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ON FORBIDDEN CONFIGURATION OF PSEUDOMODULAR LATTICES

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Abstract. We characterize the pseudomodular lattices by means of a forbidden configuration.

Keywords: forbidden configuration; pseudomodular lattice; semimodular lattice

MSC 2020: 06C10, 06C99

1. Introduction and preliminaries

Dress and Lovász in [4] studied full algebraic matroids of finite ranks for modularity of the flats. They characterized the existence of flats using the rank function and quasi-intersection. Björner and Lovász in [2] introduced a class of pseudomodular lattices as a generalization of modular lattices to contain full algebraic combinatorial geometries; see also [3]. A semimodular lattice L of finite length is said to be pseudomodular if every pair of elements of L has a pseudointersection. The class of pseudomodular lattices forms a subclass of the class of semimodular lattices and contains all modular lattices of finite length. Characterizations of classess of lattices by means of the non-existence of certain sublattices called forbidden configurations are available in the literature, such as the classess of distributive lattices, modular lattices, semimodular lattices, etc. In this paper, we establish a characterization by means of a forbidden sublattice for the class of pseudomodular lattices.

We give here some definitions and notations for ready reference; see Birkhoff [1], Grätzer [5], Haskins and Gudder [6], Stern [7], etc.

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Let P be a nonempty poset and $x, y \in P$. If $x \leqslant y$, then the length of an interval $[x, y]$, denoted by $lt[x, y]$, is the supremum of the lengths of the chains in $[x, y]$. The *height* or *rank* $r(x)$ of an element x of a poset P bounded below is the length of the interval $[0, x]$.

A lattice L is (*upper*) *semimodular* if $a \wedge b$ is a lower cover of a. Then b is a lower cover of $a \vee b$, for $a, b \in L$. A lattice L is said to be modular if the following condition (M) holds.
(M): $c \vee (a \wedge b) = (c \wedge a) \vee b$ for all $a, b, c \in L$ with $c \leqslant b$.

Definition 1.1 ([1]). The *graded poset* is defined as a poset P with a function $g\colon P \to \mathbb{Z}$ such that:

(i) $x > y$ implies $g[x] > g[y]$ and
(ii) if $x \prec y$, then $g[x] = g[y] + 1$.

Note that any semimodular lattice of finite length is graded by its rank function. Following is the definition due to Björner and Lovász [2], see also [3].

Definition 1.2 ([2]). Let L be a semimodular lattice of finite length, and denote by $r(x)$ the rank function (height function) of L. For each $x, y \in L$ let $P_{x,y} = \{z \leqslant y\colon r(x \vee z) - r(z) = r(x \vee y) - r(y)\}$. If the set $P_{x,y}$ has a unique least element, then we call this the *pseudointersection* of x and y and denote it by $x\rceil y$.

A semimodular lattice of finite length is called *pseudomodular* if every pair of its elements has the pseudointersection.

Remark 1.3 ([2]). The set $P_{x,y}$ lies in the interval $[x \wedge y, y]$ and is dual order ideal in $[x \wedge y, y]$.

Lemma 1.4 ([2]). *For any two elements x and y in a semimodular lattice L, the following are equivalent:*

(i) *x and y form a modular pair, i.e., $r(x \vee y) + r(x \wedge y) = r(x) + r(y)$.*
(ii) *$x\rceil y$ exists and $x\rceil y \leqslant x$.*
(iii) *$x\rceil y$ exists and $x\rceil y = x$.*
(iv) *$x \wedge y \in P_{x,y}$.*

Lemma 1.5 ([2]). *For any two elements x and y of a semimodular lattice L, the following are equivalent:*

(i) *$x\rceil y$ exists, i.e., $P_{x,y}$ has a unique least element.*
(ii) *$P_{x,y}$ is closed under meets.*
(iii) *If $u, v, z \in P_{x,y}$ and z covers u and v, then $u \wedge v \in P_{x,y}$.*

A subset I of a poset P is an order ideal if $x \in I$ and $y \leqslant x$ imply $y \in I$.

2. Main result

We have a forbidden characterization of pseudomodular lattices in the following theorem. In what follows, a sublattice S of a lattice L is said to be *cover-preserving* in L if for $a, b \in S$, $a \prec b$ in S implies $a \prec b$ in L, see [7].

