Kybernetika

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Kybernetika, Vol. 60 (2024), No. 5, 624–651

Persistent URL: `http://dml.cz/dmlcz/152718`

K Y B E R N E T I K A — V O L U M E *6 0* (2 0 2 4) , N U M B E R 5 , P A G E S 6 2 4 – 6 5 1

NEW CONSTRUCTIONS OF UNI-NULLNORMS ON CERTAIN CLASSES OF BOUNDED LATTICES BY CLOSURE (INTERIOR) OPERATORS

Tao Wu

The primary aim of this article is to put forward new classes of uni-nullnorms on certain classes of bounded lattices via closure (interior) operators. Due to the new classes of uninorms combining both a t-norm T and a t-conorm S by various kinds of closure operators or interior operators, the relationships and properties among the same class of uninorms on L, we obtain new classes of uni-nullnorms on L via closure (interior) operators. The constructions of uni-nullnorms on some certain classes of bounded lattices can provide another different perspective of t-norms and the dual of t-norms, uninorms and some other associative aggregation operations on bounded lattices. That is, the constructions seem to be the ordinal like sum constructions, but not limited to the ordinal like sum constructions.

Keywords: uni-nullnorm, nullnorm, closure operator, construction, bounded lattice, ordinal like sum construction

Classification: 03B52, 06B20, 03E72

1. INTRODUCTION

According to the perspective of the School of Bourbaki, there are three mother structures (posets, topological spaces and algebraic structures) in mathematics from which all other mathematical structures can be generated, and which are not reducible one to the other. The theory of representations of Boolean algebras [51] has shown that three mother structures when combining together can create some interesting and meaningful results. Partial researches can be seen in [2, 30, 32, 47, 52, 55, 56, 57, 58, 66, 67]. In this article we mainly use uninorms combining a t-norm T (a t-conorm S) and closure operators (interior operators) to construct uni-nullnorms on certain special classes of bounded lattices.

As an extension of the triangular norms and triangular conorms [38], Yager and Rybalov [61] introduced uninorms, then Fodor et al. [28, 29] systematically investigated them that are particular aggregation operations with the neutral element e belonging to the real unit interval $[0, 1]$. The operations such as $U_{\min}$ and $U_{\max}$ are constructed from triangular norms and triangular conorms by ordinal sums from the point perspective of algebraic structure, they are significant in fuzzy logics, operations researchs, decision

DOI: 10.14736/kyb-2024-5-0624

making and game theories [20, 22, 26, 31, 33, 39, 53]. Some related researches for uninorms on the real unit interval can be seen in [6, 18, 19, 45, 46, 48]. In the last few years, various kinds of aggregation operations on a bounded lattice have been investigated by researchers. The constructions of triangular norms (triangular conorms) on posets or bounded lattices have been proposed and studied in [16, 17, 62]. Karaçal and Mesiar [36] provided one method and its dual to construct uninorms on L and proved the existence of uninorms on L. In [4] they presented one method and its dual for constructing uninorms on L combining both a triangular norm and a triangular conorm. Some interesting and meaningful construction methods of uninorms on L were proposed by Çaylı et al., further details, please refer to [7, 8, 9, 10, 15, 16, 36, 43, 44].

By the positions of absorbing elements laying anywhere on the real unit interval [0, 1], Mas et al. proposed t-operators [41] (nullnorms [6, 11, 12, 13, 50, 64]). Uni-nullnorms [49] were the combinations of uninorms and nullnorms, take some special values for uni-nullnorms, we can get both nullnorms and uninorms. Uni-nullnorms and null-uninorms were dual under a strong negation. In [54] Wang et al. put forward two construction methods for uni-nullnorms on L and illustrated the existence of uni-nullnorms on L. Zhang, Ouyang and De Baets [63] presented the constructions of the uni-nullnorm on L via a disjunctive uninorm and a t-norm, they generalized the second construction method for uni-nullnorms in [54]. Ertuğrul, Kesicioğlu and Karaçal [23] extended two construction methods of uni-nullnorms [54] in different ways.

Inspired by the work [63] and the constructions of uninorms on L [65], furthermore, in the concluding paragraph of [63], Zhang, Ouyang and De Baets wrote "For future research, it might be interesting to consider how to construct a uni-nullnorm (resp. null-uninorm) on a bounded lattice via a non-disjunctive uninorm and a t-norm (resp. a non-conjunctive uninorm and a t-conorm)". Constructing new uninorms on a bounded lattice L combining both a t-norm T and a t-conorm S may be too difficult, even troubled. With the rapid developments of uninorms on bounded lattices, we obtain the following results: A class of uninorm $\mathbf{U} = (F, G, F^{op}, G^{op})$, F denotes the disjunctive uninorm by the closure operator cl, G denotes the corresponding conjunctive uninorm by cl, F^{op} denotes the dual uninorm of F. All disjunctive (conjunctive) uninorms constructed by closure (interior) operators can be written in this form. Once constructing one uninorm of the four uninorms on bounded lattices via closure (interior) operators, the other three can be obtained accordingly. F and G can be one-to-one correspondences. And in some extreme cases, the four uninorms may be degenerated into only one uninorm and its dual. Observe that the methods for disjunctive uni-nullnorm on L in [23] extends the methods in [54] with certain additional condition, but the researchers did not construct a conjunctive uni-nullnorm on L. And structures are the weapons of the mathematician. A question arises: Are there any other different ways that generate new (conjunctive) uni-nullnorms on L different from the existing uni-nullnorms on L? That is to say, if those uni-nullnorms exist, what are their specific structures? It is well known that the constructions of t-norms on posets by pseduo-inverses of monotone functions are the special cases of the constructions of t-norms on posets via Galois connections in essence, and closure (interior) operators can be obtained by Galois connections. We obtain some new classes of uni-nullnorms on L. Using various closure (interior) operators to construct uninorms, we propose other generalized methods that can produce infinitely many (conjunctive) uni-nullnorms on L. The constructions of uni-nullnorms on bounded lattices can provide another different perspective of t-norms (t-conorms), uninorms and some other associated aggregation operations on bounded lattices. That is, the constructions seem to be the ordinal like sum constructions, but not limited to the ordinal like sum constructions.

The structure of this paper is as follows. Section 2 lists some basic background. Some construction methods for uninorms on L are introduced. In Section 3, investigate more generalized constructions of (conjunctive) uni-nullnorms on L with a closure operator or an interior operator, the concrete constructions of uni-nullnorms on bounded lattices has been obtained. The end are some conclusions and further work.

2. PRELIMINARIES

Assume that the readers know some basic results about basic theory of t-norms (t-conorms) [38]. Next just give some fundamental results on uninorms, nullnorms and uni-nullnorms on a bounded lattice.

A lattice is a nonempty set $(L, \leq)$ such that any two elements p and q have an infimum, denoted by $p \wedge q$, similarly a supremum denoted by $p \vee q$. For $p, q \in L$, denote $q \geq p$ if $p \leq q$ holds. $p < q$ means that $p \leq q$ and $p \neq q$. If neither $p \leq q$ nor $q \leq p$, then p is incomparable with q, and write $p \parallel q$. Denote the set of elements which are incomparable with p but comparable with q by I_{p^q}, $I_{p^q} = \{x \in L | x \parallel p, x \nparallel q\}$. Similarly, $I_{q^p} = \{x \in L | x \parallel q, x \nparallel p\}$, $I_{p,q} = \{x \in L | x \parallel p, x \parallel q\}$. And $I_e = \{x \in L | x \parallel e\}$.

A bounded lattice $(L, \leq, \wedge, \vee)$ is the lattice has a greatest element 1 and a smallest element 0. Let L be a lattice and $p, q \in L$ with $p \leq q$. The subinterval $[p, q]$ is a sublattice of L defined by $[p, q] = \{x \in L \mid p \leq x \leq q\}$. Other subintervals such as $[p, q[$ and $]p, q[$ can be defined similarly. Other information please see [3].

Definition 2.1. (Aşıcı [1], Karaçal and Khadjiev [35], Karaçal and Mesiar [36]) An operation $T : L^2 \to L$ (resp. $S : L^2 \to L$) is called a t-norm (resp. a t-conorm) on L if it is commutative, associative, and non-decreasing in each variable, and it has the neutral element $1 \in L$ (resp. $0 \in L$), that is, $T(1, p) = p$ (resp. $S(0, p) = p$) for arbitrary $p \in L$.

Definition 2.2. (Bodjanova and Kalina [4], Karaçal and Mesiar [36], Mas at al. [42]) An operation $U : L^2 \to L$ is called a uninorm on L if it is commutative, associative, and non-decreasing in each variable, and an element $e \in L$ called the neutral element exists such that $U(e, p) = p$ for arbitrary $p \in L$.

Definition 2.3. (Karaçal at al. [37], Mas et al. [42]) An operation $V : L^2 \to L$ is called a nullnorm on L if it is commutative, associative, and non-decreasing in each variable, there is an element $a \in L$ such that $V(p, 0) = p$ for arbitrary $p \leq a$, $V(p, 1) = p$ for arbitrary $p \geq a$.

Definition 2.4. (Wang at el. [54], Zhang et al. [63]) A binary operation $F : L^2 \to L$ is called a uni-nullnorm on L if it is commutative, associative, and non-decreasing in each variable, and there exist the elements $e, a \in L$ such that $0 \leq e < a \leq 1, F(e, p) = p$ for arbitrary $p \in [0, a]$ and $F(p, 1) = p$ for arbitrary $p \in [a, 1]$.

Observe that e is a neutral element of F in $[0, a]$. It can be easily obtained that $F(p, a) = a$ for arbitrary $p \in [e, 1]$ by Definition 2.4, so a is an absorbing element of F in $[e, 1]$. That is, e is a partial neutral element and a is a partial absorbing element of F, call e the neutral element and a the absorbing element of uni-nullnorm F in this paper. For a uni-nullnorm, if $p \in I_{e^a}$, then $p < a$. If $p \in I_{a^e}$, then $e < p$.

Proposition 2.5. (Wang et al. [54], Zhang et al. [63]) Let $e, a \in L \setminus \{0, 1\}$ and F be a uni-nullnorm with the neutral element e and the absorbing element a on L. Then
(i) $U = F \mid_{[0,a]^2} : [0, a]^2 \to [0, a]$ is a uninorm.
(ii) $V = F \mid_{[e,1]^2} : [e, 1]^2 \to [e, 1]$ is a nullnorm.
(iii) $U = F \mid_{[0,e]^2} : [0, e]^2 \to [0, e]$ is a t-norm.
(iv) $U = F \mid_{[e,a]^2} : [e, a]^2 \to [e, a]$ is a t-conorm.
(v) $U = F \mid_{[a,1]^2} : [a, 1]^2 \to [a, 1]$ is a t-norm.

Definition 2.6. (Dvořák and Holčapek [21], Everett [27], Han and Zhao [32], Ouyang and Zhang [43], Rosenthal [47]) Let L be a lattice. A mapping $cl : L \to L$ is called a closure operator on L if, for arbitrary $p, q \in L$, $(cl1)$ $p \le cl(p)$; $(cl2)$ $cl(p \vee q) = cl(p) \vee cl(q)$; $(cl3)$ $cl(cl(p)) = cl(p)$.

Example 2.7. (Dvořák and Holčapek [21], Han and Zhao [32], Ouyang and Zhang [43], Rosenthal [47]) Let L be a bounded lattice and $a, b \in L$ be given, where $a \le b$. Then the mapping $cl : L \to L$ defined by the following are all closure operators: for arbitrary $p \in L$, (i) $cl(p) = p$; (ii) $cl(p) = 1$; (iii) $cl(p) = p \vee a$; (iv) $cl(p) = a \to p$; (v) $cl(p) = (p \to a) \to a$; (vi) (f, g) is a Galois connection, $cl = g \circ f$. In (iv)(v), $(\wedge, \to)$ forms a Galois connection, for the case of t-norm T, $(T, \to)$ forms a Galois connection and cl needs to satisfy $(cl2)$; (vii) $cl_b^{\vee}(p) = \begin{cases} p & p \le b, \\ p \vee b & \text{otherwise;} \end{cases}$ (viii) $cl_b^{1}(p) = \begin{cases} p & p \le b, \\ 1 & \text{otherwise;} \end{cases}$ (ix) $cl_{a,b}^{\vee}(p) = \begin{cases} p & p \le a, \\ p \vee b & \text{otherwise.} \end{cases}$ We say that two closure operators cl_1 and cl_2 commute on L provided that $cl_1 \circ cl_2 = cl_2 \circ cl_1$. And $cl_b^{\vee} \circ cl_a^{\vee} = cl_a^{\vee} \circ cl_b^{\vee}$. Specially, $cl_1 = cl_2$, closure operators cl_1 and cl_2 commute on L.

