ and $A_{\mathrm{SS}}, A_{\mathrm{ST}} \in \mathbb{R}$, where the point c represents the distance from the origin in Minkowski plane $\mathbb{R}^2_1$. Additionally, in [16], Cauchy-length formulas to the envelope of a family of lines and the Holditch-type theorems for the length of the enveloping trajectories are given. In this paper, the orthogonal projection formulas on the line are examined. The distance between line and point is formulated for timelike and spacelike lines.

2. Fundamental notations

A space $\mathbb{R}^n$ is called the semi-Euclidean space with the scalar product function

$$\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{n-\nu} x_i y_i - \sum_{i=n-\nu+1}^{n} x_i y_i$$

where $\vec{x} = (x_i)$, $\vec{y} = (y_i) \in \mathbb{R}^n$, $1 \leqslant i \leqslant n$ and ν is an integer with $0 \leqslant \nu \leqslant n$. It is denoted by $\mathbb{R}^n_\nu = (\mathbb{R}^n, \langle \cdot, \cdot \rangle_\nu)$ (see [11]).

Let $\vec{x} \in \mathbb{R}^2_1$ then $\vec{x}$ is called the spacelike vector if $\langle \vec{x}, \vec{x} \rangle > 0$ or $\vec{x} = 0$, $\vec{x}$ is called the timelike vector if $\langle \vec{x}, \vec{x} \rangle < 0$, $\vec{x}$ is called the null vector if $\langle \vec{x}, \vec{x} \rangle = 0$ and $\vec{x} \neq 0$ (see [4], [11], [13]). The norm of $\vec{x} \in \mathbb{R}^2_1$ is defined as $\|\vec{x}\| = \sqrt{|\langle \vec{x}, \vec{x} \rangle|}$. If $\langle \vec{x}, \vec{y} \rangle = 0$, the vectors $\vec{x}$ and $\vec{y}$ are called perpendicular in the Lorentz sense. Let $\vec{x} = (x_1, x_2) \in \mathbb{R}^2_1$ be a timelike vector and $\vec{e} = (0, 1)$. Then, $\vec{x}$ is a future pointing (positive) timelike vector if $\langle \vec{x}, \vec{e} \rangle < 0$, $\vec{x}$ is a past pointing (negative) timelike vector if $\langle \vec{x}, \vec{e} \rangle > 0$, see [4].

Let $\vec{x} = (x_1, x_2) \in \mathbb{R}^2_1$ be a spacelike vector and $\vec{E} = (1, 0)$. Thus, $\vec{x}$ is called the vector oriented in the same direction with $\vec{E}$, if $\langle \vec{x}, \vec{E} \rangle > 0$ and $\vec{x}$ is called the vector oriented in the opposite direction with $\vec{E}$, if $\langle \vec{x}, \vec{E} \rangle < 0$. Let $\vec{x}, \vec{y} \in \mathbb{R}^2_1$ be future pointing (past pointing) timelike vectors with $\operatorname{sgn} x_2 = \operatorname{sgn} y_2$. Then, $\vec{x} + \vec{y}$ is the future pointing (past pointing) timelike vector (see [4]). Let $\vec{x}, \vec{y} \in \mathbb{R}^2_1$ be spacelike vectors oriented in the same (opposite) direction with $\vec{E} = (1, 0)$ and $\operatorname{sgn} x_1 = \operatorname{sgn} y_1$ ($\operatorname{sgn} x_1 \neq \operatorname{sgn} y_1$). Then, $\vec{x} + \vec{y}$ is a spacelike vector oriented in the same (opposite) direction with $\vec{E} = (1, 0)$.

Theorem 2.1 ([10]). *Let $\vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2) \in \mathbb{R}^2_1$ be spacelike and timelike unit vectors with $\operatorname{sgn} x_1 = \operatorname{sgn} y_2$, respectively. Then, $\langle \vec{x}, \vec{y} \rangle = \sinh \theta$ where θ be the oriented angle from $\vec{x}$ to $\vec{y}$.*

Theorem 2.2 ([11]). *Let $\vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2) \in \mathbb{R}^2_1$ be timelike unit vectors. Then, $\langle \vec{x}, \vec{y} \rangle = \cosh \theta$ where θ is the oriented angle from $\vec{x}$ to $\vec{y}$.*

Theorem 2.3 ([10]). *Let $\vec{x} = (x_1, x_2), \vec{y} = (y_1, y_2) \in \mathbb{R}^2_1$ spacelike and timelike unit vectors with $\operatorname{sgn} x_1 \neq \operatorname{sgn} y_2$, respectively. Then, $\langle \vec{x}, \vec{y} \rangle = -\sinh \theta$ where θ is the oriented angle from $\vec{x}$ to $\vec{y}$.*

Definition 2.1 ([4]). Let A, B, C be three non-collinear points. The triangle $\overset{\triangle}{ABC}$ is called a timelike pure triangle such that $\overrightarrow{AB}$, $\overrightarrow{BC}$ are future pointing timelike vectors in the Minkowski plane $\mathbb{R}^2_1$.

Definition 2.2. Let A, B, C be three non-collinear points such that $\overrightarrow{AB}$, $\overrightarrow{BC}$ be spacelike vectors oriented in the same (different) direction with the vector $\vec{E}=(1,0)$ and $\overrightarrow{AC}$ be spacelike vectors oriented in the same (different) direction with the vector $\vec{E}=(1,0)$. In this way, the triangle $\overset{\triangle}{ABC}$ is called a spacelike pure triangle. Triangles other than the pure timelike triangle and the spacelike pure triangle are called non-pure triangles.

In this study, the following abbreviations are used to understand between which vectors are the angles:

T for a future pointing timelike vector,
t for a past pointing timelike vector,
S for a spacelike vector oriented in the same direction with $\vec{E}=(1,0)$,
s for a spacelike vector oriented in the different direction with $\vec{E}=(1,0)$.

Definition 2.3 ([4]). Let A, B, C be three non-collinear points and $A_{\mathrm{TT}} = \angle(\overrightarrow{AB}, \overrightarrow{AC})$. The area of the timelike pure triangle $\overset{\triangle}{ABC}$ is denoted by

$$S_{\mathrm{TT}} = \frac{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\| \sinh A_{\mathrm{TT}}}{2}.$$

Definition 2.4 ([10]). Let $\overrightarrow{AB}$, $\overrightarrow{AC}$ be two linearly independent spacelike vectors. The area of the spacelike pure triangle $\overset{\triangle}{ABC}$ is denoted by

$$S_{\mathrm{SS}} = \frac{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\| \sinh A_{\mathrm{SS}}}{2}$$

where $A_{\mathrm{SS}} = \angle(\overrightarrow{AB}, \overrightarrow{AC})$.

Definition 2.5 ([10]). Let $\overrightarrow{AB}$ be a spacelike vector and $\overrightarrow{AC}$ a timelike vector. The area of the non-pure triangle $\overset{\triangle}{ABC}$ is denoted by

$$S_{\mathrm{ST}} = \frac{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\| \cosh A_{\mathrm{ST}}}{2}$$

where $A_{\mathrm{ST}} = \angle(\overrightarrow{AB}, \overrightarrow{AC})$.

Definition 2.6 ([11]). The matrix that provides both the point rotation and axis rotation in the Lorentz sense is called the Lorentz rotation matrix such that $\mathbf{A}(\theta) = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$, $\theta \in \mathbb{R}$, in the Minkowski plane $\mathbb{R}^2_1$. The Lorentz rotation matrix is provided as $\mathbf{A}^{-1} = \varepsilon \mathbf{A}^t \varepsilon$ such that $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. So, the Lorentz rotation matrix $\mathbf{A}$ is orthogonal.