Theorem 2.1. *Let L be a semimodular lattice of finite length. Then L is pseudomodular if and only if L does not contain a cover preserving sublattice isomorphic to the lattice as depicted in Figure* 1.

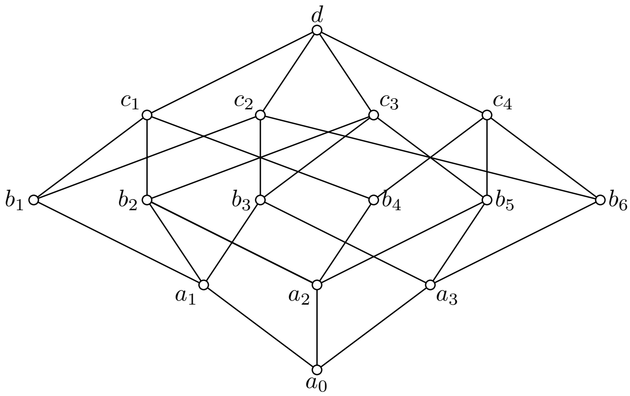

Figure 1.

P r o o f of Theorem 2.1. Let L be a semimodular lattice of finite length. If L contains a sublattice isomorphic to the lattice as depicted in Figure 1, then the elements c_4, b_4, b_6 belong to P_{b_1,c_4}. Thus $r(b_1 \vee c_4) - r(c_4) = r(b_1 \vee b_4) - r(b_4) = r(b_1 \vee b_6) - r(b_6)$, but $r(b_1 \vee (b_4 \wedge b_6)) - r(b_4 \wedge b_6) \neq r(b_1 \vee b_6) - r(b_6)$. Therefore $b_4 \wedge b_6 \notin P_{b_1,c_4}$, which implies that P_{b_1,c_4} does not have the least element and so the pseudointersection of b_1 and c_4 does not exist. Therefore, L is not a pseudomodular lattice.

Conversely, suppose that L is a semimodular lattice of finite length which is not pseudomodular. Then there exists a pair of elements $x, y \in L$ such that $P_{x,y} = \{z \leqslant y \colon r(x \vee z) - r(z) = r(x \vee y) - r(y)\}$ does not have the least element. Or equivalently, we have a pair x, y in L whose meet does not belong to $P_{x,y}$.

Consider a pair z_1, z_2 in $P_{x,y}$ with minimal height whose meet does not belong to $P_{x,y}$. Since $y, z_1, z_2 \in P_{x,y}$, we have $r(x \vee y) - r(y) = r(x \vee z_1) - r(z_1) = r(x \vee z_2) - r(z_2)$. Without loss of generality we assume that $lt[x \wedge y, x \vee y]$ is minimum, i.e., for $u, v \in L$, if $lt[u \wedge v, u \vee v] < lt[x \wedge y, x \vee y]$, then $P_{u,v}$ has the least element. We also assume that $x, y \in L$ is a pair such that for $x \wedge y < u < x$, the set $P_{u,y}$ has the least element.

If $x \leqslant y$, then x, y becomes a modular pair and by Lemma 1.4, $x \wedge y \in P_{x,y}$, which is nothing but the pseudointersection of x and y, a contradiction to the assumption.

Similarly, if $x > y$, then also by Lemma 1.4, we have a contradiction. Consequently, we must have $x \parallel y$.

If $x \wedge y \prec x$, then by semimodularity we have $y \prec x \vee y$ and thus $r(x) - r(x \wedge y) = 1 = r(x \vee y) - r(y)$ and $P_{x,y} = \{z \leqslant y \colon r(x \vee z) - r(z) = r(x \vee y) - r(y)\} = \{z \leqslant y \colon r(x \vee z) - r(z) = 1\}$. Also, as $x \wedge y \leqslant y$ and $r(x \vee (x \wedge y)) - r(x \wedge y) = r(x) - r(x \wedge y) = 1$, we have $x \wedge y \in P_{x,y}$ and so $x \wedge y$ is the least element of $P_{x,y}$, a contradiction, and therefore we must have $x \wedge y \nprec x$.