Definition 2.8. (Dvořák and Holčapek [21], Han and Zhao [32], Ouyang and Zhang [43], Rosenthal [47]) Let L be a lattice. A mapping $int : L \to L$ is called an interior operator on L if, for arbitrary $p, q \in L$, $(int1)$ $int(p) \le p$; $(int2)$ $int(p \wedge q) = int(p) \wedge int(q)$; $(int3)$ $int(int(p)) = int(p)$.

Example 2.9. (Dvořák and Holčapek [21], Han and Zhao [32], Ouyang and Zhang [43], Rosenthal [47]) Let L be a bounded lattice and $a, b \in L$ be given, where $a \le b$. Then the mapping $int : L \to L$ defined by the following are all interior operators: for arbitrary $p \in L$, (i) $int(p) = 0$; (ii) $int(p) = p$; (iii) $int(p) = p \wedge a$; (iv) (f, g) is a Galois connection, $int = f \circ g$ satisfies $(int2)$; (v) $int_b^{\wedge}(p) = \begin{cases} p & p \ge b, \\ p \wedge b & \text{otherwise;} \end{cases}$ (vi) $int_b^{0}(p) = \begin{cases} p & p \ge b, \\ 0 & \text{otherwise;} \end{cases}$ (vii) $int_{a,b}^{\wedge}(p) = \begin{cases} p & p \ge b, \\ p \wedge a & \text{otherwise.} \end{cases}$ We say that two interior operators int_1 and int_2 commute on L provided that $int_1 \circ int_2 = int_2 \circ int_1$. And $int_b^\wedge \circ int_a^\wedge = int_a^\wedge \circ int_b^\wedge$. Specially, $int_1 = int_2$, interior operators int_1 and int_2 commute on L.

Note that the closure operator (interior operator) in [3, 47] where $(cl2)$ $p \le q \Rightarrow cl(p) \le cl(q)$ $(p \le q \Rightarrow int(p) \le int(q))$ is different from the above Definition 2.6 (Definition 2.8), without explanation we use the closure operator (interior operator) in [27]. Some authors, following [52, 66], call the closure operators nucleus. For additional background and information, please refer to [38].

Inspired by [63] and the constructions of uninorms in [65]. This paper is presented by new classes of uni-nullnorms on bounded lattices via a (conjunctive) uninorm and a t-norm as the main line. It is difficult to satisfy monotonicity and associativity of a uninorm U. For the framework of a uninorm U, we can not provide the unified forms, uninorms on L generalize $U_{\min}$ and $U_{\max}$ in most situations. A uninorm U varies as we deal with incomparable elements with e. For the possible order relations among p, q and r, where $p \in]0, e[, q \in I_e, r \in]e, 1[$, we obtain that $p < q$, or $p \parallel q$, or $q < r$, or $q \parallel r$.

By the results in [65]: A class of uninorm $\mathbf{U} = (F, G, F^{op}, G^{op})$, F denotes the disjunctive uninorm by the closure operator cl, G denotes the corresponding conjunctive uninorm by cl, F^{op} denotes the dual uninorm of F. The same class of uninorms in this paper can be written in this form. When constructing one uninorm of the four uninorms, the other three can be obtained accordingly. F and G can be one-to-one correspondences. The above observable properties can be used to construct different classes of uninorms.

The uninorm on L and its dual in the following can be obtained by closure (interior) operators. For additional background, please refer to [4, 43]. We can obtain the other uninorms on L in the following two propositions.

Proposition 2.10. (Çaylı [14]) Let L be a bounded lattice, T be a t-norm on $[0, e]$, S be a t-conorm on $[e, 1]$, $cl : L \to L$ be a closure operator and $int : L \to L$ be an interior operator, where $e \in L \setminus \{0, 1\}$. The operation $U^0_{cl,1}$ defined by

$$U^0_{cl,1}(p, q) = \begin{cases} T(p, q) & (p, q) \in [0, e]^2, \\ p \wedge q & (p, q) \in [0, e[\times (L \setminus [0, e[) \cup (L \setminus [0, e[) \times [0, e[, \\ q & (p, q) \in \{e\} \times (L \setminus [0, e]), \\ p & (p, q) \in (L \setminus [0, e]) \times \{e\}, \\ cl(p) \vee cl(q) & (p, q) \in (L \setminus [0, e])^2, \end{cases}$$

is a uninorm on L with the neutral element e iff $p > q$ for arbitrary $p \in I_e, q \in [0, e[$.

The operation $U^1_{int,1}$ defined by

$$U^1_{int,1}(p, q) = \begin{cases} S(p, q) & (p, q) \in [e, 1]^2, \\ p \vee q & (p, q) \in]e, 1] \times (L \backslash]e, 1]) \cup (L \backslash]e, 1]) \times]e, 1], \\ q & (p, q) \in \{e\} \times (L \setminus [e, 1]), \\ p & (p, q) \in (L \setminus [e, 1]) \times \{e\}, \\ int(p) \wedge int(q) & (p, q) \in (L \setminus [e, 1])^2, \end{cases}$$

is a uninorm on L with the neutral element e iff $p < q$ for arbitrary $p \in I_e, q \in]e, 1]$.

Proposition 2.11. (Çaylı [14]) Let L be a bounded lattice, T be a t-norm on $[0, e]$, S be a t-conorm on $[e, 1]$, where $e \in L \setminus \{0, 1\}$. The operation U_c^0 defined by

$$U_c^0(p,q) = \begin{cases} T(p,q) & (p,q) \in [0,e]^2, \\ p \wedge q & (p,q) \in [0,e[\times(L \setminus [0,e[) \cup (L \setminus [0,e[) \times [0,e[, \\ q & (p,q) \in \{e\} \times (L \setminus [0,e]), \\ p & (p,q) \in (L \setminus [0,e]) \times \{e\}, \\ S(p \vee e, q \vee e) & (p,q) \in (L \setminus [0,e])^2, \end{cases}$$

is a uninorm on L with neutral element e iff $p > q$ for arbitrary $p \in I_e, q \in [0, e[$.

The operation U_d^1 defined by

$$U_d^1(p,q) = \begin{cases} S(p,q) & (p,q) \in [e,1]^2, \\ p \vee q & (p,q) \in]e,1] \times (L \backslash]e,1]) \cup (L \backslash]e,1]) \times]e,1], \\ q & (p,q) \in \{e\} \times (L \setminus [e,1]), \\ p & (p,q) \in (L \setminus [e,1]) \times \{e\}, \\ T(p \wedge e, q \wedge e) & (p,q) \in (L \backslash]e,1])^2, \end{cases}$$

is a uninorm on L with neutral element e iff $p < q$ for arbitrary $p \in I_e, q \in]e, 1]$.

By replacing partial t-norms or t-conorms of nullnorms via closure (interior) operators, we can construct nullnorms on L. Theorem 2.12 can be obtained by [25].

Theorem 2.12. (Ertuğrul [25]) Let L be a bounded lattice, $cl : [0, a] \to [0, a]$ be a closure operator, $int : [a, 1] \to [a, 1]$ be an interior operator, S be a t-conorm on $[0, a]$ and T be a t-norm on $[a, 1]$, where $a \in L \setminus \{0, 1\}$.

(1) The operation $V_{cl,1} : L^2 \to L$ is a nullnorm on L with the absorbing element a, where

$$V_{cl,1}(p,q) = \begin{cases} T(p,q) & (p,q) \in [a,1]^2, \\ cl(p \wedge a) \vee cl(q \wedge a) & (p,q) \in]0,a]^2 \cup [0,a] \times I_a \cup I_a \times [0,a] \cup (I_a)^2, \\ p \vee q & (p,q) \in [0,a] \times \{0\} \cup \{0\} \times [0,a], \\ a & \text{otherwise.} \end{cases}$$

(2) The operation $V_{int,1} : L^2 \to L$ is a nullnorm on L with the absorbing element a, where

$$V_{int,1}(p,q) = \begin{cases} S(p,q) & (p,q) \in [0,a]^2, \\ int(p \vee a) \wedge int(q \vee a) & (p,q) \in [a,1[^2 \cup [a,1] \times I_a \cup I_a \times [a,1] \cup (I_a)^2, \\ p \wedge q & (p,q) \in [a,1] \times \{1\} \cup \{1\} \times [a,1], \\ a & \text{otherwise.} \end{cases}$$

3. CONSTRUCTIONS OF UNI-NULLNORMS ON BOUNDED LATTICES

Next construct new classes of uni-nullnorms on L by the following theorems. In the following, the index sets between closure operators and interior operators may not be one-to-one correspondences. The following is presented by the closure operators as the main line. Denote $cl_1(x) \vee cl_1(y)$ in the following figures by $(\triangle)$, denote $cl_2(x) \vee cl_2(y)$ in the following figures by $(\triangle\triangle)$.

3.1. Some constructions of uni-nullnorms on L via closure (interior) operators

Closure (interior) operators are significant for the constructions of uninorms on L, closure (interior) operators can also play a vital place in the constructions of uni-nullnorms on L. Next we obtain some constructions of uni-nullnorms on L via closure (interior) operators.

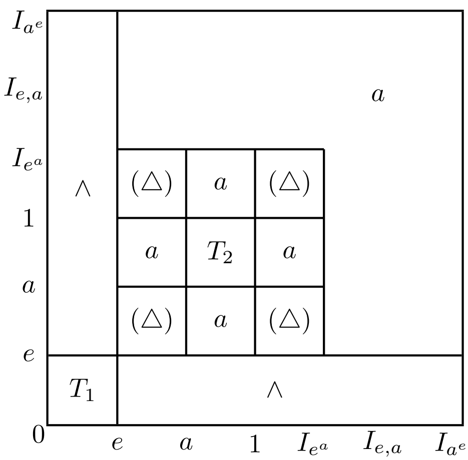

Fig. 1. $F^{T,S,0}_{cl,1}$ in Theorem 3.1(1).

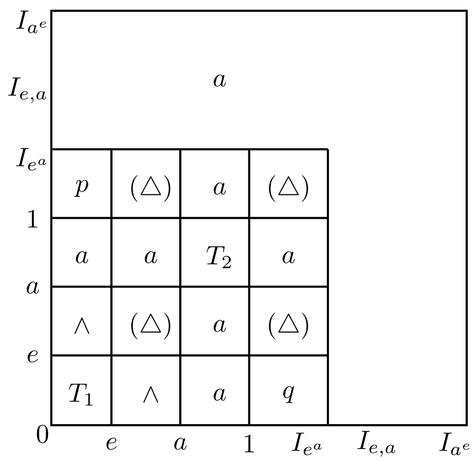

Fig. 2. $F^{T,S,a}_{cl,2}$ in Theorem 3.1(2).

Theorem 3.1. Let L be a bounded lattice, $cl : ([e,a] \cup I_{e^a}) \to ([e,a] \cup I_{e^a})$ be a closure operator, T_1 be a t-norm on $[0,e]$ and T_2 be a t-norm on $[a,1]$, where $e, a \in L \setminus \{0,1\}$.

(1) The operation $F^{T,S,0}_{cl,1}$ defined by

$$F^{T,S,0}_{cl,1}(p,q) = \begin{cases} T_1(p,q) & (p,q) \in [0,e]^2, \\ cl(p) \vee cl(q) & (p,q) \in]e,a]^2 \cup]e,a] \times I_{e^a} \cup I_{e^a} \times]e,a] \cup (I_{e^a})^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ p \wedge q & (p,q) \in [0,e[\times (L \setminus [0,e[) \cup (L \setminus [0,e[) \times [0,e[, \\ p \vee q & (p,q) \in [e,a] \times \{e\} \cup \{e\} \times [e,a], \\ p & (p,q) \in I_{e^a} \times \{e\}, \\ q & (p,q) \in \{e\} \times I_{e^a}, \\ a & \text{otherwise,} \end{cases}$$

is a uni-nullnorm on L iff $p < q$ for arbitrary $p \in [0, e[, q \in I_{e^a} \cup I_{e,a}$.