Definition 2.7. The equation of the timelike line passing through any point $P = (x_0, y_0)$ and perpendicular to the normal $\vec{n} = (a, b)$ is obtained as $\langle \overrightarrow{PX}, \vec{n} \rangle = ax - by + c = 0$ where $c = -(ax_0 - by_0)$ and $X = (x, y)$ is a representative point on the line.

Definition 2.8. The equation of the spacelike line passing through any point $P = (x_0, y_0)$ and perpendicular to the normal $\vec{n} = (b, a)$ is obtained as $\langle \overrightarrow{PX}, \vec{n} \rangle = bx - ay + c = 0$ where $c = -(bx_0 - ay_0)$ and $X = (x, y)$ is a representative point on the line.

3. On $\mathbb{R}^3_2$

3.1. Orthogonality in the Minkowski space $\mathbb{R}^3_2$.

Theorem 3.1. *Two timelike vectors can be orthogonal in the Minkowski space $\mathbb{R}^3_2$.*

Proof. Let $\vec{x} = (x_1, x_2, x_3), \vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_2$ be two timelike vectors. In that case, $\langle \vec{x}, \vec{x} \rangle = x_1^2 - x_2^2 - x_3^2 < 0$. Thus,

$$x_1^2 < x_2^2 + x_3^2 \tag{3.1}$$

and $\langle \vec{y}, \vec{y} \rangle = y_1^2 - y_2^2 - y_3^2 < 0$. So, we can write

$$y_1^2 < y_2^2 + y_3^2. \tag{3.2}$$

Let us assume that $\vec{x} \perp \vec{y}$, namely $\langle \vec{x}, \vec{y} \rangle = x_1y_1 - x_2y_2 - x_3y_3 = 0$. From here, we obtain

$$x_1y_1 = x_2y_2 + x_3y_3. \tag{3.3}$$

If the equations (3.1) and (3.2) are multiplied, we find $x_1^2y_1^2 < x_2^2y_2^2 + x_2^2y_3^2 + x_3^2y_2^2 + x_3^2y_3^2$. Also, if the equation (3.3) is squared and substituted into the last equation, we have

$$(x_3y_2 - x_2y_3)(x_2y_3 - x_2y_1) = -(x_3y_2 - x_2y_3)^2 < 0.$$

In this way, there is no contradiction. Therefore, it is shown that two timelike vectors can be perpendicular to each other in the Minkowski space $\mathbb{R}^3_2$. □

Example 3.1. Let $\vec{x} = (1, 1, 1)$, $\vec{y} = (1, -2, 3)$ be two timelike vectors in the Minkowski space $\mathbb{R}^3_2$. Then,

$$\langle \vec{x}, \vec{y} \rangle = \langle (1, 1, 1), (1, -2, 3) \rangle = 0.$$

Two timelike vectors cannot be orthogonal in the Minkowski space $\mathbb{R}^3_1$, see [4].

Theorem 3.2. *A timelike vector and a null vector can be orthogonal in the Minkowski space* $\mathbb{R}^3_2$.

P r o o f. Let $\vec{x} = (x_1, x_2, x_3), \vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3_1$ be a timelike and null vector, respectively. In that case,

$$\langle \vec{x}, \vec{x} \rangle = x_1^2 - x_2^2 - x_3^2 < 0.$$

Thus, we have

$$x_1^2 < x_2^2 + x_3^2 \tag{3.4}$$

and $\langle \vec{y}, \vec{y} \rangle = y_1^2 - y_2^2 - y_3^2 = 0$. So, we can write

$$y_1^2 = y_2^2 + y_3^2. \tag{3.5}$$

Let us assume that $\vec{x} \perp \vec{y}$, namely $\langle \vec{x}, \vec{y} \rangle = x_1 y_1 - x_2 y_2 - x_3 y_3 = 0$. From here, we obtain

$$x_1 y_1 = x_2 y_2 + x_3 y_3. \tag{3.6}$$

Here, from the inequality (3.4) multiplied by y_1^2, we find $x_1^2 y_1^2 < y_1^2 (x_2^2 + x_3^2)$. Also, if we substitute in the last equation, we have

$$(x_2 y_3 - x_3 y_2)(x_3 y_2 - x_2 y_3) = -(x_2 y_3 - x_3 y_2)^2 < 0.$$

In this way, it is seen that there is no contradiction. Therefore, it is shown that a timelike vector and a null vector can be perpendicular to each other in the Minkowski space $\mathbb{R}^3_2$. □

E x a m p l e 3.2. Let $\vec{x} = (1, 1, 3)$ be a timelike vector and $\vec{y} = (1, 1, 0)$ be the null vector in the Minkowski space $\mathbb{R}^3_2$. Then,

$$\langle \vec{x}, \vec{y} \rangle = \langle (1, 1, 3), (1, 1, 0) \rangle = 0.$$

A timelike vector and the null vector cannot be orthogonal in the Minkowski space $\mathbb{R}^3_1$, see [4]. Similarly, the following theorems can be proved.

Theorem 3.3. *A timelike vector and a spacelike vector can be orthogonal in the Minkowski space* $\mathbb{R}^3_2$.

A timelike vector and a spacelike vector can be orthogonal in the Minkowski space $\mathbb{R}^3_1$, see [4].

Theorem 3.4. *A spacelike vector and the null vector cannot be orthogonal in the Minkowski space* $\mathbb{R}^3_2$.

A spacelike vector and the null vector can be orthogonal in the Minkowski space $\mathbb{R}^3_1$, see [4].

4. The Gram-Schmidt method for Minkowski space $\mathbb{R}^3_2$

Theorem 4.1. *Let $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^3_2$ be timelike vectors and $\vec{x}_3 \in \mathbb{R}^3_2$ be a spacelike vector. Let $\mathbb{R}^3_2 = \mathrm{sp}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$. If $\vec{y}_1 = \vec{x}_1$,*

$$\vec{y}_2 = \vec{x}_2 - \frac{\langle \vec{x}_2, \vec{y}_1\rangle}{\langle \vec{y}_1, \vec{y}_1\rangle}\vec{y}_1, \quad \vec{y}_3 = \vec{x}_3 - \frac{\langle \vec{x}_3, \vec{y}_1\rangle}{\langle \vec{y}_1, \vec{y}_1\rangle}\vec{y}_1 - \frac{\langle \vec{x}_3, \vec{y}_2\rangle}{\langle \vec{y}_2, \vec{y}_2\rangle}\vec{y}_2,$$

then $\mathbb{R}^3_2 = \mathrm{sp}\{\vec{y}_1, \vec{y}_2, \vec{y}_3\}$. Also, $\vec{y}_1$, $\vec{y}_2$ are timelike vectors, $\vec{y}_3$ is a spacelike vector.