Consider an element q such that $x \wedge y < q < x$ and without loss of generality, we consider $x \wedge y < q \prec x$. If $x \wedge y \prec z_1$, then by semimodularity we have $q \prec q \vee z_1$ and also $x \prec x \vee z_1$. Therefore, $r(x \vee (x \wedge y)) - r(x \wedge y) = r(x \vee z_1) - r(z_1)$ and consequently, $x \wedge y \in P_{x,y}$, a contradiction, and so, we must have $x \wedge y \nprec z_1$. Similarly, we have to have $x \wedge y \nprec z_2$.

Now, since $x \wedge y \nprec z_1$ and $x \wedge y \nprec z_2$, there exist q_1 and q_2 such that $z_1 \wedge z_2 < q_1 < z_1$ and $z_1 \wedge z_2 < q_2 < z_2$ and without loss of generality, we consider $x \wedge y < q_1 \prec z_1$ and $x \wedge y < q_2 \prec z_2$.

Consider the set $\{x,\ y,\ x \vee y,\ x \wedge y,\ z_1,\ z_2,\ x \vee z_1,\ x \vee z_2,\ q,\ q_1,\ q_2,\ q \vee q_1,\ q \vee q_2,\ q_1 \vee q_2,\ q \vee q_1 \vee q_2\}$ and we contend that these elements are distinct and also the set forms a cover preserving sublattice of L. Note that by the choice x ,y, $x \vee y$, $x \wedge y$, z_1, z_2, $x \vee z_1$, $x \vee z_2$, q, q_1 and q_2 are distinct elements. For the other elements, we have the following.

Claim 2.2. $(x \vee z_1) \wedge (x \vee z_2) = x$.

Proof. Suppose that $(x \vee z_1) \wedge (x \vee z_2) > x$. As $q_1 \prec z_1$, by semimodularity we have $q_1 \vee x \prec z_1 \vee x$. If $q_1 \vee x \prec z_1 \vee x$, then $r(q_1 \vee x) - r(q_1) = r(z_1 \vee x) - r(z_1) = r(x \vee y) - r(y) = r(x \vee z_2) - r(z_2)$, which implies $q_1 \in P_{x,y}$, a contradiction, and so we must have $(x \vee z_1) \wedge (x \vee z_2) = x$. □

Claim 2.3. $x \wedge y = z_1 \wedge z_2$.

Proof. Suppose $x \wedge y < z_1 \wedge z_2$. If $x_1 = x \vee (z_1 \wedge z_2)$, then we have $x_1 \leqslant (x \vee z_1) \wedge (x \vee z_2)$, which gives $x \vee y = x_1 \vee y$, $x \vee z_1 = x_1 \vee z_1$ and $x \vee z_2 = x_1 \vee z_2$. It follows that $r(x_1 \vee y) - r(y) = r(x_1 \vee z_1) - r(z_1) = r(x_1 \vee z_2) - r(z_2)$ and so $y, z_1, z_2 \in P_{x_1,y}$. Since $lt[x_1 \wedge y, x_1 \vee y] < lt[x \wedge y, x \vee y]$, we have $z_1 \wedge z_2 \in P_{x_1,y}$. Thus, $r(x_1 \vee (z_1 \wedge z_2)) - r(z_1 \wedge z_2) = r(x_1 \vee y) - r(y) = r(x \vee y) - r(y)$. Moreover, $z_1 \wedge z_2 \in P_{x,y}$, a contradiction, and so we must have $x \wedge y = z_1 \wedge z_2$. □

Claim 2.4. $z_1 \vee z_2 = y$.

P r o o f. Suppose that $z_1 \vee z_2 < y$. If $y_1 = z_1 \vee z_2$, then $z_1 < y_1 < y$. Since $P_{x,y}$ is a dual order ideal, we have $y_1 \in P_{x,y}$ and so $r(x \vee y_1) - r(y_1) = r(x \vee y) - r(y)$, which gives $x \vee y \neq x \vee y_1$. Now consider the interval $[x \wedge y_1, x \vee y_1]$. Since $lt[x \wedge y_1, x \vee y_1] < lt[x \wedge y, x \vee y]$ and $z_1, z_2, y_1 \in P_{x,y_1}$, we have $z_1 \wedge z_2 \in P_{x,y_1}$. Thus $r(x \vee (z_1 \wedge z_2)) - r(z_1 \wedge z_2) = r(x \vee y_1) - r(y_1) = r(x \vee y) - r(y)$, which implies $z_1 \wedge z_2 \in P_{x,y}$, a contradiction, and so $z_1 \vee z_2 = y$. □

Claim 2.5. $(x \vee z_1) \vee (x \vee z_2) = x \vee y$.