(2) $cl : ([e, a] \cup I_{e^a}) \to ([e, a] \cup I_{e^a})$ satisfies $cl(p) \vee cl(q) \in ([e, a[\cup I_{e^a})$ for arbitrary $p, q \in ([e, a[\cup I_{e^a})$. The operation $F_{cl,2}^{T,S,a}$ is a uni-nullnorm on L, where

$$F_{cl,2}^{T,S,a}(p,q) = \begin{cases} T_1(p,q) & (p,q) \in [0,e]^2, \\ cl(p) \vee cl(q) & (p,q) \in]e,a]^2 \cup]e,a] \times I_{e^a} \cup I_{e^a} \times]e,a] \cup (I_{e^a})^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ p \wedge q & (p,q) \in [0,e[\times[e,a[\cup[e,a[\times[0,e[, \\ p \vee q & (p,q) \in]e,a] \times \{e\} \cup \{e\} \times]e,a], \\ p & (p,q) \in I_{e^a} \times \{e\} \cup [0,e[\times I_{e^a}, \\ q & (p,q) \in \{e\} \times I_{e^a} \cup I_{e^a} \times [0,e[, \\ a & \text{otherwise.} \end{cases}$$

Proof. (1) Necessity: For the case: $p < q$ for arbitrary $p \in [0, e[, q \in I_{e^a} \cup I_{e,a}$, take $r = 1$, by $F_{cl,1}^{T,S,0}(F_{cl,1}^{T,S,0}(p, q), 1) = F_{cl,1}^{T,S,0}(p \wedge q, 1) = p \wedge q$, $F_{cl,1}^{T,S,0}(p, F_{cl,1}^{T,S,0}(q, 1)) = F_{cl,1}^{T,S,0}(p, a) = p$, then $p < q$ for arbitrary $p \in [0, e[, q \in I_{e^a} \cup I_{e,a}$.

Sufficiency: By the definition of $F_{cl,1}^{T,S,0}$, it is easy to get that $F_{cl,1}^{T,S,0}$ is commutative, $F_{cl,1}^{T,S,0}(p, e) = p$ for arbitrary $p \in [0, a]$, $F_{cl,1}^{T,S,0}(p, a) = a$ for arbitrary $p \in [e, 1]$, $F_{cl,1}^{T,S,0}(p, 1) = p$ for arbitrary $p \in [a, 1]$, take steps to prove monotonicity and associativity.

(i) Monotonicity. For $p, q, r \in L$ with $p \leq q$, take steps to check $F_{cl,1}^{T,S,0}(p, r) \leq F_{cl,1}^{T,S,0}(q, r)$. For $p, q \in L$ are simultaneous elements of $[0, e]$, or $]e, a]$, or $[a, 1]$, or I_{e^a}, or $I_{e,a}$, or I_{a^e}, the proof is obvious. Next we prove the remaining.

1. $p \in [0, e[$, $q \in L \setminus [0, e[$.
   1.1 $r \in [0, e[$. $F_{cl,1}^{T,S,0}(p, r) = T_1(p, r) \leq r = q \wedge r = F_{cl,1}^{T,S,0}(q, r)$.
   1.2 $r \in L \setminus [0, e[$. By $p < q$ for arbitrary $p \in [0, e[, q \in I_{e^a} \cup I_{e,a}$, $F_{cl,1}^{T,S,0}(q, r) \in [e, 1]$, then $F_{cl,1}^{T,S,0}(p, r) = p \wedge r = p \leq F_{cl,1}^{T,S,0}(q, r)$.

2. $p = e$, $q \in]e, a] \cup]a, 1] \cup I_{a^e}$.
   2.1 $r \in [0, e[$. $F_{cl,1}^{T,S,0}(p, r) = r = q \wedge r = F_{cl,1}^{T,S,0}(q, r)$.
   2.2 $r \in]e, a]$. $F_{cl,1}^{T,S,0}(p, r) = r \leq cl(q) \vee cl(r) \leq F_{cl,1}^{T,S,0}(q, r)$.
   2.3 $r \in]a, 1]$. $F_{cl,1}^{T,S,0}(p, r) = a \leq F_{cl,1}^{T,S,0}(q, r)$.
   2.4 $r \in I_{e^a}$. $F_{cl,1}^{T,S,0}(p, r) = r \leq cl(q) \vee cl(r) \leq F_{cl,1}^{T,S,0}(q, r)$.
   2.5 $r \in I_{e,a} \cup I_{a^e}$. $F_{cl,1}^{T,S,0}(p, r) = a = F_{cl,1}^{T,S,0}(q, r)$.
   2.6 $r = e$. By $F_{cl,1}^{T,S,0}(q, r) \in]e, a]$, then $F_{cl,1}^{T,S,0}(p, r) = e \leq F_{cl,1}^{T,S,0}(q, r)$.

3. $p \in]e, a[$, $q \in]a, 1] \cup I_{a^e}$.
   3.1 $r \in [0, e[$. $F_{cl,1}^{T,S,0}(p, r) = p \wedge r = r = q \wedge r = F_{cl,1}^{T,S,0}(q, r)$.
   3.2 $r \in]e, a[\cup I_{e^a}$. $F_{cl,1}^{T,S,0}(p, r) = cl(p) \vee cl(r) \leq a = F_{cl,1}^{T,S,0}(q, r)$.
   3.3 $r \in [a, 1]$. $F_{cl,1}^{T,S,0}(p, r) = a \leq F_{cl,1}^{T,S,0}(q, r)$.

3.4 $r \in I_{e,a} \cup I_{a^e}$. $F_{cl,1}^{T,S,0}(p,r) = a = F_{cl,1}^{T,S,0}(q,r)$.
3.5 $r = e$. $F_{cl,1}^{T,S,0}(p,r) = p \leq a = F_{cl,1}^{T,S,0}(q,r)$.

4. $p \in I_{e^a}$, $q \in]e,a] \cup]a,1] \cup I_{e,a} \cup I_{a^e}$.
4.1 $r \in [0,e[$. $F_{cl,1}^{T,S,0}(p,r) = p \wedge r = r = q \wedge r = F_{cl,1}^{T,S,0}(q,r)$.
4.2 $r \in]e,a] \cup I_{e^a}$. By $F_{cl,1}^{T,S,0}(q,r) \in [cl(q) \vee cl(r), a]$, $F_{cl,1}^{T,S,0}(p,r) = cl(p) \vee cl(r) \leq cl(q) \vee cl(r) \leq F_{cl,1}^{T,S,0}(q,r)$.
4.3 $r \in]a,1]$. $F_{cl,1}^{T,S,0}(p,r) = a \leq F_{cl,1}^{T,S,0}(q,r)$.
4.4 $r \in I_{e,a} \cup I_{a^e}$. $F_{cl,1}^{T,S,0}(p,r) = a = F_{cl,1}^{T,S,0}(q,r)$.
4.5 $r = e$. By $F_{cl,1}^{T,S,0}(q,r) \in]e,a]$, then $F_{cl,1}^{T,S,0}(p,r) = p \leq F_{cl,1}^{T,S,0}(q,r)$.

5. $p \in I_{e,a}$, $q \in]a,1] \cup I_{a^e}$.
5.1 $r \in [0,e[$. $F_{cl,1}^{T,S,0}(p,r) = p \wedge r = r = q \wedge r = F_{cl,1}^{T,S,0}(q,r)$.
5.2 $r \in L \setminus [0,e[$. $F_{cl,1}^{T,S,0}(p,r) = a \leq F_{cl,1}^{T,S,0}(q,r)$.

6. $p \in I_{a^e}$, $q \in]a,1]$.
6.1 $r \in [0,e[$. $F_{cl,1}^{T,S,0}(p,r) = p \wedge r = r = q \wedge r = F_{cl,1}^{T,S,0}(q,r)$.
6.2 $r \in L \setminus [0,e[$. $F_{cl,1}^{T,S,0}(p,r) = a \leq F_{cl,1}^{T,S,0}(q,r)$.

(ii) Associativity. If $\{p,q,r\} \subseteq \{e,a\}$, or $\{p,q,r\} \subseteq [0,e[\cup]e,a[\cup]a,1]$, it is easy to prove that $F_{cl,1}^{T,S,0}(F_{cl,1}^{T,S,0}(p,q),r) = F_{cl,1}^{T,S,0}(p, F_{cl,1}^{T,S,0}(q,r))$. We need to check the rest.

1. $\{p,q\} \subseteq [0,e[, \{r\} \subseteq L \setminus [0,e[$.
$L = F_{cl,1}^{T,S,0}(F_{cl,1}^{T,S,0}(p,q),r) = T_1(p,q) = F_{cl,1}^{T,S,0}(p,q) = F_{cl,1}^{T,S,0}(p, F_{cl,1}^{T,S,0}(q,r)) = R$.

2. $\{p\} \subseteq [0,e[$, $\{q,r\} \subseteq L \setminus [0,e[$.
$L = F_{cl,1}^{T,S,0}(p,r) = p = F_{cl,1}^{T,S,0}(p, F_{cl,1}^{T,S,0}(q,r)) = R$.

3. $\{p,q\} \subseteq]e,a[$, $\{r\} \subseteq I_{e^a}$, or $\{p\} \subseteq]e,a[$, $\{q,r\} \subseteq I_{e^a}$, or $\{p,q,r\} \subseteq I_{e^a}$.
$L = F_{cl,1}^{T,S,0}(cl(p) \vee cl(q), r) = cl(p) \vee cl(q) \vee cl(r) = F_{cl,1}^{T,S,0}(p, cl(q) \vee cl(r)) = R$.

4. $\{p,q\} \subseteq]e,a[$, $\{r\} \subseteq I_{e,a} \cup I_{a^e}$, or $\{p\} \subseteq]e,a[$, $\{q\} \subseteq I_{e^a}$, $\{r\} \subseteq I_{e,a} \cup I_{a^e}$, or $\{p,q\} \subseteq I_{e^a}$, $\{r\} \subseteq I_{e,a} \cup I_{a^e}$.
$L = F_{cl,1}^{T,S,0}(cl(p) \vee cl(q), r) = a = F_{cl,1}^{T,S,0}(p,a) = R$.

5. $\{p\} \subseteq]e,a[$, $\{q,r\} \subseteq I_{e,a} \cup I_{a^e}$.
$L = F_{cl,1}^{T,S,0}(a,r) = a = F_{cl,1}^{T,S,0}(p,a) = R$.

6. $\{p,q\} \subseteq]a,1]$, $\{r\} \subseteq I_{e^a} \cup I_{e,a} \cup I_{a^e}$.
$L = F_{cl,1}^{T,S,0}(T_2(p,q),r) = a = F_{cl,1}^{T,S,0}(p,a) = R$.

7. $\{p\} \subseteq]a,1]$, $\{q,r\} \subseteq I_{e^a} \cup I_{e,a} \cup I_{a^e}$, by $F_{cl,1}^{T,S,0}(q,r) \in]e,a] \cup I_{e^a}$.
$L = F_{cl,1}^{T,S,0}(a,r) = a = F_{cl,1}^{T,S,0}(p, F_{cl,1}^{T,S,0}(q,r)) = R$.

8. $\{p\} \subseteq I_{e^a}$, $\{q,r\} \subseteq I_{e,a} \cup I_{a^e}$, or $\{p\} \subseteq]e,a]$, $\{q\} \subseteq]a,1]$, $\{r\} \subseteq I_{e^a} \cup I_{e,a} \cup I_{a^e}$, or $\{p,q,r\} \subseteq I_{e,a} \cup I_{a^e}$.
$L = F_{cl,1}^{T,S,0}(a,r) = a = F_{cl,1}^{T,S,0}(p,a) = R$.

(2) It can be proved analogously to (1). □

Theorem 3.1 can be equivalently to characterize it as the following proposition.

Proposition 3.2. Consider the construction of the uni-nullnorm from Theorem 3.1 denoted as $F = F^{T,S,0}_{cl,1}$. F is a uni-nullnorm on L iff $p < q$ for arbitrary $p \in [0, e[$ and $q \in I_{e^a} \cup I_{e,a}$. Using this condition, L can be expressed as the union of the following three sets: $L_1 = [0, e], L_2 = [e, a] \cup I_{e^a}$ and $L_3 = [a, 1] \cup I_{e,a} \cup I_{a^e}$, where $L_1 \cap L_2 = \{e\}$ and $L_2 \cap L_3 = \{a\}$. Consider t-norms T_1 on $[0, e]$ and T_2 on $[a, 1]$, $cl : ([e, a] \cup I_{e^a}) \to ([e, a] \cup I_{e^a})$ be a closure operator. Moreover, consider the t-conorm $S = \vee$ on $[e, a]$. To get the construction of F, in the first step, extend S from $[e, a]$ to L_2 and T_2 from $[a, 1]$ to L_3 in an appropriate way. Proceed as follows:

1. The extension of S (formally restricted to $cl(L_2)$) to L_2 can be given as

$$\tilde{S}(p,q) = \begin{cases} S(cl(p), cl(q)) & p, q \in L_2 \setminus \{e\}, \\ p & q = e, \\ q & p = e. \end{cases}$$

By redefining $cl' : L_2 \to L_2$ as $cl'(p) = cl(p \vee e)$. It is easy to verify that cl' is again a closure operator. Comparing with the approach in [21]. e is the minimum (not the bottom) element of L_2, so it seems to be a slight generalization.