Proof. Let $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^3_2$ be timelike vectors and $\vec{x}_3 \in \mathbb{R}^3_2$ be a spacelike vector. Let us define the vectors

$$\begin{aligned} \vec{y}_1 &= \vec{x}_1, \\ \vec{y}_2 &= \lambda'_2 \vec{y}_1 + \vec{x}_2, \\ \vec{y}_3 &= \lambda'_3 \vec{y}_1 + \lambda''_3 \vec{y}_2 + \vec{x}_3. \end{aligned} \tag{4.1}$$

In this case, it is clear that $\vec{y}_1$ is a timelike vector. So, we can write $\langle \lambda'_2 \vec{y}_1 + \vec{x}_2, \vec{y}_1\rangle = \lambda'_2\langle \vec{y}_1, \vec{y}_1\rangle + \langle \vec{x}_2, \vec{y}_1\rangle = 0$ for $\langle \vec{y}_2, \vec{y}_1\rangle = 0$. Here, since $\vec{y}_1$ is a timelike vector, we have $\langle \vec{y}_1, \vec{y}_1\rangle < 0$. Then, we find

$$\lambda'_2 = -\frac{\langle \vec{x}_2, \vec{y}_1\rangle}{\langle \vec{y}_1, \vec{y}_1\rangle}.$$

If λ'_2 is written in the equation (4.1), we obtain

$$\vec{y}_2 = \vec{x}_2 - \frac{\langle \vec{x}_2, \vec{y}_1\rangle}{\langle \vec{y}_1, \vec{y}_1\rangle}\vec{y}_1.$$

Now, let us show that $\vec{y}_2$ is a timelike vector: We have

$$\langle \vec{y}_2, \vec{y}_2\rangle = \langle \vec{x}_2, \vec{x}_2\rangle - \frac{\langle \vec{x}_2, \vec{y}_1\rangle}{\langle \vec{y}_1, \vec{y}_1\rangle}\langle \vec{y}_1, \vec{x}_2\rangle - \frac{\langle \vec{x}_2, \vec{y}_1\rangle}{\langle \vec{y}_1, \vec{y}_1\rangle}\langle \vec{x}_2, \vec{y}_1\rangle + \Big(\frac{\langle \vec{x}_2, \vec{y}_1\rangle}{\langle \vec{y}_1, \vec{y}_1\rangle}\Big)^2\langle \vec{y}_1, \vec{y}_1\rangle.$$

If necessary algebraic operations are done, we can write

$$\langle \vec{y}_2, \vec{y}_2\rangle = \langle \vec{x}_2, \vec{x}_2\rangle - \frac{(\langle \vec{x}_2, \vec{y}_1\rangle)^2}{\langle \vec{y}_1, \vec{y}_1\rangle}.$$

Here, we have $\langle \vec{x}_2, \vec{y}_1\rangle^2 = \|\vec{x}_2\|^2\|\vec{y}_1\|^2\cosh^2\theta$. Then, we find $\langle \vec{x}_2, \vec{y}_1\rangle^2 \geqslant \|\vec{x}_2\|^2 \times \|\vec{y}_1\|^2 = \langle \vec{x}_2, \vec{x}_2\rangle\langle \vec{y}_1, \vec{y}_1\rangle$ for $\cosh^2\theta \geqslant 1$. It follows that

$$\frac{\langle \vec{x}_2, \vec{y}_1\rangle^2}{\langle \vec{y}_1, \vec{y}_1\rangle} \geqslant \langle \vec{x}_2, \vec{x}_2\rangle \quad \text{and} \quad \langle \vec{y}_2, \vec{y}_2\rangle = \langle \vec{x}_2, \vec{x}_2\rangle - \frac{(\langle \vec{x}_2, \vec{y}_1\rangle)^2}{\langle \vec{y}_1, \vec{y}_1\rangle} \leqslant 0.$$

So, $\vec{y}_2$ is a timelike vector. Moreover, it should be $\langle \vec{y}_3, \vec{y}_2\rangle = 0$ and $\langle \vec{y}_3, \vec{y}_1\rangle = 0$. Therefore, we found $\langle \vec{y}_3, \vec{y}_1\rangle = \lambda'_3\langle \vec{y}_1, \vec{y}_1\rangle + \lambda''_3\langle \vec{y}_2, \vec{y}_1\rangle + \langle \vec{x}_3, \vec{y}_1\rangle = 0$ for $\langle \vec{y}_3, \vec{y}_1\rangle = 0$.

We have $\langle \vec{y}_2, \vec{y}_1 \rangle = 0$. Since $\vec{y}_1$ is a timelike vector, we have $\langle \vec{y}_1, \vec{y}_1 \rangle < 0$. Then, we obtain

$$\lambda_3' = -\frac{\langle \vec{x}_3, \vec{y}_1 \rangle}{\langle \vec{y}_1, \vec{y}_1 \rangle}.$$

Also, $\langle \vec{y}_3, \vec{y}_2 \rangle = \lambda_3' \langle \vec{y}_1, \vec{y}_2 \rangle + \lambda_3'' \langle \vec{y}_2, \vec{y}_2 \rangle + \lambda_3'' \langle \vec{x}_3, \vec{y}_2 \rangle = 0$ for $\langle \vec{y}_3, \vec{y}_2 \rangle = 0$. We have $\langle \vec{y}_2, \vec{y}_1 \rangle = 0$. Moreover, since $\vec{y}_2$ is a timelike vector, we have $\langle \vec{y}_2, \vec{y}_2 \rangle < 0$. Then, we found

$$\lambda_3'' = -\frac{\langle \vec{x}_3, \vec{y}_2 \rangle}{\langle \vec{y}_2, \vec{y}_2 \rangle}.$$

So, we obtain

$$\vec{y}_3 = \vec{x}_3 - \frac{\langle \vec{x}_3, \vec{y}_1 \rangle}{\langle \vec{y}_1, \vec{y}_1 \rangle} \vec{y}_1 - \frac{\langle \vec{x}_3, \vec{y}_2 \rangle}{\langle \vec{y}_2, \vec{y}_2 \rangle} \vec{y}_2.$$

Let us show that the vector $\vec{y}_3$ is spacelike. We have

$$\langle \vec{y}_3, \vec{y}_3 \rangle = \langle \vec{x}_3, \vec{x}_3 \rangle - \frac{\langle \vec{x}_3, \vec{y}_2 \rangle^2}{\langle \vec{y}_2, \vec{y}_2 \rangle} - \frac{\langle \vec{x}_3, \vec{y}_1 \rangle^2}{\langle \vec{y}_1, \vec{y}_1 \rangle}.$$

Since $\vec{x}_3$ is a spacelike vector, $\vec{y}_1$ and $\vec{y}_2$ are timelike vectors, we have $\langle \vec{y}_3, \vec{y}_3 \rangle > 0$. So, $\vec{y}_3$ is a spacelike vector. Let us check that $\langle \vec{y}_3, \vec{y}_1 \rangle = 0, \langle \vec{y}_3, \vec{y}_2 \rangle = 0$ and $\langle \vec{y}_2, \vec{y}_1 \rangle = 0$. We have

$$\langle \vec{y}_3, \vec{y}_2 \rangle = \langle \vec{x}_3, \vec{y}_2 \rangle - \frac{\langle \vec{x}_3, \vec{y}_1 \rangle}{\langle \vec{y}_1, \vec{y}_1 \rangle} \langle \vec{y}_1, \vec{y}_2 \rangle - \frac{\langle \vec{x}_3, \vec{y}_2 \rangle}{\langle \vec{y}_2, \vec{y}_2 \rangle} \langle \vec{y}_2, \vec{y}_2 \rangle.$$

If necessary algebraic operations are done, $\langle \vec{y}_3, \vec{y}_2 \rangle = 0$ is obtained. Similarly, other equations can be found. □