P r o o f. Observe that $(x \vee z_1) \vee (x \vee z_2) = x \vee (z_1 \vee z_2) = x \vee y$. □

Claim 2.6. $q \vee y = x \vee y$.

P r o o f. Suppose that $q \vee y < x \vee y$. As $q \prec x$, by semimodularity we have $q \vee y \prec x \vee y$. In this case, $q \vee z_1 \prec x \vee z_1$ and $q \vee z_2 \prec x \vee z_2$. If $q \vee z_1 = x \vee z_1$ or $q \vee z_2 = x \vee z_2$, then this implies that $q \vee z_1 \vee y = x \vee z_1 \vee y = q \vee y = x \vee y$, which is not possible and so, $r(x \vee y) - r(y) = r(x \vee z_1) - r(z_1) = r(x \vee z_2) - r(z_2)$. Also, we have $r(q \vee y) - r(y) = r(q \vee z_1) - r(z_1) = r(q \vee z_2) - r(z_2)$, which implies $y, z_1, z_2 \in P_{q,y}$. Since $lt[q \wedge y, q \vee y] < lt[x \wedge y, x \vee y]$, we have $z_1 \wedge z_2 \in P_{q,y}$ and so, $r(q \vee (z_1 \wedge z_2)) - r(z_1 \wedge z_2) = r(q \vee y) - r(y) = r(x \vee y) - r(y) - 1$. Consequently, we have $r(q \vee (z_1 \wedge z_2)) + 1 - r(z_1 \wedge z_2) = r(x \vee y) - r(y)$. Now, since $q \prec x$ and $q \vee (z_1 \wedge z_2) = q$, we have $r(q \vee (z_1 \wedge z_2)) + 1 = r(x \vee (z_1 \wedge z_2))$, which gives $r(x \vee (z_1 \wedge z_2)) - r(z_1 \wedge z_2) = r(x \vee y) - r(y)$. This implies that $z_1 \wedge z_2 \in P_{x,y}$, a contradiction, and so we must have $y \vee q = x \vee y$. □

Claim 2.7. $q \vee z_1 = x \vee z_1$.

P r o o f. Suppose that $q \vee z_1 < x \vee z_1$ and consider a chain of length n in $[z_1, x \vee z_1]$: $z_1 \prec p_1 \prec p_2 \prec \ldots \prec p_{n-1} = q \vee z_1 \prec x \vee z_1$. By semimodularity, we have a chain in $[y, x \vee y]$: $y = z_1 \vee y \prec p_1 \vee y \preceq p_2 \vee y \preceq \ldots \preceq p_n = q \vee z_1 \vee y = x \vee z_1 \vee y$ which is of length at most n, a contradiction to the fact that $r(q \vee y) - r(y) = r(q \vee z_1) - r(z_1)$. So we must have $q \vee z_1 = x \vee z_1$. □

Claim 2.8. $q \vee z_2 = x \vee z_2$.

P r o o f. Is similar to that of Claim 2.7. □

Claim 2.9. $x \vee q_1 = x \vee z_1$.

P r o o f. Suppose that $x \vee q_1 < x \vee z_1$. As $q_1 \prec z_1$, by semimodularity we have $x \vee q_1 \prec x \vee z_1$. This gives $r(x \vee z_1) - r(z_1) = r(x \vee q_1) - r(q_1)$. Thus, $q_1 \in P_{x,y}$, a contradiction, and so we must have $x \vee q_1 = x \vee z_1$. □

Claim 2.10. $x \vee q_2 = x \vee z_2$.

P r o o f. Is similar to that of Claim 2.9. □

Claim 2.11. $q_1 \vee z_2 = y$.

