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# ORBIT-CONE CORRESPONDENCE FOR THE PROALGEBRAIC COMPLETION OF NORMAL TORIC VARIETIES

GENARO HERNANDEZ-MADA AND HUMBERTO ABRAHAM MARTINEZ-GIL

ABSTRACT. We prove that there is an orbit-cone correspondence for the proalgebraic completion of normal toric varieties, which is analogous to the classical orbit-cone correspondence for toric varieties.

## 1. INTRODUCTION

Let  $N$  be a finite rank free abelian group (also called a lattice) and  $\mathcal{F}$  a fan in  $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ . It is well-known that there is a toric variety  $X^{\mathcal{F}}$  associated to  $\mathcal{F}$ . Moreover a classical result states that if  $T_N$  is the torus of  $X^{\mathcal{F}}$  then there is a bijective correspondence between the  $T_N$ -orbits in  $X^{\mathcal{F}}$  and the cones  $\sigma$  in  $\mathcal{F}$ . (See for example [2], Theorem 3.2.6).

Now consider  $(I, \leq)$  a directed poset and a projective system of complex algebraic varieties

$$f_{ij}: V_j \rightarrow V_i, \quad i, j \in I, \quad i \leq j.$$

This system is called **proalgebraic variety**, and as pointed out in [1] this concept plays an important role in dynamical systems. In the cases that we will study, we can take projective limit and obtain **laminations**, (also called **solenoidal manifolds** in [6]). These have been used, for example in [4] and [6], to obtain results in conformal dynamics.

In this paper we consider a projective system of toric varieties. Namely, given a normal toric variety  $X$ , for each  $m, n \in \mathbb{N}$  such that  $n \mid m$ , we can define

$$p_{n,m}: \mathbb{C} \rightarrow \mathbb{C}$$

by  $z \mapsto z^{m/n}$ , and these induce

$$q_{n,m}: X \rightarrow X,$$

which define a projective system. The projective limit, denoted by  $X_{\mathbb{Q}}$  is called the **proalgebraic completion** of  $X$ . The first part of [1] is devoted to the study of the proalgebraic completion of toric varieties. More particularly, in Theorem 1 they

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give a description of its vector bundle category. For a more general treatment of proalgebraic completions, we refer to [3].

The main result of this paper is an analogous result to the one described in the first paragraph. Namely, we will prove that there is a *proalgebraic torus* acting on the proalgebraic completion of the toric variety  $X^{\mathcal{F}}$  (defined by the fan  $\mathcal{F}$ ), and there is a bijective correspondence of its orbits with the cones of  $\mathcal{F}$ . This correspondence satisfies similar properties to the classical one. See Theorem 3 for the precise statement and Theorem 1 for the comparison with the classical result.

Now let us give an outline of this paper. In Section 2 we recall some concepts regarding toric varieties, as well as the precise statement of the classical theorem that inspires this work (Theorem 1). In section 3 we give the basic definitions regarding proalgebraic completions of normal toric varieties. Namely, we define the algebraic solenoid, which allows to define the proalgebraic torus and finally the proalgebraic completion of a normal toric variety. For this we define it first for an affine toric variety and we use this for the general case. Finally in Section 4 we prove our main result.

## 2. PRELIMINARIES ON TORIC VARIETIES

Given a lattice  $N$  with dual  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$  and a rational strongly convex polyhedral cone  $\sigma \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ , we have the classical definition as in [5] of the affine normal toric variety

$$X^{\sigma} = \text{Hom}_{sg}(S^{\sigma}, \mathbb{C}),$$

where  $S^{\sigma} \subset M$  is the semigroup associated to  $\sigma$ . In particular, each point of  $X^{\sigma}$  is a homomorphism of semigroups  $S^{\sigma} \rightarrow \mathbb{C}$ . In this setting, the torus of  $X^{\sigma}$  is the topological group

$$T_N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*).$$

If  $\mathcal{F}$  is a fan that lies in the vector space  $N \otimes_{\mathbb{Z}} \mathbb{R}$ , then for every  $\sigma_1, \sigma_2 \in \mathcal{F}$  and  $\tau = \sigma_1 \cap \sigma_2$  there are natural injective morphisms

$$\text{Hom}_{sg}(S^{\sigma_1}, \mathbb{C}) \hookrightarrow \text{Hom}_{sg}(S^{\tau}, \mathbb{C}) \hookrightarrow \text{Hom}_{sg}(S^{\sigma_2}, \mathbb{C}).$$