5. On $\mathbb{R}_1^2$

5.1. Hyperbolic cosine formulas on the Minkowski plane $\mathbb{R}_1^2$.

5.1.1. Hyperbolic cosine formulas for the timelike-pure riangle.

Theorem 5.1 ([4]). *Let $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$ be future pointing timelike vectors and $B_{\mathrm{TT}} = \angle(\overrightarrow{AB}, \overrightarrow{BC})$. Then, the timelike pure triangle $\overset{\Delta}{ABC}$ provides $b^2 = a^2 + 2ac \cosh B_{\mathrm{TT}} + c^2$ where $\|\overrightarrow{AB}\| = c$, $\|\overrightarrow{AC}\| = b$, $\|\overrightarrow{BC}\| = a$.*

5.1.2. Hyperbolic cosine formulas for the spacelike-pure triangle.

Theorem 5.2 ([10]). *Let $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$ be three linearly independent spacelike vectors. Then, the triangle $\overset{\Delta}{ABC}$ provides $b^2 = a^2 \pm 2ac \cosh B_{\mathrm{SS}} + c^2$ where $B_{\mathrm{SS}} = \angle(\overrightarrow{AB}, \overrightarrow{BC})$ and $\|\overrightarrow{AB}\| = c$, $\|\overrightarrow{AC}\| = b$, $\|\overrightarrow{BC}\| = a$.*

5.1.3. Hyperbolic cosine formulas for the non-pure triangle.

Theorem 5.3. *Let $\overrightarrow{AB}$ be a future pointing (past pointing) timelike vector, $\overrightarrow{BC}$ a past pointing (future pointing) timelike vector and $\overrightarrow{AC}$ be a spacelike vector. Let $A_{\mathrm{TS}} = \angle(\overrightarrow{AB}, \overrightarrow{AC})$, $B_{\mathrm{Tt}} = \angle(\overrightarrow{AB}, \overrightarrow{BC})$, $C_{\mathrm{St}} = \angle(\overrightarrow{AC}, \overrightarrow{BC})$ and $\|\overrightarrow{AB}\| = c$, $\|\overrightarrow{AC}\| = b$, $\|\overrightarrow{BC}\| = a$. Then, the triangle $\overset{\triangle}{ABC}$ provides the following equations:*

(1) (i) *When $\overrightarrow{AC}$ is oriented in the same direction with $\vec{E} = (1, 0)$, if $\overrightarrow{AB}$ is a future pointing timelike vector, then $a^2 = -b^2 + 2bc \sinh A_{\mathrm{ST}} + c^2$, if $\overrightarrow{AB}$ is a past pointing timelike vector, then $a^2 = -b^2 - 2bc \sinh A_{\mathrm{St}} - c^2$.*

(ii) *When $\overrightarrow{AC}$ is oriented in a different direction with $\vec{E} = (1, 0)$, if $\overrightarrow{AB}$ is a future pointing timelike vector, then $a^2 = -b^2 - 2bc \sinh A_{\mathrm{sT}} - c^2$, if $\overrightarrow{AB}$ is a past pointing timelike vector, then $a^2 = -b^2 + 2bc \sinh A_{\mathrm{st}} - c^2$.*

(2) (i) *If $\overrightarrow{AB}$ is a future pointing vector and $\overrightarrow{BC}$ is a past pointing vector, then $b^2 = -a^2 + 2ac \cosh B_{\mathrm{Tt}} - c^2$,*

(ii) *if $\overrightarrow{AB}$ is a past pointing vector and $\overrightarrow{BC}$ is a future pointing vector, then $b^2 = -a^2 + 2ac \cosh B_{\mathrm{tT}} - c^2$.*

(3) (i) *When $\overrightarrow{AC}$ is oriented in the same direction with $\vec{E} = (1, 0)$, if $\overrightarrow{BC}$ is a past pointing timelike vector, then $c^2 = -b^2 - 2ab \sinh C_{\mathrm{St}} + a^2$, if $\overrightarrow{BC}$ is a future pointing timelike vector, then $c^2 = -b^2 + 2ab \sinh C_{\mathrm{ST}} + a^2$.*

(ii) *When $\overrightarrow{AC}$ is oriented in a different direction with $\vec{E} = (1, 0)$, if $\overrightarrow{BC}$ is a past pointing timelike vector, then $c^2 = -b^2 + 2ab \sinh C_{\mathrm{st}} + a^2$, if $\overrightarrow{BC}$ is a future pointing timelike vector, then $c^2 = -b^2 - 2ab \sinh C_{\mathrm{sT}} + a^2$.*

P r o o f. (1) Since $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, we can write

$$\langle \overrightarrow{BC}, \overrightarrow{BC} \rangle = \langle \overrightarrow{AC} - \overrightarrow{AB}, \overrightarrow{AC} - \overrightarrow{AB} \rangle = \langle \overrightarrow{AC}, \overrightarrow{AC} \rangle - 2\langle \overrightarrow{AC}, \overrightarrow{AB} \rangle + \langle \overrightarrow{AB}, \overrightarrow{AB} \rangle.$$

Then, we obtain

$$a^2 = -b^2 + 2\langle \overrightarrow{AC}, \overrightarrow{AB} \rangle + c^2 \tag{5.1}$$

where $\langle \overrightarrow{BC}, \overrightarrow{BC} \rangle = -\|\overrightarrow{BC}\|^2$, $\langle \overrightarrow{AC}, \overrightarrow{AC} \rangle = \|\overrightarrow{AC}\|^2$, $\langle \overrightarrow{AB}, \overrightarrow{AB} \rangle = -\|\overrightarrow{AB}\|^2$ and $\|\overrightarrow{AB}\| = c$, $\|\overrightarrow{AC}\| = b$, $\|\overrightarrow{BC}\| = a$.

Here, let us examine the orientation cases of the spacelike vector $\overrightarrow{AC}$ in the same or different direction with the vector $\vec{E} = (1, 0)$:

(i) Let $\overrightarrow{AC}$ be a spacelike vector oriented in the same direction with $\vec{E} = (1, 0)$. If $\overrightarrow{AB}$ is a future pointing timelike vector, we have

$$\sinh A_{\mathrm{ST}} = \frac{\langle \overrightarrow{AC}, \overrightarrow{AB} \rangle}{\|\overrightarrow{AC}\| \|\overrightarrow{AB}\|}$$

according to Theorem 2.1. If this last equation is written in the equation (5.1), we found $a^2 = -b^2 + 2bc \sinh A_{\mathrm{ST}} + c^2$. If $\overrightarrow{AB}$ is a past pointing timelike vector, we have

$$\sinh A_{\mathrm{St}} = -\frac{\langle \overrightarrow{AC}, \overrightarrow{AB} \rangle}{\|\overrightarrow{AC}\| \|\overrightarrow{AB}\|}$$

according to Theorem 2.3. If this last equation is written in equation (5.1), we found $a^2 = -b^2 - 2bc \sinh A_{\mathrm{St}} - c^2$.