P r o o f. Suppose that $q_1 \vee z_2 < y$. As $q_1 \prec z_1$, by semimodularity we have $q_1 \vee z_2 \prec y$. If $y_1 = q_1 \vee z_2$, then $z_2 < y_1 < y$. Since $P_{x,y}$ is a filter, we have $y_1 \in P_{x,y}$, which implies $r(x \vee y_1) - r(y_1) = r(x \vee y) - r(y)$. Also, we have $q_1 \leqslant y_1 \leqslant x \vee y_1$ and $x \leqslant x \vee y_1$, therefore $x \vee q_1 \leqslant x \vee y_1$. Since $x \vee q_1 = x \vee z_1$, we have $z_1 \leqslant x \vee z_1 \leqslant x \vee y_1$. Also $z_2 \leqslant y_1 \leqslant x \vee y_1$ and therefore $z_1 \vee z_2 \leqslant x \vee y_1$. However, we have $z_1 \vee z_2 = y < x \vee y_1$, a contradiction, and so we must have $q_1 \vee z_2 = y$. □

Claim 2.12. $q_2 \vee z_2 = y$.

P r o o f. Is similar to that of Claim 2.11. □

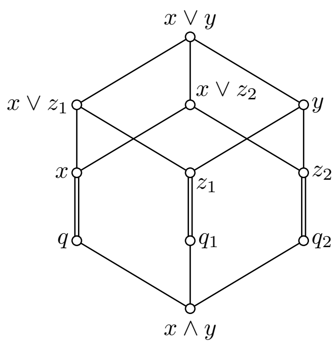

Figure 2.

Claim 2.13. $q \vee q_1 < x \vee z_1$.

P r o o f. Suppose that $q \vee q_1 = x \vee z_1$. Note that $x \wedge y \not\prec q$ and $x \wedge y \not\prec q_1$; otherwise, $q_1 \prec q \vee q_1$, which is not true since $q_1 \prec z_1 < x \vee z_1$. Therefore there exists p_1 such that $x \wedge y \prec p_1 < q$. We have $p_1 \parallel z_1$, $p_1 \parallel q_1$ and $p_1 \vee q_1 \parallel z_1$. Now, if $(p_1 \vee q_1) \vee z_1 = x \vee z_1$, then by semimodularity we have $p_1 \vee q_1 \prec x \vee z_1$ and $z_1 \prec x \vee z_1$ and so $r(x \vee z_1) - r(z_1) = 1$. Also, as $q \vee y = x \vee y$, $q \vee z_1 = x \vee z_1$, $q \vee z_2 = x \vee z_2$ and $q < x$ and so, by assumption, $P_{q,y}$ must have the least element. We have $y, z_1, z_2 \in P_{q,y}$, $z_1 \wedge z_2 \in P_{q,y}$ but $r(q \vee (x \wedge y)) - r(x \wedge y) \neq 1$, a contradiction, and so we must have $q \vee q_1 < x \vee z_1$. □

Claim 2.14. $q \vee q_2 < x \vee z_2$.

P r o o f. Is similar to that of Claim 2.13. □

Claim 2.15. $x \wedge y \prec q$.

P r o o f. Suppose there exists an element p such that $x \wedge y \prec p < q$. It follows that $p \vee q_1 \leqslant q \vee q_1$. If $p \vee q_1 = q \vee q_1$, then by semimodularity we have $q_1 \prec q \vee q_1$, which is not true. Therefore $p \vee q_1 < q \vee q_1$ and by semimodularity we have $q_1 \prec p \vee q_1$ and similarly, $q_2 \prec p \vee q_2$. In this case, $(p \vee q_1) \vee z_1 \leqslant x \vee z_1$. We consider the following subcases:

(i) Suppose $(p \vee q_1) \vee z_1 = x \vee z_1$. By semimodularity we have $z_1 \prec x \vee z_1$ and $y \prec x \vee y$ and therefore $r(x \vee z_1) - r(z_1) = r(x \vee y) - r(y) = 1$. Since $r(x \vee y) - r(y) = r(x \vee z_2) - r(z_2)$, we have $r(x \vee z_2) - r(z_2) = 1$ and hence $z \prec x \vee z_2$. We also have $q \vee y = x \vee y$, $q \vee z_1 = x \vee z_1$, $q \vee z_2 = x \vee z_2$. Thus $y_1, z_1, z_2 \in P_{q,y}$ and by assumption, $P_{q,y}$ must have the least element, which gives $z_1 \wedge z_2 \in P_{q,y}$. Therefore $r(q \vee y) - r(y) = r(q \vee (z_1 \wedge z_2)) - r(z_1 \wedge z_2) = 1$, a contradiction to the fact that $r(q) - r(x \wedge y) > 1$, and therefore $(p \vee q_1) \vee z_1 \neq x \vee z_1$. Similarly, $(p \vee q_2) \vee z_2 \neq x \vee z_2$.