2. The generalization of T_2 to L_3 can be done directly as

$$\tilde{T_2}(p,q) = \begin{cases} T_2(p,q) & p, q \in [a, 1], \\ a & \text{otherwise}. \end{cases}$$

Note that the extension $\tilde{T_2}$ is an example of an operation, for which 1 is not the neutral element, and $\tilde{T_2}$ is no longer a t-norm on L_3 (L_3 is a poset, but not a lattice).

Now, three operations $T_1, \tilde{S}$, and $\tilde{T_2}$ have been obtained. In the second step, construct a uninorm U on $M = L_1 \cup L_2$ from T_1 and $\tilde{S}$ in the conjunctive form, which can be given as follows:

$$U(p,q) = \begin{cases} T_1(p,q) & p, q \in L_1, \\ \tilde{S}(p,q) & p, q \in L_2, \\ p \wedge q & \text{otherwise}. \end{cases}$$

Since $p \leq q$ for arbitrary $p \in L_1$ and $q \in L_2$, one can simply verify the monotonicity of U, and U should be a uninorm on M (i.e., M endowed with U is a partial ordered monoid).

3. Therefore, two operations U and $\tilde{T_2}$ have been obtained. In the third step, construct the uni-nullnorm F on L from the operations U and $\tilde{T_2}$ as follows (denote $L_1^- = L_1 \setminus \{e\}$)

$$F(p,q) = \begin{cases} U(p,q) & p, q \in M, \\ \tilde{T_2}(p,q) & p, q \in L_3, \\ p \wedge q & (p,q) \in L_1^- \times (L \setminus L_1^-) \cup (L \setminus L_1^-) \times L_1^-, \\ a & \text{otherwise}. \end{cases}$$

For Theorem 3.1(2), as a is the absorbing element of $F_{cl,2}^{T,S,a}$, add the condition: $cl : ([e,a] \cup I_{e^a}) \to ([e,a] \cup I_{e^a})$ satisfies $cl(p) \vee cl(q) \in ([e,a[\cup I_{e^a})$ for arbitrary $p,q \in ([e,a[\cup I_{e^a})$ (The reason is similar to Remark 3.8), at this moment cl play a key role. $F_{cl,2}^{T,S,a}$ is given by the same structure as $F_{cl,1}^{T,S,0}$, only have the following changes, $F_{cl,2}^{T,S,a}(p,q) = p$ for $(p,q) \in [0,e[\times I_{e^a}$, $F_{cl,2}^{T,S,a}(p,q) = a$ for $(p,q) \in [0,e[\times([a,1] \cup I_{e,a} \cup I_{a^e})$, and omit the condition $p < q$ for arbitrary $p \in [0,e[$ and $q \in I_{e^a} \cup I_{e,a}$.

Similar to Theorem 3.1, we can obtain the following proposition.

Proposition 3.3. Let L be a bounded lattice, $int : ([0,e] \cup I_{e^a}) \to ([0,e] \cup I_{e^a})$ be an interior operator, S be a t-conorm on $[e,a]$, and T_2 be a t-norm on $[a,1]$, where $e,a \in L \setminus \{0,1\}$.

(1) The operation $F_{int,1}^{T,S,0} : L^2 \to L$ defined by

$$F_{int,1}^{T,S,0}(p,q) = \begin{cases} int(p) \wedge int(q) & (p,q) \in [0,e[^2 \cup [0,e[\times I_{e^a} \cup I_{e^a} \times [0,e[\cup (I_{e^a})^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ p \wedge q & (p,q) \in [0,e[\times(L \setminus ([0,e[\cup I_{e^a})) \\ & \cup (L \setminus ([0,e[\cup I_{e^a})) \times [0,e[, \\ p & (p,q) \in I_{e^a} \times [e,a[, \\ q & (p,q) \in [e,a[\times I_{e^a}, \\ a & \text{otherwise,} \end{cases}$$

is a uni-nullnorm on L iff $p < q$ for arbitrary $p \in [0,e[, q \in I_{e,a}$.

(2) $S : [e,a]^2 \to [e,a]$ satisfies $S(p,q) \in [e,a[$ for arbitrary $p,q \in [e,a[$. The operation $F_{int,2}^{T,S,a} : L^2 \to L$ is a uni-nullnorm on L, where

$$F_{int,2}^{T,S,a}(p,q) = \begin{cases} int(p) \wedge int(q) & (p,q) \in [0,e[^2 \cup [0,e[\times I_{e^a} \cup I_{e^a} \times [0,e[\cup (I_{e^a})^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ p \wedge q & (p,q) \in [0,e[\times[e,a[\cup[e,a[\times[0,e[, \\ p & (p,q) \in I_{e^a} \times [e,a[, \\ q & (p,q) \in [e,a[\times I_{e^a}, \\ a & \text{otherwise.} \end{cases}$$

Proof. It can be proved analogously to Theorem 3.1(1). □

3.2. Uni-nullnorms on L with a uninorm and a t-norm by closure (interior) operators

Next we introduce some new constructions of (conjunctive) uni-nullnorms on L.

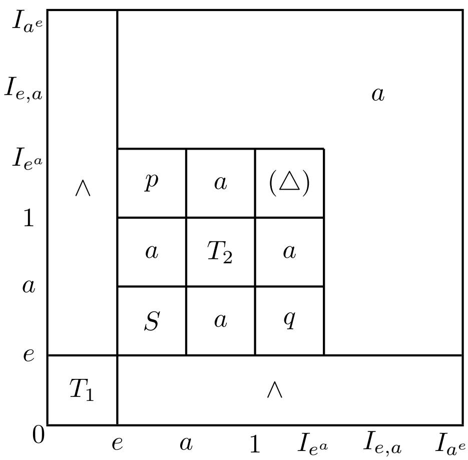

Fig. 3. $F_{cl,3}^{T,S,0}$ in Theorem 3.4(1).

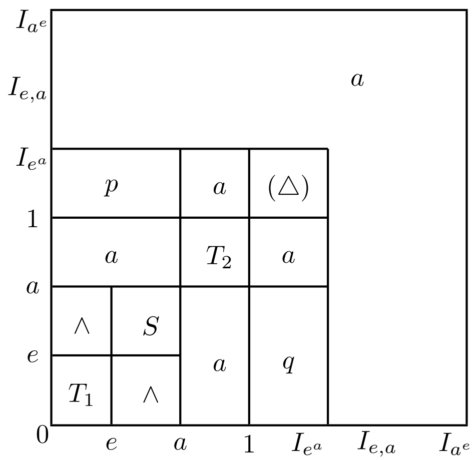

Fig. 4. $F_{cl,4}^{T,S,a}$ in Theorem 3.4(2).

Theorem 3.4. Let L be a bounded lattice, T_1 be a t-norm on $[0, e]$, T_2 be a t-norm on $[a, 1]$, S be a t-conorm on $[e, a]$, $cl : L \to L$ be a closure operator and $cl(p) \vee cl(q) \in I_{e^a}$ for arbitrary $p, q \in I_{e^a}$, where $e, a \in L \setminus \{0, 1\}$.

(1) The operation $F_{cl,3}^{T,S,0} : L^2 \to L$ defined by

$$F_{cl,3}^{T,S,0}(p,q) = \begin{cases} T_1(p,q) & (p,q) \in [0,e]^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ cl(p) \vee cl(q) & (p,q) \in I_{e^a} \times I_{e^a}, \\ p \wedge q & (p,q) \in [0,e[\times (L \setminus [0,e[) \cup (L \setminus [0,e[) \times [0,e[, \\ p & (p,q) \in]e,a] \times I_{e^a} \cup I_{e^a} \times \{e\}, \\ q & (p,q) \in I_{e^a} \times]e,a] \cup \{e\} \times I_{e^a}, \\ a & \text{otherwise,} \end{cases}$$

is a uni-nullnorm on L iff $p < q$ for arbitrary $p \in [0, e[, q \in I_{e^a} \cup I_{e,a}$, $p < q$ for arbitrary $p \in I_{e^a}, q \in]e, a]$.

(2) $S : [e, a]^2 \to [e, a]$ satisfies $S(p, q) \in [e, a[$ for arbitrary $p, q \in [e, a[$. The operation $F_{cl,4}^{T,S,a} : L^2 \to L$ defined by

$$F_{cl,4}^{T,S,a}(p,q) = \begin{cases} T_1(p,q) & (p,q) \in [0,e]^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ cl(p) \vee cl(q) & (p,q) \in I_{e^a} \times I_{e^a}, \\ p \wedge q & (p,q) \in [0,e[\times [e,a[\cup [e,a[\times [0,e[, \\ p & (p,q) \in ([0,e[\cup]e,a]) \times I_{e^a} \cup I_{e^a} \times \{e\}, \\ q & (p,q) \in I_{e^a} \times ([0,e[\cup]e,a]) \cup \{e\} \times I_{e^a}, \\ a & \text{otherwise,} \end{cases}$$

is a uni-nullnorm on L iff $p < q$ for arbitrary $p \in I_{e^a}, q \in]e, a]$.

P r o o f. (1) Necessity: For arbitrary $p \in [0, e[$, $q \in I_{e^a} \cup I_{e,a}$, by $F_{cl,3}^{T,S,0}(p, F_{cl,3}^{T,S,0}(q, a)) = F_{cl,3}^{T,S,0}(p, a) = p$, $F_{cl,3}^{T,S,0}(F_{cl,3}^{T,S,0}(p, q), a) = F_{cl,3}^{T,S,0}(p \wedge q, a) = p \wedge q$, because $F_{cl,3}^{T,S,0}$ is associative, then $p < q$ for arbitrary $p \in [0, e[$, $q \in I_{e^a} \cup I_{e,a}$. If $]e, a[= \emptyset$, then $p < q$ for arbitrary $p \in I_{e^a}$, $q \in]e, a]$. Consider $]e, a[\neq \emptyset$, assume that $p \parallel q$ for $p \in I_{e^a}$, $q \in]e, a[$, by $F_{cl,3}^{T,S,0}(e, p) = p$, $F_{cl,3}^{T,S,0}(q, p) = q$, this is a contradiction with $F_{cl,3}^{T,S,0}(e, p) \leq F_{cl,3}^{T,S,0}(q, p)$ because of monotonicity of $F_{cl,3}^{T,S,0}$, then $p < q$ for arbitrary $p \in I_{e^a}$, $q \in]e, a]$.

Sufficiency: By the definition of $F_{cl,3}^{T,S,0}$, it is easy to get that $F_{cl,3}^{T,S,0}$ is commutative, $F_{cl,3}^{T,S,0}(p, e) = p$ for arbitrary $p \in [0, a]$, $F_{cl,3}^{T,S,0}(p, a) = a$ for arbitrary $p \in [e, 1]$, $F_{cl,3}^{T,S,0}(p, 1) = p$ for arbitrary $p \in [a, 1]$, we need to prove monotonicity and associativity.

(i) Monotonicity. For $p, q, r \in L$ with $p \leq q$, we need to prove $F_{cl,3}^{T,S,0}(p, r) \leq F_{cl,3}^{T,S,0}(q, r)$. For $p, q \in L$ are simultaneous elements of $[0, e]$, or $[e, a]$, or $[a, 1]$, or I_{e^a}, or $I_{e,a}$, or I_{a^e}, the proof is obvious. Next we prove the remaining cases.

1. $p \in [0, e[$, $q \in L \setminus [0, e[$.

1.1 $r \in [0, e[$. $F_{cl,3}^{T,S,0}(p, r) = T_1(p, r) \leq r = q \wedge r = F_{cl,3}^{T,S,0}(q, r)$.

1.2 $r \in L \setminus [0, e[$. By $F_{cl,3}^{T,S,0}(q, r) \in [e, a] \cup I_{e^a}$, then $F_{cl,3}^{T,S,0}(p, r) = p \wedge r = p \leq F_{cl,3}^{T,S,0}(q, r)$.

2. $p \in [e, a[$, $q \in [a, 1] \cup I_{a^e}$.

2.1 $r \in [0, e[$. $F_{cl,3}^{T,S,0}(p, r) = p \wedge r = r = q \wedge r = F_{cl,3}^{T,S,0}(q, r)$.

2.2 $r \in L \setminus [0, e[$. $F_{cl,3}^{T,S,0}(p, r) \leq a \leq F_{cl,3}^{T,S,0}(q, r)$.

3. $p \in I_{e^a}$.

3.1 $q \in]e, a] \cup]a, 1]$.

3.1.1 $r \in [0, e[$. $F_{cl,3}^{T,S,0}(p, r) = p \wedge r = r = q \wedge r = F_{cl,3}^{T,S,0}(q, r)$.