(ii) Let $\overrightarrow{AC}$ be a spacelike vector oriented in a different direction with the vector $\vec{E} = (1, 0)$. If $\overrightarrow{AB}$ is a future pointing timelike vector, we can write

$$\sinh A_{\mathrm{sT}} = -\frac{\langle \overrightarrow{AC}, \overrightarrow{AB} \rangle}{\|\overrightarrow{AC}\| \|\overrightarrow{AB}\|}$$

according to Theorem 2.3. If this last equation is written in equation (5.1), we found $a^2 = -b^2 - 2bc \sinh A_{\mathrm{sT}} - c^2$. If $\overrightarrow{AB}$ is a past pointing vector,

$$\sinh A_{\mathrm{st}} = \frac{\langle \overrightarrow{AC}, \overrightarrow{AB} \rangle}{\|\overrightarrow{AC}\| \|\overrightarrow{AB}\|}$$

according to Theorem 2.1. If this last equation is written in equation (5.1), we found $a^2 = -b^2 + 2bc \sinh A_{\mathrm{st}} - c^2$. Other conditions are proved similarly. □

Theorem 5.4 ([10]). *Let $\overrightarrow{AB}$, $\overrightarrow{AC}$ be two linearly independent spacelike vectors and $\overrightarrow{BC}$ be timelike vector. Then, the non-pure triangle $\overset{\Delta}{ABC}$ provides $b^2 = -a^2 \pm 2ac \sinh B_{\mathrm{ST}} + c^2$ where $B = \angle(\overrightarrow{AB}, \overrightarrow{BC})$ and $\|\overrightarrow{AB}\| = c$, $\|\overrightarrow{AC}\| = b$, $\|\overrightarrow{BC}\| = a$.*

Theorem 5.5. *Let $\overrightarrow{AB}$, $\overrightarrow{AC}$ be future pointing (past pointing) timelike vectors and $\overrightarrow{BC}$ be a past pointing (future pointing) timelike vector. Let $A_{\mathrm{TT}} = \angle(\overrightarrow{AB}, \overrightarrow{AC})$, $B_{\mathrm{Tt}} = \angle(\overrightarrow{AB}, \overrightarrow{BC})$, $C_{\mathrm{Tt}} = \angle(\overrightarrow{AC}, \overrightarrow{BC})$. Then, the non-pure triangle $\overset{\Delta}{ABC}$ provides the following equations:*

(1) $a^2 = b^2 - 2bc \cosh A_{\mathrm{TT}} + c^2$,

(2) $b^2 = a^2 - 2ac \cosh B_{\mathrm{Tt}} + c^2$,

(3) $c^2 = a^2 + 2ab \cosh C_{\mathrm{Tt}} + b^2$,

where $\|\overrightarrow{AB}\| = c$, $\|\overrightarrow{AC}\| = b$, $\|\overrightarrow{BC}\| = a$.

Theorem 5.6. *Let $\overrightarrow{AB}$ be a spacelike vector oriented in the same (different) direction with $\vec{E}=(1,0)$, $\overrightarrow{BC}$ be a spacelike vector oriented in the different (same) direction with $\vec{E}=(1,0)$ and $\overrightarrow{AC}$ be a timelike vector. Let $A_{\mathrm{ST}}=\angle(\overrightarrow{AB},\overrightarrow{AC})$, $B_{\mathrm{Ss}}=\angle(\overrightarrow{AB},\overrightarrow{BC})$, $C_{\mathrm{Ts}}=\angle(\overrightarrow{AC},\overrightarrow{BC})$ and $\|\overrightarrow{AB}\|=c$, $\|\overrightarrow{AC}\|=b$, $\|\overrightarrow{BC}\|=a$. Then, the non-pure triangle $\overset{\triangle}{ABC}$ provides the following equations:*

(1) (i) *When $\overrightarrow{AC}$ is a future pointing timelike vector, if $\overrightarrow{AB}$ is a spacelike vector oriented in the same direction with $\vec{E}=(1,0)$, then $a^2=-b^2+2bc\sinh A_{\mathrm{TS}}+c^2$, if $\overrightarrow{AB}$ is a spacelike vector oriented in a different direction with $\vec{E}=(1,0)$, then $a^2=-b^2-2bc\sinh A_{\mathrm{Ts}}+c^2$.*

(ii) *When $\overrightarrow{AC}$ is a past pointing timelike vector, if $\overrightarrow{AB}$ is a spacelike vector oriented in the same direction with $\vec{E}=(1,0)$, then $a^2=-b^2-2bc\sinh A_{\mathrm{tS}}+c^2$, if $\overrightarrow{AB}$ is a spacelike vector oriented in a different direction with $\vec{E}=(1,0)$, then $a^2=-b^2+2bc\sinh A_{\mathrm{ts}}+c^2$.*

(2) (i) *If $\overrightarrow{AB}$ is a spacelike vector oriented in a same direction with $\vec{E}=(1,0)$ and $\overrightarrow{BC}$ is a spacelike vector oriented in a different direction with $\vec{E}=(1,0)$, then $b^2=-a^2+2ac\cosh B_{\mathrm{Ss}}-c^2$.*

(ii) *If $\overrightarrow{AB}$ is a spacelike vector oriented in a different direction with $\vec{E}=(1,0)$ and $\overrightarrow{BC}$ is a spacelike vector oriented in the same direction with $\vec{E}=(1,0)$, then $b^2=-a^2+2ac\cosh B_{\mathrm{sS}}-c^2$.*

(3) (i) *When $\overrightarrow{AC}$ is a future pointing timelike vector, if $\overrightarrow{AB}$ is a spacelike vector oriented in a different direction with $\vec{E}=(1,0)$, then $c^2=-b^2-2ab\sinh C_{\mathrm{Ts}}+a^2$, if $\overrightarrow{AB}$ is a spacelike vector oriented in the same with $\vec{E}=(1,0)$, then $c^2=-b^2+2ab\sinh C_{\mathrm{TS}}+a^2$.*

(ii) *When $\overrightarrow{AC}$ is a past pointing timelike vector, if $\overrightarrow{AB}$ is a spacelike vector oriented in a different direction with $\vec{E}=(1,0)$, then $c^2=-b^2+2ab\sinh C_{\mathrm{ts}}+a^2$, if $\overrightarrow{AB}$ is a spacelike vector oriented in the same direction with $\vec{E}=(1,0)$, then $c^2=-b^2-2ab\sinh C_{\mathrm{tS}}+a^2$.*

Theorem 5.7. *Let $\overrightarrow{AB}$, $\overrightarrow{AC}$ be future pointing (past pointing) timelike vectors and $\overrightarrow{BC}$ be a spacelike vector. Let $A_{\mathrm{TT}}=\angle(\overrightarrow{AB},\overrightarrow{AC})$, $B_{\mathrm{TS}}=\angle(\overrightarrow{AB},\overrightarrow{BC})$, $C_{\mathrm{TS}}=\angle(\overrightarrow{AC},\overrightarrow{BC})$ and $\|\overrightarrow{AB}\|=c$, $\|\overrightarrow{AC}\|=b$, $\|\overrightarrow{BC}\|=a$. Then, the non-pure triangle $\overset{\triangle}{ABC}$ provides the following equations:*

(1) $a^2=-b^2+2bc\cosh A_{\mathrm{TT}}-c^2$.