(ii) Suppose $(p \vee q_1) \vee z_1 < x \vee z_1$. Let $p_1 = (p \vee q_1) \vee z_1$. By semimodularity we have $z_1 \prec p_1$. If $p_1 \vee y = x \vee y$, then $y \prec x \vee y$, a contradiction to the fact that $r(x \vee y) - r(y) = r(x \vee z_1) - r(z_1)$, and therefore $p_1 \vee y < x \vee y$. By semimodularity we have $y \prec p_1 \vee y$. Similarly, for $p_2 = (p \vee q_2) \vee z_2$, we have $y \prec p_2 \vee y$.

In this case, $p_1 \vee y = p_2 \vee y = (p \vee q_2 \vee z_2) \vee y = (p \vee z_2) \vee y = p \vee y$. Let $y_1 = p_1 \vee y = p_2 \vee y = p \vee y$. Then $x \vee y_1 = x \vee y$, $x \wedge y < x \wedge y_1$ and $x \wedge y_1 \geqslant p$. Therefore $lt[x \wedge y_1, x \vee y_1] < lt[x \wedge y, x \vee y]$. As $r(x \vee y_1) - r(y_1) = r(x \vee p_1) - r(p_1) = r(x \vee z_1) - r(p_1) = r(x \vee p_2) - r(p_2) = r(x \vee z_2) - r(p_2)$, we have $y_1, p_1, p_2 \in P_{x,y_1}$. Hence P_{x,y_1} has the least element, which gives $p_1 \wedge p_2 \in P_{x,y_1}$. Thus $r(x \vee y_1) - r(y_1) = r(x \vee (p_1 \wedge p_2)) - r(p_1 \wedge p_2)$ and we have $p_1 \wedge p_2 \geqslant p$. Also, $p_1 \wedge p_2 \leqslant x \vee z_1$ and $p_1 \wedge p_2 \leqslant x \vee z_2$, which gives $p_1 \wedge p_2 \leqslant (x \vee z_1) \wedge (x \vee z_2)$, and so $p_1 \wedge p_2 \leqslant x$. In this case, $q \vee y = q \vee y_1 = x \vee y$, $q \vee z_1 = q \vee p_1 = x \vee z_1$, $q \vee z_2 = q \vee p_2 = x \vee z_2$ and $q \wedge y_1 > x \wedge y$, and so $y_1, p_1, p_2 \in P_{q,y_1}$. By assumption, $P_{q_1,y}$ has the least element and so $p_1 \wedge p_2 \in P_{q,y}$. Thus $r(q \vee (p_1 \wedge p_2)) - r(p_1 \wedge p_2) = r(x \vee z_1) - r(p_1)$. Since $(x \vee (p_1 \wedge p_2)) - r(p_1 \wedge p_2) = r(x \vee z_1) - r(p_1)$, which implies that $r(q \vee (p_1 \wedge p_2)) - r(p_1 \wedge p_2) = (x \vee (p_1 \wedge p_2)) - r(p_1 \wedge p_2)$, we have a contradiction as $q \prec x$. Thus, in each of the cases we get a contradiction and consequently we must have $x \wedge y \prec q$. □

Also, since $q_1 \prec z_1$, $q_2 \prec z_2$ and $q \prec x$, by semimodularity we have $q \vee q_2 \prec x \vee z_2$ and $q \vee q_1 \prec x \vee z_1$.

Claim 2.16. $x \wedge y \prec q_1$.