3.1.2 $r \in]e, a]$. By the value of $F_{cl,3}^{T,S,0}(q, r)$ is $S(q, r)$ or a, then $F_{cl,3}^{T,S,0}(p, r) = r \leq F_{cl,3}^{T,S,0}(q, r)$.

3.1.3 $r \in I_{e^a}$. By the value of $F_{cl,3}^{T,S,0}(q, r)$ is q or a, $m < n$ for arbitrary $m \in I_{e^a}$, $n \in]e, a]$, then $F_{cl,3}^{T,S,0}(p, r) = cl(p) \vee cl(r) \leq F_{cl,3}^{T,S,0}(q, r)$.

3.1.4 $r \in]a, 1] \cup I_{e,a} \cup I_{a^e}$. $F_{cl,3}^{T,S,0}(p, r) = a \leq F_{cl,3}^{T,S,0}(q, r)$.

3.1.5 $r \in \{e\}$. By $F_{cl,3}^{T,S,0}(q, r) \in]e, a]$, $m < n$ for arbitrary $m \in I_{e^a}$, $n \in]e, a]$, then $F_{cl,3}^{T,S,0}(p, r) = p \leq F_{cl,3}^{T,S,0}(q, r)$.

3.2 $q \in I_{e,a} \cup I_{a^e}$.

3.2.1 $r \in [0, e[$. $F_{cl,3}^{T,S,0}(p, r) = p \wedge r = r = q \wedge r = F_{cl,3}^{T,S,0}(q, r)$.

3.2.2 $r \in L \setminus [0, e[$. $F_{cl,3}^{T,S,0}(p, r) \leq a = F_{cl,3}^{T,S,0}(q, r)$.

4. $p \in I_{e,a}$, $q \in I_{a^e} \cup]a, 1]$.

4.1 $r \in [0, e[$. $F_{cl,3}^{T,S,0}(p, r) = p \wedge r = r = q \wedge r = F_{cl,3}^{T,S,0}(q, r)$.

4.2 $r \in L \setminus [0, e[$. $F_{cl,3}^{T,S,0}(p, r) = a \leq F_{cl,3}^{T,S,0}(q, r)$.

5. $p \in I_{a^e}$, $q \in]a, 1]$.

5.1 $r \in [0, e[$. $F_{cl,3}^{T,S,0}(p, r) = p \wedge r = r = q \wedge r = F_{cl,3}^{T,S,0}(q, r)$.

5.2 $r \in L \setminus [0, e[$. $F_{cl,3}^{T,S,0}(p, r) = a \leq F_{cl,3}^{T,S,0}(q, r)$.

(ii) Associativity. If one of the elements p, q and r equals e or a, or $\{p, q, r\} \subseteq [0, e[\cup]e, a[\cup]a, 1]$, it is clear that $F_{cl,3}^{T,S,0}(F_{cl,3}^{T,S,0}(p, q), r) = F_{cl,3}^{T,S,0}(p, F_{cl,3}^{T,S,0}(q, r))$. We need to check the rest.

1. $\{p, q, r\} \subseteq]e, a[\cup]a, 1] \cup I_{e,a} \cup I_{a^e}$, it is obvious that $F_{cl,3}^{T,S,0}$ in the domains is a nullnorm that is associative, then $L = F_{cl,3}^{T,S,0}(F_{cl,3}^{T,S,0}(p, q), r) = F_{cl,3}^{T,S,0}(p, F_{cl,3}^{T,S,0}(q, r)) = R$.

2. $\{p, q\} \subseteq [0, e[, \{r\} \subseteq L \setminus [0, e[$.
$L = F_{cl,3}^{T,S,0}(T_1(p, q), r) = T_1(p, q) = F_{cl,3}^{T,S,0}(p, q) = R$.

3. $\{p\} \subseteq [0, e[, \{q, r\} \subseteq L \setminus [0, e[$.
$L = F_{cl,3}^{T,S,0}(p, r) = p = F_{cl,3}^{T,S,0}(p, F_{cl,3}^{T,S,0}(q, r)) = R$.

4. $\{p, q, r\} \subseteq I_{e^a}$.
$L = F_{cl,3}^{T,S,0}(cl(p) \vee cl(q), r) = cl(p) \vee cl(q) \vee cl(r) = F_{cl,3}^{T,S,0}(p, cl(q) \vee cl(r)) = R$.

5. $\{p, q\} \subseteq]e, a[, \{r\} \subseteq I_{e^a}$.
$L = F_{cl,3}^{T,S,0}(S(p, q), r) = S(p, q) = F_{cl,3}^{T,S,0}(p, q) = R$.

6. $\{p\} \subseteq]e, a[, \{q, r\} \subseteq I_{e^a}$.
$L = F_{cl,3}^{T,S,0}(p, r) = p = F_{cl,3}^{T,S,0}(p, cl(q) \vee cl(r)) = R$.

7. $\{p\} \subseteq]a, 1] \cup I_{e,a} \cup I_{a^e}, \{q, r\} \subseteq I_{e^a}$.
$L = F_{cl,3}^{T,S,0}(a, r) = a = F_{cl,3}^{T,S,0}(p, cl(q) \vee cl(r)) = R$.

8. $\{p\} \subseteq]e, a[, \{q\} \subseteq]a, 1] \cup I_{e,a} \cup I_{a^e}, \{r\} \subseteq I_{e^a}$, or $\{p\} \subseteq I_{e^a}, \{q, r\} \subseteq I_{e,a} \cup I_{a^e}$, or $\{p\} \subseteq I_{e^a}, \{q\} \subseteq I_{e,a} \cup I_{a^e}, \{r\} \subseteq]a, 1]$.
$L = F_{cl,3}^{T,S,0}(a, r) = a = F_{cl,3}^{T,S,0}(p, a) = R$.

9. $\{p\} \subseteq I_{e^a}, \{q, r\} \subseteq]a, 1]$.
$L = F_{cl,3}^{T,S,0}(a, r) = a = F_{cl,3}^{T,S,0}(p, T_2(q, r)) = R$.

(2) It can be proved analogously to (1). □

Theorem 3.4 can be equivalently to characterize it as the following proposition.

Proposition 3.5. Consider the construction of the uni-nullnorm from Theorem 3.4 denoted as $F = F_{cl,3}^{T,S,0}$. Let $cl : L \to L$ be a closure operator and $cl(p) \vee cl(q) \in I_{e^a}$ for arbitrary $p, q \in I_{e^a}$, F is a uni-nullnorm on L iff $p < q$ for arbitrary $p \in [0, e[$ and $q \in I_{e^a} \cup I_{e,a}$, $m < n$ for arbitrary $m \in I_{e^a}, n \in]e, a]$. Using these conditions, L can be expressed as the union of the following four sets: $L_1 = [0, e], L_2 = [e, a]$, $L_3 = [a, 1] \cup I_{e,a} \cup I_{a^e}$ and $L_4 = I_{e^a}$, where clearly $L_1 \cap L_2 = \{e\}$ and $L_2 \cap L_3 = \{a\}$. Consider t-norms T_1 on $[0, e]$ and T_2 on $[a, 1]$, the t-conorm S on $[e, a]$. Extend T_2 from $[a, 1]$ to L_3 in an appropriate way. We can proceed as follows:

1. The extension of the t-norm T_2 to L_3 can be done directly as

$$\tilde{T_2}(p, q) = \begin{cases} T_2(p, q) & p, q \in [a, 1], \\ a & \text{otherwise.} \end{cases}$$

Note that the extension $\tilde{T_2}$ is an example of an operation, for which 1 is not the neutral element, and $\tilde{T_2}$ is no longer a t-norm on L_3 (L_3 is a poset, but not a lattice).

Now, three operations T_1, S, and $\tilde{T_2}$ have been obtained. In the first step, construct a uninorm U on $M = L_1 \cup L_2 \cup L_4$ from T_1 and S in the conjunctive form, which can be given as follows:

$$U(p,q) = \begin{cases} T_1(p,q) & p,q \in L_1, \\ S(p,q) & p,q \in L_2, \\ cl(p) \vee cl(q) & p,q \in L_4, \\ p & (p,q) \in]e,a] \times I_{e^a} \cup I_{e^a} \times \{e\}, \\ q & (p,q) \in I_{e^a} \times]e,a] \cup \{e\} \times I_{e^a}, \\ p \wedge q & \text{otherwise.} \end{cases}$$

The above uninorm U is precisely the structure of $U_{cl,3}^{T,S,0}$ in [65], and U should be a slight change, U is a uninorm on M (i. e., M endowed with U is a partial ordered monoid).

2. Therefore, two operations U and $\tilde{T_2}$ have been obtained. In the second step, we can construct the uni-nullnorm F on L from the operations U and $\tilde{T_2}$ as follows (denote $L_1^- = L_1 \setminus \{e\}$)

$$F(p,q) = \begin{cases} U(p,q) & p,q \in M, \\ \tilde{T_2}(p,q) & p,q \in L_3, \\ p \wedge q & (p,q) \in L_1^- \times (L \setminus L_1^-) \cup (L \setminus L_1^-) \times L_1^-, \\ a & \text{otherwise.} \end{cases}$$

For Theorem 3.4(2), as a is the absorbing element of $F_{cl,4}^{T,S,a}$, add the condition: $S : [e,a]^2 \to [e,a]$ satisfies $S(p,q) \in [e,a[$ for arbitrary $p,q \in [e,a[$ (See Remark 3.8). $F_{cl,4}^{T,S,a}$ is given by the same structure as $F_{cl,3}^{T,S,0}$, only have the following changes, $F_{cl,4}^{T,S,a}(p,q) = p$ for $(p,q) \in [0,e[\times I_{e^a}$, $F_{cl,4}^{T,S,a}(p,q) = a$ for $(p,q) \in [0,e[\times([a,1] \cup I_{e,a} \cup I_{a^e})$.

The following proposition follows from Theorem 3.4 and Corollary 4.7 [65].

Proposition 3.6. Let L be a bounded lattice, T_1 be a t-norm on $[0,e]$, T_2 be a t-norm on $[a,1]$, S be a t-conorm on $[e,a]$, and $int : L \to L$ be an interior operator, where $e,a \in L \setminus \{0,1\}$.

(1) The operation $F_{int,3}^{T,S,0} : L^2 \to L$ defined by

$$F_{int,3}^{T,S,0}(p,q) = \begin{cases} T_1(p,q) & (p,q) \in [0,e]^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ int(p) \wedge int(q) & (p,q) \in I_{e^a} \times I_{e^a}, \\ p \wedge q & (p,q) \in [0,e[\times(L \setminus [0,e[) \cup (L \setminus [0,e[) \times [0,e[, \\ p & (p,q) \in]e,a] \times I_{e^a} \cup I_{e^a} \times \{e\}, \\ q & (p,q) \in I_{e^a} \times]e,a] \cup \{e\} \times I_{e^a}, \\ a & \text{otherwise,} \end{cases}$$

is a uni-nullnorm on L iff $p < q$ for arbitrary $p \in [0, e[, q \in I_{e^a} \cup I_{e,a}$, $m < n$ for arbitrary $m \in I_{e^a}, n \in]e, a]$, $int(p) \wedge int(q) \in I_{e^a}$ for arbitrary $p, q \in I_{e^a}$.

(2) If $int(p) \wedge int(q) \in I_{e^a}$ for arbitrary $p, q \in I_{e^a}$, $S : [e, a]^2 \to [e, a]$ satisfies $S(p, q) \in [e, a[$ for arbitrary $p, q \in [e, a[$, then the operation $F^{T,S,a}_{int,4} : L^2 \to L$ defined by

$$F^{T,S,a}_{int,4}(p,q) = \begin{cases} T_1(p,q) & (p,q) \in [0,e]^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ int(p) \wedge int(q) & (p,q) \in I_{e^a} \times I_{e^a}, \\ p \wedge q & (p,q) \in [0,e[\times[e,a[\cup[e,a[\times[0,e[, \\ p & (p,q) \in ([0,e[\cup]e,a]) \times I_{e^a} \cup I_{e^a} \times \{e\}, \\ q & (p,q) \in I_{e^a} \times ([0,e[\cup]e,a]) \cup \{e\} \times I_{e^a}, \\ a & \text{otherwise,} \end{cases}$$

is a uni-nullnorm on L iff $m < n$ for arbitrary $m \in I_{e^a}, n \in]e, a]$.

The following proposition derives from Theorem 3.4 and Theorem 3.10 [65].

Proposition 3.7. Let L be a bounded lattice, T_1 be a t-norm on $[0, e]$, T_2 be a t-norm on $[a, 1]$, S be a t-conorm on $[e, a]$ such that $S(p, q) \in [e, a[$ for arbitrary $p, q \in [e, a[$, and $cl : L \to L$ be a closure operator, where $e, a \in L \setminus \{0, 1\}$.