(2) (i) *When $\overrightarrow{BC}$ is a spacelike vector oriented in the same direction with $\vec{E}=(1,0)$, if $\overrightarrow{AB}$ is a future pointing timelike vector, then $b^2=c^2+2ac\sinh B_{\mathrm{TS}}-a^2$, if $\overrightarrow{AB}$ is a past pointing timelike vector, then $b^2=c^2-2ac\sinh B_{\mathrm{tS}}-a^2$.*

(ii) *When $\overrightarrow{BC}$ is a spacelike vector oriented in a different direction with $\vec{E} = (1,0)$, if $\overrightarrow{AB}$ is a future pointing timelike vector, then $b^2 = c^2 - 2ac\sinh B_{\mathrm{Ts}} - a^2$, if $\overrightarrow{AB}$ is a past pointing timelike vector, then $b^2 = c^2 - 2ac\sinh B_{\mathrm{ts}} - a^2$.*

(3) (i) *When $\overrightarrow{BC}$ is a spacelike vector oriented in the same direction with $\vec{E} = (1,0)$, if $\overrightarrow{AC}$ is a future pointing timelike vector, then $c^2 = b^2 - 2ab\sinh C_{\mathrm{TS}} - a^2$, if $\overrightarrow{AC}$ is a past pointing timelike vector, then $c^2 = b^2 + 2ab\sinh C_{\mathrm{tS}} - a^2$.*

(ii) *When $\overrightarrow{BC}$ is a spacelike vector oriented in a different direction with $\vec{E} = (1,0)$, if $\overrightarrow{AC}$ is a future pointing timelike vector, then $c^2 = b^2 + 2ab\sinh C_{\mathrm{Ts}} - a^2$, if $\overrightarrow{AC}$ is a past pointing timelike vector, then $c^2 = b^2 - 2ab\sinh C_{\mathrm{ts}} - a^2$.*

5.2. Pedoe inequality for triangle in Minkowski plane $\mathbb{R}^2_1$.

5.2.1. Pedoe inequality for timelike pure triangle.

Theorem 5.8. *Let $\overset{\triangle}{ABC}$ be a timelike pure triangle with side lengths $a = \|\overrightarrow{BC}\|$, $b = \|\overrightarrow{CA}\|$, $c = \|\overrightarrow{AB}\|$and the area* S. *Let $X\overset{\triangle}{Y}Z$ be a timelike pure triangle with side lengths $x = \|\overrightarrow{YZ}\|$, $y = \|\overrightarrow{ZX}\|$, $z = \|\overrightarrow{XY}\|$ and the area* T. *Then, the timelike triangles $\overset{\triangle}{ABC}$ and $X\overset{\triangle}{Y}Z$ are similar if and only if the following inequality are satisfied:*

$$a^2(3z^2 + x^2 - y^2) + b^2(y^2 - x^2 - z^2) + c^2(3x^2 + z^2 - y^2) > 16\mathrm{ST}.$$

P r o o f. The sides are homogeneous in the timelike pure triangle $X\overset{\triangle}{Y}Z$. In this case, the inequality is invariant under any transformation in the timelike pure triangle $X\overset{\triangle}{Y}Z$. For this, the points $Y = B$ and $Z = C$ can be selected on the line X, with the point A in the same hemisphere as the line BC.

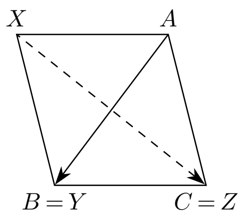

Figure 1. Pedoe inequality for two triangle.

We have $\overrightarrow{AB} + \overrightarrow{BX} = \overrightarrow{AX}$ (see Figure 1). Then, $\overrightarrow{AX}$ is a timelike vector. Thus, the timelike pure triangle $\overset{\triangle}{ABX}$ is obtained. From Theorem 5.1, we have

$$\|\overrightarrow{AX}\|^2 = \|\overrightarrow{AB}\|^2 + \|\overrightarrow{XB}\|^2 + 2\|\overrightarrow{AB}\|\|\overrightarrow{XB}\|\cosh(\sphericalangle ABX) \tag{5.2}$$

for triangle $\overset{\triangle}{ABX}$. Since $\|\overrightarrow{AB}\| = c$, $B = Y$ and $Z = C$, we obtain $\|\overrightarrow{XB}\| = \|\overrightarrow{XY}\| = z$ and $\overset{\triangle}{XBC} = \overset{\triangle}{XYZ}$. Also, we have $\sphericalangle ABX = \sphericalangle ABC - \sphericalangle XBC$. It follows that

$$\begin{aligned}(5.3)\qquad \cosh(\sphericalangle ABX) &= \cosh(\sphericalangle ABC - \sphericalangle XBC) = \cosh(\sphericalangle ABC - \sphericalangle XYZ)\\ &= \cosh(\sphericalangle ABC)\cosh(\sphericalangle XYZ) - \sinh(\sphericalangle ABC)\sinh(\sphericalangle XYZ).\end{aligned}$$

According to Theorem 5.1, we have

$$b^2 = c^2 + a^2 + 2ac\cosh B, \quad y^2 = x^2 + z^2 + 2xz\cosh Y$$

and we can write

$$S = \frac{1}{2}ac\sinh B, \quad T = \frac{1}{2}xz\sinh Y$$

from Theorem 2.4. If these equations are written in equation (5.3), we found

$$\cosh(\sphericalangle ABX) = \frac{(b^2 - c^2 - a^2)}{2ac}\frac{(y^2 - x^2 - z^2)}{2xz} - \frac{2\mathrm{S}}{ac}\frac{2\mathrm{T}}{xz}.$$

If this last equation is written in equation (5.2), we obtain

$$\|\overrightarrow{AX}\|^2 = \frac{a^2(3z^2 + x^2 - y^2) + b^2(y^2 - x^2 - z^2) + c^2(3x^2 + z^2 - y^2) - 16\mathrm{ST}}{2ax}.$$

Since $\overrightarrow{AX}$ is a timelike vector, we have $\|\overrightarrow{AX}\|^2 > 0$. Thus, we obtain

$$a^2(3z^2 + x^2 - y^2) + b^2(y^2 - x^2 - z^2) + c^2(3x^2 + z^2 - y^2) - 16\mathrm{ST} > 0.$$

$\square$

Similarly, Pedoe inequalities for spacelike triangle and non-pure triangle types are as follows.

5.2.2. Pedoe inequality for spacelike pure triangle.

Theorem 5.9. *Let $\overset{\triangle}{ABC}$ be a spacelike pure triangle with side lengths $a = \|\overrightarrow{BC}\|$, $b = \|\overrightarrow{CA}\|$, $c = \|\overrightarrow{AB}\|$ and the area* S. *Let $\overset{\triangle}{XYZ}$ be a spacelike pure triangle with side lengths $x = \|\overrightarrow{YZ}\|$, $y = \|\overrightarrow{ZX}\|$, $z = \|\overrightarrow{XY}\|$ and the area* T. *Then, the spacelike triangles $\overset{\triangle}{ABC}$ and $\overset{\triangle}{XYZ}$ are similar if and only if the following inequalities are satisfied:*

(1) $a^2(3z^2 + x^2 - y^2) + b^2(y^2 - x^2 - z^2) + c^2(3x^2 + z^2 - y^2) \geqslant 16\mathrm{ST}$,
(2) $a^2(-x^2 + y^2 + z^2) + b^2(x^2 + z^2 - y^2) + c^2(x^2 - z^2 + y^2) \geqslant 16\mathrm{ST}$.

5.2.3. Pedoe inequality for non-pure triangle.

(1) Let $\overset{\Delta}{ABC}$ be a non-pure triangle such that $\overrightarrow{AB}$, $\overrightarrow{CA}$ are spacelike vectors and $\overrightarrow{BC}$ is a timelike vector with side lengths $a = \|\overrightarrow{BC}\|$, $b = \|\overrightarrow{CA}\|$, $c = \|\overrightarrow{AB}\|$ and area SS. Let $\overset{\Delta}{XYZ}$ be a non-pure triangle such that $\overrightarrow{XY}$, $\overrightarrow{ZX}$ are spacelike vectors and $\overrightarrow{YZ}$ is a timelike vector with side lengths $x = \|\overrightarrow{YZ}\|$, $y = \|\overrightarrow{ZX}\|$, $z = \|\overrightarrow{XY}\|$ and area T. Then, the non-pure triangles $\overset{\Delta}{ABC}$ and $\overset{\Delta}{XYZ}$ are similar if and only if the following inequality is satisfied:

$$a^2(3z^2 - y^2 - x^2) + b^2(z^2 - x^2 - y^2) + c^2(3x^2 + y^2 - z^2) + 16\mathrm{ST} > 0.$$

(2) Let $\overset{\Delta}{ABC}$ be a non-pure triangle such that $\overrightarrow{AB}$ is a future pointing (past pointing) timelike vector, $\overrightarrow{BC}$ is a past pointing (future pointing) vector and $\overrightarrow{CA}$ is a spacelike vector with side lengths $a = \|\overrightarrow{BC}\|$, $b = \|\overrightarrow{CA}\|$, $c = \|\overrightarrow{AB}\|$ and area S.