P r o o f. Suppose there exists p_1 such that $x \wedge y \prec p_1 < q_1$. By semimodularity, we have $q \prec q \vee p_1$, $x \prec x \vee p_1$, $z_2 \prec p_1 \vee z_2$ and $q \vee q_2 \prec q \vee q_2 \vee p_1$. Let $z_2' = p_1 \vee z_2$ and we have $x \vee z_2 \prec z_2' \vee x$. Let $x_1 = x \vee p_1$ and we have $lt[x_1 \wedge y, x_1 \vee y] < lt[x \wedge y, x \vee y]$.

Since $z_2' > z_2$ and $P_{x,y}$ is a dual order ideal, we have $z_2' \in P_{x,y}$. Thus $r(x \vee z_2') - r(z_2') = 1$. Also we have $x_1 \vee y = x \vee y$, $x_1 \vee z_1 = x \vee z_1$ and $x_1 \vee z_2' = x \vee z_2'$. It follows that $y, z_1, z_2' \in P_{x_1,y}$ and so $z_1 \wedge z_2' \in P_{x_1,y}$. Thus $r(x_1 \vee (z_1 \wedge z_2')) - r(z_1 \wedge z_2') = 1$. Since $q_1 > z_1 \wedge z_2' \geqslant p_1$ and $x_1 \vee (z_1 \wedge z_2') = x \vee z_1$, we have $r(x_1 \vee (z_1 \wedge z_2')) - r(z_1 \wedge z_2') > 1$, a contradiction, and so we must have $x \wedge y \prec q_1$. □

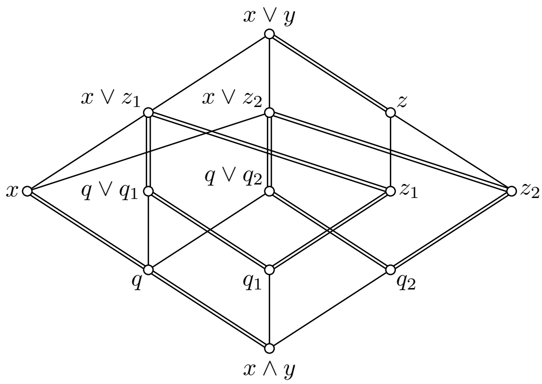

Figure 3.

Claim 2.17. $x \wedge y \prec q_2$.

Proof. Is similar to that of Claim 2.16. □

Claim 2.18. $q_1 \vee q_2 < y$.

Proof. Suppose that $q_1 \vee q_2 = y$, since $x \wedge y \prec q_1, q_2$ implies $x \wedge y = q_1 \wedge q_2$ and by semimodularity, $q_1 \prec y$ and $q_2 \prec y$, which is not true and so we must have $q_1 \vee q_2 < y$. □

Now, $x \wedge y \prec q_1, q_2$ implies $x \wedge y = q_1 \wedge q_2$ and by semimodularity, $q_1 \prec q_1 \vee q_2$ and $q_2 \prec q_1 \vee q_2$. Also, $x \wedge y \prec q, q_1$ implies $x \wedge y = q_1 \wedge q$ and so by semimodularity, $q \prec q \vee q_1$ and $q_1 \prec q \vee q_1$. Similarly, $x \wedge y \prec q, q_2$ implies $x \wedge y = q_2 \wedge q$, so by semimodularity, $q \prec q \vee q_2$ and $q_2 \prec q \vee q_2$. Now, $q_1 \prec q_1 \vee q_2$ and $q_2 \prec q_1 \vee q_2$ implies $q_1 = (q_1 \vee q_2) \wedge z_1$ and $q_2 = (q_1 \vee q_2) \wedge z_2$ and by semimodularity, $z_1 \prec (q_1 \vee q_2) \vee z_1 = y$, $z_2 \prec (q_1 \vee q_2) \vee z_2 = y$ and $q_1 \vee q_2 \prec (q_1 \vee q_2) \vee z_2 = y$.

Claim 2.19. $(q_1 \vee q_2) \vee q < x \vee y$.