(1) The operation $F^{T,S,0}_{cl,5} : L^2 \to L$ defined by

$$F^{T,S,0}_{cl,5}(p,q) = \begin{cases} T_1(p,q) & (p,q) \in [0,e]^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ p \wedge q & (p,q) \in [0,e[\times(L \setminus [0,e[) \cup (L \setminus [0,e[) \times [0,e[, \\ q & (p,q) \in [e,a[\times I_{e^a}, \\ p & (p,q) \in I_{e^a} \times [e,a[, \\ a & \text{otherwise,} \end{cases}$$

is a uni-nullnorm on L iff $p < q$ for arbitrary $p \in [0, e[, q \in I_{e^a} \cup I_{e,a}$, $m \parallel n$ for arbitrary $m \in I_{e^a}, n \in [e, a[$.

(2) The operation $F^{T,S,a}_{cl,6} : L^2 \to L$ defined by

$$F^{T,S,a}_{cl,6}(p,q) = \begin{cases} T_1(p,q) & (p,q) \in [0,e]^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ p \wedge q & (p,q) \in [0,e[\times[e,a[\cup[e,a[\times[0,e[, \\ q & (p,q) \in [e,a[\times I_{e^a} \cup I_{e^a} \times [0,e[, \\ p & (p,q) \in I_{e^a} \times [e,a[\cup[0,e[\times I_{e^a}, \\ a & \text{otherwise,} \end{cases}$$

is a uni-nullnorm on L iff $m \parallel n$ for arbitrary $m \in I_{e^a}, n \in [e, a[$.

Remark 3.8. For Theorem 3.4(2), $S(p,q) \in [e,a[$ for arbitrary $p,q \in [e,a[$ can not be omitted. Consider the following case: Assume that $p,q \in [e,a[$, take $S = S_D$, $r \in [0,e[$, by the associativity of $F_{cl,4}^{T,S,a}$, and $F_{cl,4}^{T,S,a}(r, F_{cl,4}^{T,S,a}(p,q)) = F_{cl,4}^{T,S,a}(r,a) = a$, $F_{cl,4}^{T,S,a}(F_{cl,4}^{T,S,a}(r,p),q) = F_{cl,4}^{T,S,a}(r,q) = r$, this is a contradiction.

Proposition 3.9. Let L be a bounded lattice, T_2 be a t-norm on $[a,1]$, S be a t-conorm on $[e,a]$ such that $S(p,q) \in [e,a[$ for arbitrary $p,q \in [e,a[$, and $int : L \to L$ be an interior operator, where $e,a \in L \setminus \{0,1\}$. Then the operation $F_{int,5}^{T,S,a} : L^2 \to L$ is a uni-nullnorm on L, where

$$F_{int,5}^{T,S,a}(p,q) = \begin{cases} int(p) \wedge int(q) & (p,q) \in [0,e[^2 \cup [0,e[\times I_{e^a} \cup I_{e^a} \times [0,e[\cup (I_{e^a})^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ p \wedge q & (p,q) \in [0,e[\times[e,a[\cup[e,a[\times[0,e[, \\ q & (p,q) \in [e,a[\times I_{e^a}, \\ p & (p,q) \in I_{e^a} \times [e,a[, \\ a & \text{otherwise.} \end{cases}$$

Proof. It can be proved analogously to Theorem 3.4(1). □

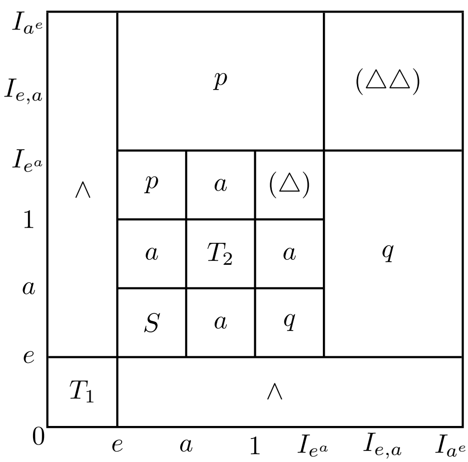

Fig. 5. $F_{cl,7}^{T,S,0}$ in Theorem 3.10(1).

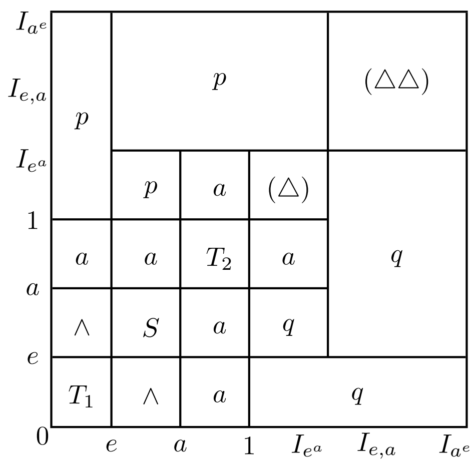

Fig. 6. $F_{cl,8}^{T,S,a}$ in Theorem 3.10(2).

Next we construct (conjunctive) uni-nullnorms via two different closure operators.

Theorem 3.10. Let L be a bounded lattice, T_1 be a t-norm on $[0,e]$, T_2 be a t-norm on $[a,1]$, S be a t-conorm on $[e,a]$, $cl_1 : L \to L$ and $cl_2 : L \to L$ be two closure operators, $p \parallel q$ for arbitrary $p \in I_{e^a}, q \in I_{e,a} \cup I_{a^e}$, and $cl_1(p) \vee cl_1(q) \in I_{e^a}$ for arbitrary $\{p,q\} \subseteq I_{e^a}$, where $e,a \in L \setminus \{0,1\}$.

(1) The operation $F^{T,S,0}_{cl,7} : L^2 \to L$ defined by

$$F^{T,S,0}_{cl,7}(p,q) = \begin{cases} T_1(p,q) & (p,q) \in [0,e]^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ cl_1(p) \vee cl_1(q) & (p,q) \in I_{e^a} \times I_{e^a}, \\ cl_2(p) \vee cl_2(q) & (p,q) \in (I_{a^e} \cup I_{e,a}) \times (I_{a^e} \cup I_{e,a}), \\ p \wedge q & (p,q) \in [0,e[\times(L \setminus [0,e[) \cup (L \setminus [0,e[) \times [0,e[, \\ p & (p,q) \in (([e,a[\cup[a,1]) \times (I_{e^a} \cup I_{e,a} \cup I_{a^e}) \\ & \setminus([a,1] \times I_{e^a} \cup \{e\} \times I_{e^a})) \cup I_{e^a} \times (\{e\} \cup I_{e,a} \cup I_{a^e}), \\ q & (p,q) \in ((I_{e^a} \cup I_{e,a} \cup I_{a^e}) \times ([e,a[\cup[a,1]) \\ & \setminus(I_{e^a} \times [a,1] \cup I_{e^a} \times \{e\})) \cup (\{e\} \cup I_{e,a} \cup I_{a^e}) \times I_{e^a}, \\ a & \text{otherwise}, \end{cases}$$

is a uni-nullnorm on L iff $p < q$ for arbitrary $p \in [0,e[, q \in I_{e^a} \cup I_{e,a}$, $m < n$ for arbitrary $m \in I_{e^a}, n \in]e,a]$, $p \parallel q$ for arbitrary $p \in]e,a], q \in I_{a^e}$, $m \parallel n$ for arbitrary $m \in [a,1[, n \in I_{a^e} \cup I_{e,a}$, $cl_2(p) \vee cl_2(q) \in I_{e,a} \cup I_{a^e}$ for arbitrary $\{p,q\} \subseteq I_{e,a} \cup I_{a^e}$.

(2) $S : [e,a]^2 \to [e,a]$ satisfies $S(p,q) \in [e,a[$ for arbitrary $p,q \in [e,a[$. The operation $F^{T,S,a}_{cl,8} : L^2 \to L$ defined by

$$F^{T,S,a}_{cl,8}(p,q) = \begin{cases} T_1(p,q) & (p,q) \in [0,e]^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ cl_1(p) \vee cl_1(q) & (p,q) \in I_{e^a} \times I_{e^a}, \\ cl_2(p) \vee cl_2(q) & (p,q) \in (I_{a^e} \cup I_{e,a}) \times (I_{a^e} \cup I_{e,a}), \\ p \wedge q & (p,q) \in [0,e[\times[e,a[\cup[e,a[\times[0,e[, \\ p & (p,q) \in (([0,e[\cup[e,a[\cup[a,1]) \times (I_{e^a} \cup I_{e,a} \cup I_{a^e}) \\ & \setminus([a,1] \times I_{e^a} \cup \{e\} \times I_{e^a})) \cup I_{e^a} \times (\{e\} \cup I_{e,a} \cup I_{a^e}), \\ q & (p,q) \in ((I_{e^a} \cup I_{e,a} \cup I_{a^e}) \times ([0,e[\cup[e,a[\cup[a,1]) \\ & \setminus(I_{e^a} \times [a,1] \cup I_{e^a} \times \{e\})) \cup (\{e\} \cup I_{e,a} \cup I_{a^e}) \times I_{e^a}, \\ a & \text{otherwise}, \end{cases}$$

is a uni-nullnorm on L iff $m < n$ for arbitrary $m \in I_{e^a}, n \in]e,a]$, $p \parallel q$ for arbitrary $p \in]e,a], q \in I_{a^e}$, $m \parallel n$ for arbitrary $m \in [a,1[, n \in I_{a^e} \cup I_{e,a}$, $cl_2(p) \vee cl_2(q) \in I_{e,a} \cup I_{a^e}$ for arbitrary $\{p,q\} \subseteq I_{e,a} \cup I_{a^e}$.

Proof. (2) We prove $F^{T,S,a}_{cl,8}$ is a uni-nullnorm on L. Necessity: For the cases: $m < n$ for arbitrary $m \in I_{e^a}, n \in]e,a]$, the proof is similar to the proof of necessity in Theorem 3.4(1), next we prove the remaining cases. Assume that $m < n$ for some $m \in]e,a[, n \in I_{a^e}$, by $F^{T,S,a}_{cl,8}(e,m) = m$, $F^{T,S,a}_{cl,8}(e,n) = e$, this is a contradiction, then $p \parallel q$ for arbitrary $p \in]e,a], q \in I_{a^e}$. Assume that $m > n$ for some $m \in]a,1[, n \in I_{a^e} \cup I_{e,a}$, by $F^{T,S,a}_{cl,8}(1,m) = m$, $F^{T,S,a}_{cl,8}(1,n) = 1$, this contradicts with the monotonicity of $F^{T,S,a}_{cl,8}$, then $m \parallel n$

for arbitrary $m \in [a,1[, n \in I_{a^e} \cup I_{e,a}$. Assume that $cl_2(p) \vee cl_2(q) \in]a,1]$ for some $\{p,q\} \subseteq I_{e,a} \cup I_{a^e}$, take $r \in [0,e[$, by $F_{cl,8}^{T,S,a}(F_{cl,8}^{T,S,a}(p,q),r) = F_{cl,8}^{T,S,a}(cl_2(p) \vee cl_2(q),r) = a$, $F_{cl,8}^{T,S,a}(p, F_{cl,8}^{T,S,a}(q,r)) = F_{cl,8}^{T,S,a}(p,r) = r$, this contradicts with the associativity of $F_{cl,8}^{T,S,a}$, then $cl_2(p) \vee cl_2(q) \in I_{e,a} \cup I_{a^e}$ for arbitrary $\{p,q\} \subseteq I_{e,a} \cup I_{a^e}$.

Sufficiency: By the definition of $F_{cl,8}^{T,S,a}$, it is easy to get that $F_{cl,8}^{T,S,a}$ is commutative, $F_{cl,8}^{T,S,a}(p,e) = p$ for arbitrary $p \in [0,a]$, $F_{cl,8}^{T,S,a}(p,a) = a$ for arbitrary $p \in [e,1]$, $F_{cl,8}^{T,S,a}(p,1) = p$ for arbitrary $p \in [a,1]$, we need to prove monotonicity and associativity.

(i) Monotonicity. For $p,q,r \in L$ with $p \leq q$, we need to prove $F_{cl,8}^{T,S,a}(p,r) \leq F_{cl,8}^{T,S,a}(q,r)$. For the elements $p,q \in L$ are simultaneous elements of $[0,e]$, or $[e,a]$, or $[a,1]$, or I_{e^a}, or $I_{e,a}$, or I_{a^e}, the proof is obvious. Next we prove the remaining cases.

1. $p \in [0,e[$, $q \in L \setminus [0,e[$.

1.1 $r \in [0,e[$. $F_{cl,8}^{T,S,a}(p,r) = T_1(p,r) \leq r \leq F_{cl,8}^{T,S,a}(q,r)$.