Let $\overset{\Delta}{XYZ}$ be a non-pure triangle such that $\overrightarrow{XY}$ is a future pointing (past pointing) timelike vector, $\overrightarrow{YZ}$ is a past pointing (future pointing) vector and $\overrightarrow{ZX}$ is a spacelike vector with side lengths $x = \|\overrightarrow{YZ}\|$, $y = \|\overrightarrow{ZX}\|$, $z = \|\overrightarrow{XY}\|$ and area T. Then, the non-pure triangles $\overset{\Delta}{ABC}$ and $\overset{\Delta}{XYZ}$ are similar if and only if the following inequality is satisfied:

$$a^2(y^2 + x^2 + 3z^2) + b^2(x^2 + y^2 + z^2) + c^2(y^2 + z^2 + 3x^2) > 16\mathrm{ST}.$$

(3) Let $\overset{\Delta}{ABC}$ be a non-pure triangle such that $\overrightarrow{AB}$ is a spacelike vector oriented in the same (different) direction with $\vec{E} = (1, 0)$, $\overrightarrow{BC}$ is a spacelike vector oriented in the different (same) direction with $\vec{E} = (1, 0)$ and $\overrightarrow{CA}$ is a timelike vector with side lengths $a = \|\overrightarrow{BC}\|$, $b = \|\overrightarrow{CA}\|$, $c = \|\overrightarrow{AB}\|$ and the area S.

Let $\overset{\Delta}{XYZ}$ be a non-pure triangle such that $\overrightarrow{XY}$ is a spacelike vector oriented in the same (different) direction with $\vec{E} = (1, 0)$, $\overrightarrow{YZ}$ is a spacelike vector oriented in the different (same) direction with $\vec{E} = (1, 0)$ and $\overrightarrow{ZX}$ is a timelike vector with side lengths $x = \|\overrightarrow{YZ}\|$, $y = \|\overrightarrow{ZX}\|$, $z = \|\overrightarrow{XY}\|$ and area T. Then, the non-pure triangles $\overset{\Delta}{ABC}$ and $\overset{\Delta}{XYZ}$ are similar if and only if the following inequality is satisfied:

$$a^2(y^2 + x^2 + 3z^2) + b^2(x^2 + y^2 + z^2) + c^2(y^2 + z^2 + 3x^2) > 16\mathrm{ST}.$$

(4) Let $\overset{\Delta}{ABC}$ be a non-pure triangle such that $\overrightarrow{AB}$, $\overrightarrow{CA}$ are future pointing (past pointing) timelike vectors and $\overrightarrow{BC}$ is a spacelike vector with side lengths $a = \|\overrightarrow{BC}\|$, $b = \|\overrightarrow{CA}\|$, $c = \|\overrightarrow{AB}\|$ and area S.

Let $\overset{\Delta}{XYZ}$ be a non-pure triangle such that $\overrightarrow{XY}$, $\overrightarrow{ZX}$ are future pointing (past pointing) timelike vectors and $\overrightarrow{YZ}$ is a spacelike vector with side lengths $x = \|\overrightarrow{YZ}\|$, $y = \|\overrightarrow{ZX}\|$, $z = \|\overrightarrow{XY}\|$ and area T. Then, the non-pure triangles $\overset{\Delta}{ABC}$ and $\overset{\Delta}{XYZ}$ are similar if and only if the following inequality is satisfied:

$$a^2(3z^2 - y^2 - x^2) + b^2(z^2 - x^2 - y^2) + c^2(3x^2 + y^2 - z^2) + 16\mathrm{ST} > 0.$$

5.3. Rotation matrix under Lorentz matrix multiplication in Minkowski plane $\mathbb{R}_1^2$.

Definition 5.1 ([6]). Let $A_1, A_2, \ldots, A_m \in \mathbb{R}_1^n$ be the row vectors of the matrices $A = [a_{ij}] \in M_{m\times n}(\mathbb{R})$ and $B_1, B_2, \ldots, B_p \in \mathbb{R}_1^m$ be the column vectors of the matrix $B = [b_{jk}] \in M_{n\times p}(\mathbb{R})$ in the space $\mathbb{R}_1^n$. The Lorentz matrix multiplication is indicated by "$\cdot_L$",

$$A \cdot_L B = \begin{bmatrix} \langle A_1, B_1\rangle_L & \langle A_1, B_2\rangle_L & \ldots & \langle A_1, B_p\rangle_L \\ \langle A_2, B_1\rangle_L & \langle A_2, B_2\rangle_L & \ldots & \langle A_2, B_p\rangle_L \\ \vdots & \vdots & & \vdots \\ \langle A_m, B_1\rangle_L & \langle A_m, B_2\rangle_L & \ldots & \langle A_m, B_p\rangle_L \end{bmatrix} = \left[\sum_{j=1}^{n-1} a_{ij}b_{jk} - a_{in}b_{nk}\right]_{m\times n}.$$

The set of $m \times n$-type matrices is represented as $M_{m\times n}(\mathbb{R})$ with the Lorentz multiplication in space $\mathbb{R}_1^n$.

5.3.1. Point rotation under Lorentz matrix multiplication. Let the point $A = (x, y)$, which makes an angle β with the axis be rotated to the point $A' = (x', y')$ by angle θ in the Minkowski plane $\mathbb{R}_1^2$. Then, we can write $x' = r\cosh(\beta + \theta)$ and $y' = r\sinh(\beta + \theta)$ where $x = r\cosh\beta$ and $y = r\sinh\beta$. By using the trigonometric property of hyperbolic functions, we have

$$x' = r\cosh(\beta + \theta) = r(\cosh\beta\cosh\theta + \sinh\beta\sinh\theta) = x\cosh\theta + y\sinh\theta$$

and

$$y' = r\sinh(\beta + \theta) = r(\sinh\beta\cosh\theta + \cosh\beta\sinh\theta) = y\cosh\theta + x\sinh\theta.$$

These equations are written in matrix form as $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{bmatrix} \cdot_L \begin{bmatrix} x \\ y \end{bmatrix}$. Thus, the matrix that provides the point rotation obtained by the Lorentz matrix multiplication is found as

$$\mathbf{A}_L = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{bmatrix}. \tag{5.4}$$

5.3.2. Axis rotation under Lorentz matrix multiplication. Let $l_1 = (1, 0)$ be spacelike vector, $l_2 = (0, 1)$ be a timelike vector and $l'_1 = (a, b)$, $l'_2 = (c, d)$. Let us rotate the axis $l_1 = (1, 0)$ by the angle θ and get the axis $l'_1 = (a, b)$. Also, let us rotate the axis $l_2 = (0, 1)$ by the angle θ and get the axis $l'_2 = (c, d)$. Here, we have

$$l'_1 = al_1 + bl_2, \quad l'_2 = cl_1 + dl_2.$$

By using the matrix (5.4) we find $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{bmatrix} \cdot_L \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for $l'_1 = (a, b)$. Thus, $a = \cosh\theta$ and $b = \sinh\theta$ can be obtained with the Lorentz matrix multiplication. Similarly, using the matrix (5.4) we find $\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{bmatrix} \cdot_L \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for $l'_2 = (c, d)$. Thus, $c = \sinh\theta$ and $d = \cosh\theta$ can be obtained with the Lorentz matrix multiplication. Therefore, we can write $\begin{bmatrix} l'_1 \\ l'_2 \end{bmatrix} = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{bmatrix} \cdot_L \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$. The matrix that provides the axis rotation obtained by the Lorentz matrix multiplication in the Lorentz plane is $\mathbf{A}_L = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{bmatrix}$.