Proof. Suppose that $(q_1 \vee q_2) \vee q = x \vee y$. Since $q \wedge (q_1 \vee q_2) \prec q$, by semimodularity we have $q_1 \vee q_2 \prec q \vee (q_1 \vee q_2)$, which is not true as $q_1 \vee q_2 \prec y \prec x \vee y$. Therefore $(q_1 \vee q_2) \vee q < x \vee y$. □

Since $q_1 \prec q \vee q_1$, by semimodularity, $(q_1 \vee q_2) \prec (q_1 \vee q_2) \vee q$. Also, $q_1 \prec q_1 \vee q_2$ implies $q_1 \vee q \prec q_1 \vee q_2 \vee q$, $q_2 \prec q_1 \vee q_2$, which further implies $q_2 \vee q \prec q_1 \vee q_2 \vee q$ and also $q_1 \vee q_2 \prec y$ implies $q_1 \vee q_2 \vee q \prec x \vee z$. Hence, L contains the following cover preserving sublattice.

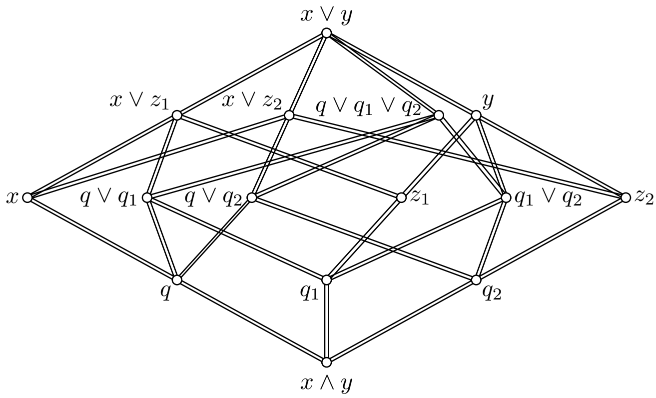

Figure 4.

□

The following result is due to Teo [8].

Corollary 2.20 ([8]). *A lattice L of finite length is not semimodular if and only if L contains a subpentagon $(a \wedge c, a, b, c, a \vee b)$ with the properties*

(i) $a \wedge c \prec a, b \prec c \prec a \vee b$, *or*

(ii) $a \wedge c \prec a, a \wedge c \prec b, c \prec a \vee b$.

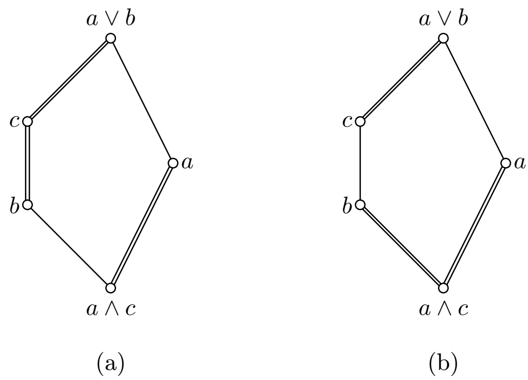

Figure 5.

Corollary 2.21. *Let L be a lattice of finite length. Then L is a pseudomodular lattice if and only if it does not contain a sublattice isomorphic to a cover preserving lattice as depicted in Figure* 4 *or Figure* 5 (a) *or Figure* 5 (b).

References

[1] *G. Birkhoff*: Lattice Theory. Colloquium Publications 25. AMS, Providence, 1967. zbl MR doi

[2] *A. Björner, L. Lovász*: Pseudomodular lattices and continuous matroids. Acta Sci. Math. *51* (1987), 295–308. zbl MR

[3] *A. Dress, W. Hochstättler, W. Kern*: Modular substructures in pseudomodular lattices. Math. Scand. *74* (1994), 9–16. zbl MR doi

[4] *A. Dress, L. Lovász*: On some combinatorial properties of algebraic matroids. Combinatorica *7* (1987), 39–48. zbl MR doi

[5] *G. Grätzer*: General Lattice Theory. Pure and Applied Mathematics 75. Academic Press, New York, 1978. zbl MR doi

[6] *L. Haskins, S. Gudder*: Heights on posets and graphs. Discrete Math. *2* (1972), 357–382. zbl MR doi

[7] *M. Stern*: Semimodular Lattices: Theory and Applications. Encyclopedia of Mathematics and Its Applications 73. Cambridge University Press, Cambridge, 1999. zbl MR doi

[8] *K. L. Teo*: Diagrammatic characterizations of semimodular lattices of finite length. Southeast Asian Bull. Math. *12* (1988), 135–140. zbl MR

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