1.2 $r \in]a,1]$. $F_{cl,8}^{T,S,a}(p,r) = a \leq F_{cl,8}^{T,S,a}(q,r)$.

1.3 $r \in L \setminus ([0,e[\cup]a,1])$. By $F_{cl,8}^{T,S,a}(q,r) \in (L \setminus ([0,e[\cup]a,1]))$, then $F_{cl,8}^{T,S,a}(p,r) = p \leq F_{cl,8}^{T,S,a}(q,r)$.

2. $p \in [e,a[$, $q \in [a,1]$.

2.1 $r \in [0,e[$. $F_{cl,8}^{T,S,a}(p,r) = p \wedge r = r \leq a = F_{cl,8}^{T,S,a}(q,r)$.

2.2 $r \in L \setminus [0,e]$. $F_{cl,8}^{T,S,a}(p,r) \leq a \leq F_{cl,8}^{T,S,a}(q,r)$.

3. $p \in I_{e^a}$, $q \in]e,a]\cup]a,1]$.

3.1 $r \in [0,e[$. $F_{cl,8}^{T,S,a}(p,r) = r \leq F_{cl,8}^{T,S,a}(q,r)$.

3.2 $r \in]e,a]$. By the value of $F_{cl,8}^{T,S,a}(q,r)$ is $S(q,r)$ or a, then $F_{cl,8}^{T,S,a}(p,r) = r \leq F_{cl,8}^{T,S,a}(q,r)$.

3.3 $r \in]a,1]$. $F_{cl,8}^{T,S,a}(p,r) = a \leq F_{cl,8}^{T,S,a}(q,r)$.

3.4 $r \in I_{e^a}$. By the value of $F_{cl,8}^{T,S,a}(q,r)$ is q or a, $m < n$ for arbitrary $m \in I_{e^a}, n \in]e,a]$, then $F_{cl,8}^{T,S,a}(p,r) = cl_1(p) \vee cl_1(r) \leq F_{cl,8}^{T,S,a}(q,r)$.

3.5 $r \in I_{e,a} \cup I_{a^e}$. $F_{cl,8}^{T,S,a}(p,r) = p \leq q = F_{cl,8}^{T,S,a}(q,r)$.

3.6 $r = e$. By the value of $F_{cl,8}^{T,S,a}(q,r)$ is q or a, then $F_{cl,8}^{T,S,a}(p,r) = p \leq F_{cl,8}^{T,S,a}(q,r)$.

4. $p \in I_{e,a}$, $q \in I_{a^e}$.

4.1 $r \in L \setminus (I_{e,a} \cup I_{a^e})$. $F_{cl,8}^{T,S,a}(p,r) = r = F_{cl,8}^{T,S,a}(q,r)$.

4.2 $r \in I_{e,a} \cup I_{a^e}$. $F_{cl,8}^{T,S,a}(p,r) = cl_2(p) \vee cl_2(r) \leq cl_2(q) \vee cl_2(r) = F_{cl,8}^{T,S,a}(q,r)$.

5. $p = e$, $q \in]e,a] \cup [a,1] \cup I_{a^e}$.

5.1 $r \in [0,e[\cup[e,a[\cup I_{e^a}$. $F_{cl,8}^{T,S,a}(p,r) = r \leq F_{cl,8}^{T,S,a}(q,r)$.

5.2 $r \in [a,1]$. $F_{cl,8}^{T,S,a}(p,r) = a \leq F_{cl,8}^{T,S,a}(q,r)$.

5.3 $r \in I_{e,a} \cup I_{a^e}$. $F_{cl,8}^{T,S,a}(p,r) = e \leq q \leq F_{cl,8}^{T,S,a}(q,r)$.

(ii) Associativity. If one of the elements p,q and r equals e or a, or $\{p,q,r\} \subseteq [0,e[\cup]e,a[\cup]a,1]$, it is clear that $F_{cl,8}^{T,S,a}(F_{cl,8}^{T,S,a}(p,q),r) = F_{cl,8}^{T,S,a}(p, F_{cl,8}^{T,S,a}(q,r))$. We need to check the rest.

1. $\{p,q,r\} \subseteq]e,a[\cup]a,1]$, because it is easy to prove that $F_{cl,8}^{T,S,a}$ in the domains is a nullnorm, then $L = F_{cl,8}^{T,S,a}(F_{cl,8}^{T,S,a}(p,q),r) = F_{cl,8}^{T,S,a}(p, F_{cl,8}^{T,S,a}(q,r)) = R$.

2. $\{p,q\} \subseteq [0,e[$, $\{r\} \subseteq]a,1]$.
$L = F_{cl,8}^{T,S,a}(T_1(p,q),r) = a = F_{cl,8}^{T,S,a}(p,a) = R$.
3. $\{p,q\} \subseteq [0,e[$, $\{r\} \subseteq L \setminus ([0,e] \cup [a,1])$.
$L = F_{cl,8}^{T,S,a}(T_1(p,q),r) = T_1(p,q) = F_{cl,8}^{T,S,a}(p,q) = R$.
4. $\{p\} \subseteq [0,e[$, $\{q,r\} \subseteq]a,1]$.
$L = F_{cl,8}^{T,S,a}(a,r) = a = F_{cl,8}^{T,S,a}(p,T_2(q,r)) = R$.
5. $\{p\} \subseteq [0,e[$, $\{q,r\} \subseteq L \setminus ([0,e] \cup [a,1])$.
$L = F_{cl,8}^{T,S,a}(p,r) = p = R$.
6. $\{p\} \subseteq [0,e[$, $\{q\} \subseteq]a,1]$, $\{r\} \subseteq L \setminus ([0,e] \cup [a,1])$, by $F_{cl,8}^{T,S,a}(q,r) \in [a,1]$.
$L = F_{cl,8}^{T,S,a}(a,r) = a = R$.
7. $\{p,q,r\} \subseteq I_{e^a}$.
$L = F_{cl,8}^{T,S,a}(cl_1(p) \vee cl_1(q),r) = cl_1(p) \vee cl_1(q) \vee cl_1(r) = F_{cl,8}^{T,S,a}(p, cl_1(q) \vee cl_1(r)) = R$.
8. $\{p,q,r\} \subseteq I_{e,a} \cup I_{a^e}$.
$L = F_{cl,8}^{T,S,a}(cl_2(p) \vee cl_2(q),r) = cl_2(p) \vee cl_2(q) \vee cl_2(r) = F_{cl,8}^{T,S,a}(p, cl_2(q) \vee cl_2(r)) = R$.
9. $\{p,q\} \subseteq]e,a[$, $\{r\} \subseteq I_{e^a} \cup I_{e,a} \cup I_{a^e}$.
$L = F_{cl,8}^{T,S,a}(S(p,q),r) = S(p,q) = F_{cl,8}^{T,S,a}(p,q) = R$.
10. $\{p\} \subseteq]e,a[$.
10.1 $\{q,r\} \subseteq I_{e^a}$. $L = F_{cl,8}^{T,S,a}(p,r) = p = F_{cl,8}^{T,S,a}(p, cl_1(q) \vee cl_1(r)) = R$.
10.2 $\{q,r\} \subseteq I_{e,a} \cup I_{a^e}$. $L = F_{cl,8}^{T,S,a}(p,r) = p = F_{cl,8}^{T,S,a}(p, cl_2(q) \vee cl_2(r)) = R$.
10.3 $\{q\} \subseteq I_{e^a}$, $\{r\} \subseteq I_{e,a} \cup I_{a^e}$. $L = F_{cl,8}^{T,S,a}(p,r) = p = F_{cl,8}^{T,S,a}(p,q) = R$.
10.4 $\{q\} \subseteq]a,1]$, $\{r\} \subseteq I_{e^a}$. $L = F_{cl,8}^{T,S,a}(a,r) = a = F_{cl,8}^{T,S,a}(p,a) = R$.
10.5 $\{q\} \subseteq]a,1]$, $\{r\} \subseteq I_{e,a} \cup I_{a^e}$. $L = F_{cl,8}^{T,S,a}(a,r) = a = F_{cl,8}^{T,S,a}(p,q) = R$.
11. $\{p,q\} \subseteq]a,1]$.
11.1 $\{r\} \subseteq I_{e^a}$. $L = F_{cl,8}^{T,S,a}(T_2(p,q),r) = a = F_{cl,8}^{T,S,a}(p,a) = R$.
11.2 $\{r\} \subseteq I_{e,a} \cup I_{a^e}$. $L = F_{cl,8}^{T,S,a}(T_2(p,q),r) = T_2(p,q) = F_{cl,8}^{T,S,a}(p,q) = R$.
12. $\{p\} \subseteq]a,1]$.
12.1 $\{q,r\} \subseteq I_{e^a}$. $L = F_{cl,8}^{T,S,a}(a,r) = a = F_{cl,8}^{T,S,a}(p, cl_1(q) \vee cl_1(r)) = R$.
12.2 $\{q,r\} \subseteq I_{e,a} \cup I_{a^e}$. $L = F_{cl,8}^{T,S,a}(p,r) = p = F_{cl,8}^{T,S,a}(p, cl_2(q) \vee cl_2(r)) = R$.
12.3 $\{q\} \subseteq I_{e^a}$, $\{r\} \subseteq I_{e,a} \cup I_{a^e}$. $L = F_{cl,8}^{T,S,a}(a,r) = a = F_{cl,8}^{T,S,a}(p,q) = R$.
13. $\{p,q\} \subseteq I_{e^a}$, $\{r\} \subseteq I_{e,a} \cup I_{a^e}$.
$L = F_{cl,8}^{T,S,a}(cl_1(p) \vee cl_1(q),r) = cl_1(p) \vee cl_1(q) = F_{cl,8}^{T,S,a}(p,q) = R$.
14. $\{p\} \subseteq I_{e^a}$, $\{q,r\} \subseteq I_{e,a} \cup I_{a^e}$.
$L = F_{cl,8}^{T,S,a}(p,r) = p = F_{cl,8}^{T,S,a}(p, cl_2(q) \vee cl_2(r)) = R$.

(1) It can be proved analogously to (2). □

Proposition 3.11. Consider the construction of the uni-nullnorm from Theorem 3.10 denoted as $F = F_{cl,7}^{T,S,0}$. Let $cl_1 : L \to L$ and $cl_2 : L \to L$ be two closure operators, $p \parallel q$ for arbitrary $p \in I_{e^a}, q \in I_{e,a} \cup I_{a^e}$, and $cl_1(p) \vee cl_1(q) \in I_{e^a}$ for arbitrary $\{p,q\} \subseteq I_{e^a}$, F is a uni-nullnorm on L iff $p < q$ for arbitrary $p \in [0,e[, q \in I_{e^a} \cup I_{e,a}$, $m < n$ for arbitrary $m \in I_{e^a}, n \in]e,a]$, $p \parallel q$ for arbitrary $p \in]e,a], q \in I_{a^e}$, $m \parallel n$ for arbitrary $m \in [a,1[, n \in I_{a^e} \cup I_{e,a}$, $cl_2(p) \vee cl_2(q) \in I_{e,a} \cup I_{a^e}$ for arbitrary $\{p,q\} \subseteq I_{e,a} \cup I_{a^e}$.

Using these conditions, L can be expressed as the union of the following sets: $L_1 = [0,e], L_2 = [e,a]$, $L_3 = [a,1]$, $L_4 = I_{e^a}$ and $L_5 = I_{e,a} \cup I_{a^e}$, where clearly $L_1 \cap L_2 = \{e\}$ and $L_2 \cap L_3 = \{a\}$. Consider t-norms T_1 on $[0,e]$ and T_2 on $[a,1]$, the t-conorm S on $[e,a]$.

1. To get the construction of F, in the first step, construct a uninorm U on $M = L_1 \cup L_2 \cup L_4$ from T_1 and S in the conjunctive form, which can be given as follows:

$$U(p,q) = \begin{cases} T_1(p,q) & p,q \in L_1, \\ S(p,q) & p,q \in L_2, \\ cl_1(p) \vee cl_1(q) & p,q \in L_4, \\ p & (p,q) \in]e,a] \times I_{e^a} \cup I_{e^a} \times \{e\}, \\ q & (p,q) \in I_{e^a} \times]e,a] \cup \{e\} \times I_{e^a}, \\ p \wedge q & \text{otherwise.} \end{cases}$$

   The above uninorm U is precisely the structure of $U^{T,S,0}_{cl,3}$ in [65], and U should be a slight change, U is a uninorm on M (i. e., M endowed with U is a partial ordered monoid).