Notation 5.1. The matrix $\mathbf{A}_L$ provides both point rotation and axis rotation obtained by the Lorentz matrix multiplication in the Minkowski plane. The inverse of the matrix obtained by the Lorentz matrix multiplication is found as $\mathbf{A}_L^{-1} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ -\sinh\theta & -\cosh\theta \end{bmatrix}$. Here, since $\mathbf{A}_L^{-1} = \mathbf{A}_L^t$, the rotation matrix obtained by the Lorentz matrix multiplication is the orthogonal matrix.

5.4. Orthogonal projection point on timelike and spacelike line.

Definition 5.2. Let l be a line, $\vec{u}$ be the directive vector of the line l in the Minkowski plane. Then, if $\vec{u}$ is a timelike vector, l is called a timelike line, if $\vec{u}$ is a spacelike vector, l is called a spacelike line, if $\vec{u}$ is a lightlike vector, l is called a lightlike line.

Let $\vec{n}$ be normal vector of the line l. The normal vector $\vec{n}$ of the timelike line is a spacelike vector. The normal vector $\vec{n}$ of the spacelike line is a timelike vector.

Theorem 5.10. *Let $ax - by + c = 0$ be a timelike line equation. Let S be an orthogonal projection of the point P on the line. Then,*

$$\vec{s} = \vec{p} - \frac{\langle \vec{p}, \vec{n} \rangle + c}{\langle \vec{n}, \vec{n} \rangle} \vec{n}$$

where $\vec{n}$ is a normal vector of the timelike line.

P r o o f. $\overrightarrow{PS} = \lambda\vec{n}$ can be written where $\overrightarrow{PS} \parallel \vec{n}$ for $\lambda \in \mathbb{R}$. Let O be the starting point of the coordinate system where the line is located. Then, we have $\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS}$ or $\vec{s} = \vec{p} + \lambda\vec{n}$. Since the projection point of P is the point S, it is located on the line and provides the line equation. Thus, $\langle\vec{s}, \vec{n}\rangle + c = 0$. If the equation $\vec{s} = \vec{p} + \lambda\vec{n}$ is used, we obtain $\langle\vec{p} + \lambda\vec{n}, \vec{n}\rangle + c = \langle\vec{p}, \vec{n}\rangle + \lambda\langle\vec{n}, \vec{n}\rangle + c = 0$. Then, we can write $\lambda = -(\langle\vec{p}, \vec{n}\rangle + c)/\langle\vec{n}, \vec{n}\rangle$. Thus, the orthogonal projection point is obtained as

$$\vec{s} = \vec{p} + \lambda\vec{n} = \vec{p} - \frac{\langle\vec{p}, \vec{n}\rangle + c}{\langle\vec{n}, \vec{n}\rangle}\vec{n}.$$

□

Theorem 5.11. *Let $ax - by + c = 0$ be spacelike line equation. Let S be an orthogonal projection of the point P on the line. Then,*

$$\vec{s} = \vec{p} - \frac{\langle\vec{p}, \vec{n}\rangle + c}{\langle\vec{n}, \vec{n}\rangle}\vec{n}$$

where $\vec{n}$ is a normal vector of the line.

P r o o f. The proof can be done in a similar way according to the Theorem 5.10.

□

5.5. The formula for distance of any point to a timelike and spacelike line.

Theorem 5.12. *Let S be an orthogonal projection point of the point $P = (x_0, y_0)$ on the timelike line $ax - by + c = 0$. Let d be the distance of the point P from the line. Then, the distance formula is expressed as*

$$d = \|\overrightarrow{PS}\| = \frac{|ax_0 - by_0 + c|}{\sqrt{a^2 - b^2}}.$$

P r o o f. Since $d = \|\overrightarrow{PS}\|$ and $\vec{s} = \vec{p} - ((\langle\vec{p}, \vec{n}\rangle + c)/\langle\vec{n}, \vec{n}\rangle)\vec{n}$, we obtain

$$d = \|\overrightarrow{PS}\| = \|\vec{s} - \vec{p}\| = \left\|\vec{p} - \frac{\langle\vec{p}, \vec{n}\rangle + c}{\langle\vec{n}, \vec{n}\rangle}\vec{n} - \vec{p}\right\| = \left\|-\frac{\langle\vec{p}, \vec{n}\rangle + c}{\langle\vec{n}, \vec{n}\rangle}\vec{n}\right\| = \frac{|\langle\vec{p}, \vec{n}\rangle + c|}{\|\langle\vec{n}, \vec{n}\rangle\|}.$$

We have $\langle\vec{p}, \vec{n}\rangle = ax_0 - by_0$. Since the normal vector is spacelike, we have $\|\langle\vec{n}, \vec{n}\rangle\| = \sqrt{a^2 - b^2}$. Then, we obtain $d = \|\overrightarrow{PS}\| = |ax_0 - by_0 + c|/\sqrt{a^2 - b^2}$. □

E x a m p l e 5.1. Find the distance between the point $P = (1, -2)$ and the timelike line $L_{\mathrm{T}} \equiv 3x - 2y + 5 = 0$.

The distance between the point and timelike line is found as

$$d = \frac{|3 \cdot (1) - (-2) \cdot 2 + 5|}{\sqrt{3^2 - (-2)^2}} = \frac{12}{\sqrt{5}}.$$

according to Theorem 5.12.

Theorem 5.13. *Let S be orthogonal projection point of the point $P = (x_0, y_0)$ on the spacelike line $ax - by + c = 0$. Let d be distance of point P from the line. Then, the distance formula is expressed as*

$$d = \|\overrightarrow{PS}\| = \frac{|bx_0 - ay_0 + c|}{\sqrt{|b^2 - a^2|}}.$$

P r o o f. The proof can be done in a similar way according to the Theorem 5.12. $\square$

6. Conclusions

In this article, orthogonality conditions are given in the Minkowski space $\mathbb{R}^3_2$. Accordingly, a different situation is obtained from the Minkowski space $\mathbb{R}^3_1$. This difference was also used in the Gram-Schmidt method. Moreover, all non-pure triangle types were studied. Thus, it has been seen that the hyperbolic cosine formulas change according to the state of the vectors. In this study, the Pedoe inequality was obtained for all Lorentz triangle types. Thanks to this inequality, a connection was obtained between two similar triangles in the Lorentz space. Thus, it is understood that the inequalities obtained change as the types of triangles change. The importance of the inner product in obtaining the rotation matrix was stated. It was emphasized that the rotation matrix obtained by the Lorentz matrix multiplication is orthogonal. Finally, distance and projection formulas were examined on timelike and spacelike lines. The distance and projection of timelike or spacelike points to the lines are given with separate examples.

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