2. Construct a nullnorm G on $L_2 \cup L_3 \cup L_5$ from T_2 and S in the following form, which can be given as follows:

$$G(p,q) = \begin{cases} S(p,q) & p,q \in L_2, \\ T_2(p,q) & p,q \in L_3, \\ cl_2(p) \vee cl_2(q) & p,q \in L_5, \\ p & (p,q) \in ([e,a[\cup[a,1]) \times L_5, \\ q & (p,q) \in L_5 \times ([e,a[\cup[a,1]), \\ a & \text{otherwise.} \end{cases}$$

3. Therefore, two operations U and G have been obtained. In the third step, construct the uni-nullnorm F on L from the operations U and G as follows (denote $L_1^- = L_1 \setminus \{e\}$), two operations U and G have the same structure and the same restricted conditions while including t-conorm S,

$$F(p,q) = \begin{cases} U(p,q) & p,q \in M, \\ T_2(p,q) & p,q \in L_3, \\ cl_2(p) \vee cl_2(q) & p,q \in L_5, \\ p \wedge q & (p,q) \in L_1^- \times (L \setminus L_1^-) \cup (L \setminus L_1^-) \times L_1^-, \\ p & (p,q) \in ([e,a[\cup[a,1] \cup L_4) \times L_5, \\ q & (p,q) \in L_5 \times ([e,a[\cup[a,1] \cup L_4), \\ a & \text{otherwise.} \end{cases}$$

For Theorem 3.10(2), as a is the absorbing element of $F^{T,S,a}_{cl,8}$, add the condition: $S : [e,a]^2 \to [e,a]$ satisfies $S(p,q) \in [e,a[$ for arbitrary $p,q \in [e,a[$. $F^{T,S,a}_{cl,8}$ is given by the same structure as $F^{T,S,0}_{cl,7}$, only have the following changes, $F^{T,S,a}_{cl,8}(p,q) = p$ for

$(p,q) \in [0,e[\times(L_4 \cup L_5)$, $F^{T,S,a}_{cl,8}(p,q) = a$ for $(p,q) \in [0,e[\times[a,1]$.

Example 3.12. Given a bounded lattice $L = \{0, p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, e, a, 1\}$ in Figure 7. If $cl_1(x) = x$ for arbitrary $x \in L$, $cl_2(x) = x \vee p_{10}$ for arbitrary $x \in L$, T_1, T_2 and S satisfy: $T_1 = T_\wedge$, $S = S_\vee$, $T_2 = T_D$. We have the uni-nullnorm $F^{T,S,a}_{cl,8}$ in Table 1. And the uni-nullnorm F_S in Table 2. It is obvious that the two uni-nullnorms are different. The uni-nullnorm F_S [23] is as follows:

$$F_S(p,q) = \begin{cases} T_1(p,q) & p,q \in [0,e], \\ S(p,q) & p,q \in [e,a[, \\ T_2(p,q) & p,q \in [a,1], \\ S(p \vee e, q \vee e) & (p,q) \in]e,a[\times I_e \cup I_e \times]e,a[\cup I_e \times I_e, \\ p \vee q & (p,q) \in [0,e]\times]e,a[\cup]e,a[\times[0,e], \\ p & (p,q) \in I_e \times [0,e], \\ q & (p,q) \in [0,e] \times I_e, \\ a & \text{otherwise.} \end{cases}$$

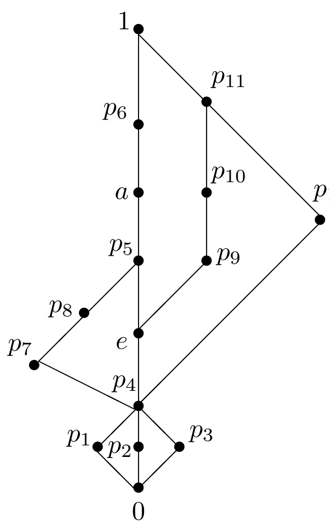

Fig. 7. The lattice L.

Remark 3.13. (i) By Theorem 3.10 and Theorem 3.9 [65], or Theorem 3.10 in this paper and Theorem 3.10 [65], we can obtain the corresponding propositions similar to above Proposition 3.6 and Proposition 3.7. $int(p) \wedge int(q)$, $int(p \wedge e) \wedge int(q \wedge e)$, $T(p \wedge e, q \wedge e)$ (where T is a t-norm [38]) and $T(p,q)$ (where T is a t-subnorm [38]) can be replaced each other, for arbitrary $\{p,q\} \subseteq [0,e[^2 \cup [0,e[\times I_{e^a} \cup I_{e^a} \times [0,e[\cup (I_{e^a})^2$. For the

F	0	p_1	p_2	p_3	p_4	e	p_5	a	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	1
0	0	0	0	0	0	0	0	a	a	0	0	0	0	0	0	a
p_1	0	p_1	0	0	p_1	p_1	p_1	a	a	p_1	p_1	p_1	p_1	p_1	p_1	a
p_2	0	0	p_2	0	p_2	p_2	p_2	a	a	p_2	p_2	p_2	p_2	p_2	p_2	a
p_3	0	0	0	p_3	p_3	p_3	p_3	a	a	p_3	p_3	p_3	p_3	p_3	p_3	a
p_4	0	p_1	p_2	p_3	p_4	p_4	p_4	a	a	p_4	p_4	p_4	p_4	p_4	p_4	a
e	0	p_1	p_2	p_3	p_4	e	p_5	a	a	p_7	p_8	e	e	e	e	a
p_5	0	p_1	p_2	p_3	p_4	p_5	p_5	a	a	p_5	p_5	p_5	p_5	p_5	p_5	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
p_6	a	a	a	a	a	a	a	a	a	a	a	p_6	p_6	p_6	p_6	p_6
p_7	0	p_1	p_2	p_3	p_4	p_7	p_5	a	a	p_7	p_8	p_7	p_7	p_7	p_7	a
p_8	0	p_1	p_2	p_3	p_4	p_8	p_5	a	a	p_8	p_8	p_8	p_8	p_8	p_8	a
p_9	0	p_1	p_2	p_3	p_4	e	p_5	a	p_6	p_7	p_8	p_{10}	p_{10}	p_{11}	p_{11}	1
p_{10}	0	p_1	p_2	p_3	p_4	e	p_5	a	p_6	p_7	p_8	p_{10}	p_{10}	p_{11}	p_{11}	1
p_{11}	0	p_1	p_2	p_3	p_4	e	p_5	a	p_6	p_7	p_8	p_{11}	p_{11}	p_{11}	p_{11}	1
p_{12}	0	p_1	p_2	p_3	p_4	e	p_5	a	p_6	p_7	p_8	p_{11}	p_{11}	p_{11}	p_{11}	1
1	a	a	a	a	a	a	a	a	p_6	a	a	1	1	1	1	1

Tab. 1. Uni-nullnorm $F_{cl,8}^{T,S,a}$ in Figure 7.

F	0	p_1	p_2	p_3	p_4	e	p_5	a	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	1
0	0	0	0	0	0	0	p_5	a	a	p_7	p_8	a	a	a	a	a
p_1	0	p_1	0	0	p_1	p_1	p_5	a	a	p_7	p_8	a	a	a	a	a
p_2	0	0	p_2	0	p_2	p_2	p_5	a	a	p_7	p_8	a	a	a	a	a
p_3	0	0	0	p_3	p_3	p_3	p_5	a	a	p_7	p_8	a	a	a	a	a
p_4	0	p_1	p_2	p_3	p_4	p_4	p_5	a	a	p_7	p_8	a	a	a	a	a
e	0	p_1	p_2	p_3	p_4	e	p_5	a	a	p_7	p_8	a	a	a	a	a
p_5	p_5	p_5	p_5	p_5	p_5	p_5	p_5	a	a	p_5	p_5	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
p_6	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	p_6
p_7	p_7	p_7	p_7	p_7	p_7	p_7	p_5	a	a	p_5	p_5	a	a	a	a	a
p_8	p_8	p_8	p_8	p_8	p_8	p_8	p_5	a	a	p_5	p_5	a	a	a	a	a
p_9	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
p_{10}	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
p_{11}	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
p_{12}	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
1	a	a	a	a	a	a	a	a	p_6	a	a	a	a	a	a	1

Tab. 2. Uni-nullnorm F_S in Figure 7.

constructions of nullnorms on L, for arbitrary $(p, q) \in]0, a]^2 \cup [0, a] \times I_a \cup I_a \times [0, a] \cup (I_a)^2$, $cl(p \wedge a) \vee cl(q \wedge a) = (p \wedge a) \vee (q \wedge a) \vee (p \wedge q)$ is reasonable. For some other constructions of nullnorms on L, for arbitrary $(p, q) \in (I_a)^2$, $cl(p \wedge a) \vee cl(q \wedge a) = (p \wedge a) \vee (q \wedge a) \vee (p \wedge q)$ may be reasonable.

(ii) For Theorem 3.10, consider the extreme case, take $e = 0$ and $cl(p) = p \vee \alpha$ for arbitrary $p \in I_a, \alpha \in I_a$, we can obtain the nullnorm $F_{\alpha}^{T,S}$ in Theorem 5 [12].

By Theorem 3.10(2) and Proposition 2.10, we can obtain the following corollary.

Corollary 3.14. Let L be a bounded lattice, T_2 be a t-norm on $[a, 1]$, S be a t-conorm on $[e, a]$ such that $S(p, q) \in [e, a[$ for arbitrary $p, q \in [e, a[$, $cl : L \to L$ be a closure operator, $int : L \to L$ be an interior operator and $int(p) \wedge int(q) \in I_{e^a}$ for arbitrary $p, q \in I_{e^a}$, and $p \parallel q$ for arbitrary $p \in I_{e^a}, q \in I_{e,a} \cup I_{a^e}$, where $e, a \in L \setminus \{0, 1\}$. The operation $F_{cl,9}^{T,S,a} : L^2 \to L$ defined by

$$F_{cl,9}^{T,S,a}(p,q) = \begin{cases} int(p) \wedge int(q) & (p,q) \in [0,e[^2 \cup [0,e[\times I_{e^a} \cup I_{e^a} \times [0,e[\cup (I_{e^a})^2, \\ S(p,q) & (p,q) \in [e,a]^2, \\ T_2(p,q) & (p,q) \in [a,1]^2, \\ cl(p) \vee cl(q) & (p,q) \in (I_{a^e} \cup I_{e,a}) \times (I_{a^e} \cup I_{e,a}), \\ p \wedge q & (p,q) \in [0,e[\times [e,a[\cup [e,a[\times [0,e[, \\ p & (p,q) \in ([0,e[\cup [e,a[\cup [a,1] \cup I_{e^a}) \times (I_{e,a} \cup I_{a^e}) \\ & \cup]e,a] \times I_{e^a} \cup I_{e^a} \times \{e\}, \\ q & (p,q) \in (I_{e,a} \cup I_{a^e}) \times ([0,e[\cup [e,a[\cup [a,1] \cup I_{e^a}) \\ & \cup I_{e^a} \times]e,a] \cup \{e\} \times I_{e^a}, \\ a & \text{otherwise}, \end{cases}$$

is a uni-nullnorm on L iff $m < n$ for arbitrary $m \in I_{e^a}, n \in]e, a]$, $p \parallel q$ for arbitrary $p \in]e, a], q \in I_{a^e}$, $m \parallel n$ for arbitrary $m \in [a, 1[, n \in I_{a^e} \cup I_{e,a}$, $cl(p) \vee cl(q) \in I_{e,a} \cup I_{a^e}$ for arbitrary $\{p, q\} \subseteq I_{e,a} \cup I_{a^e}$.

4. CONCLUSIONS AND FURTHER WORK

In this paper, we introduce some new constructions of uni-nullnorms on L via closure (interior) operators. The methods of constructing some (conjunctive) uni-nullnorms on L via closure (interior) operators are deeply investigated. The construction methods are demonstrated. The theoretical developments in this paper provide infinitely many (conjunctive) uni-nullnorms on L under given conditions. Analysing the constructions of uni-nullnorms on bounded lattices has been obtained. For future work, we will focus on obtaining more methods of constructing (conjunctive) uni-nullnorms on L, and apply the methods to get new constructions of nullnorms (t-operators) [40, 41] and other aggregation operations like null-uninorms, overlap and grouping functions [5, 59, 60] on L to analyse their algebraic structures and deal with more complex problems about lattice-valued information in mathematics and information sciences.

ACKNOWLEDGEMENT

The authors would like to express their sincere thanks to the Editors and anonymous reviewers for their most valuable comments and suggestions in improving this paper greatly. This work was partially supported by Scientific and Technologial Innovation Programs of Higher Education Institutions in Shanxi (STIP)(2022L408), Shanxi Basic Research Program (free exploration 202203021212187) and a grant of National Natural Science Foundation of China (11971286,12171294).

(Received August 16, 2023)

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