These define an equivalence relation so the affine toric varieties  $X^{\sigma}$ ,  $\sigma \in \mathcal{F}$  glue together to get the normal toric variety

$$X^{\mathcal{F}} = \bigsqcup_{\sigma \in \mathcal{F}} \text{Hom}_{sg}(S^{\sigma}, \mathbb{C}) / \sim,$$

with torus  $T_N$ . Recall that the action of  $T_N$  on  $X^{\mathcal{F}}$  is given by the map

$$(1) \quad T_N \times X^{\mathcal{F}} \rightarrow X^{\mathcal{F}}, \quad \tilde{\gamma} \cdot [\gamma] = [\tilde{\gamma}|_{S^{\sigma}} \gamma],$$

where  $\gamma: S^{\sigma} \rightarrow \mathbb{C}$ ,  $\tilde{\gamma}: M \rightarrow \mathbb{C}^*$ .

An important result in toric geometry is that the  $T_N$ -orbits of a toric variety

$X^{\mathcal{F}}$  are characterized by the **special points**. Namely, by the semigroup homomorphism (point in  $X^{\sigma} = \text{Hom}_{sg}(S^{\sigma}, \mathbb{C})$ ) defined for every  $\sigma \in \mathcal{F}$  by

$$m \in S^{\sigma} \mapsto \begin{cases} 1 & \text{if } m \in S^{\sigma} \cap \sigma^{\perp} = \sigma^{\perp} \cap M, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\sigma^{\perp}$  is the orthogonal complement of  $\sigma$  on the dual vector space of  $N \otimes_{\mathbb{Z}} \mathbb{R}$ .

**Theorem 1.** *Let  $X^{\mathcal{F}}$  be the toric variety associated to the fan  $\mathcal{F}$  on  $N \otimes_{\mathbb{Z}} \mathbb{R}$ . Then*

(a) *There is a bijective correspondence*

$$\begin{aligned} \{\text{Cones } \sigma \in \mathcal{F}\} &\longleftrightarrow \{T_N\text{-orbits on } X^{\mathcal{F}}\} \\ \sigma &\longleftrightarrow T_N \cdot \gamma^{\sigma}. \end{aligned}$$

(b) *If  $\dim N \otimes_{\mathbb{Z}} \mathbb{R} = r$ , then for every cone  $\sigma \in \mathcal{F}$  the orbit  $T_N \cdot \gamma^{\sigma}$  is an algebraic torus of dimension  $r - \dim \sigma$ .*

(c) *For every  $\sigma \in \mathcal{F}$  the affine toric variety  $X^{\sigma}$  is the union of orbits*

$$X^{\sigma} = \bigcup_{\tau \text{ face of } \sigma} T_N \cdot \gamma^{\tau}.$$

The proof is in [2] Theorem 3.2.6. Our main result is a proalgebraic version of this theorem. To get there, we will use the notion of proalgebraic completion of a toric variety from [1].

### 3. PROALGEBRAIC COMPLETION OF NORMAL TORIC VARIETIES

For every  $n, m \in \mathbb{N}$  such that  $n \mid m$  ( $n$  divides  $m$ ) there is a finite covering map

$$(2) \quad p_{n,m}: \mathbb{C}^* \rightarrow \mathbb{C}^*, \quad z \mapsto z^{m/n}.$$

This determines a projective system  $\{\mathbb{C}^*, p_{n,m}\}_{n|m}$  of covering spaces whose projective limit is the **algebraic solenoid**  $\mathbb{C}_{\mathbb{Q}}^*$  which is a topological group with the initial topology defined by the canonical projections

$$\pi_n: \mathbb{C}_{\mathbb{Q}}^* \rightarrow \mathbb{C}^*, \quad (z_j)_{j \in \mathbb{N}} \mapsto z_n.$$

Let  $\widehat{\mathbb{Z}}$  be the profinite completion of the integers. Then the algebraic solenoid is a 2-dimensional **lamination** in the sense of [4] with **transversal space**  $\widehat{\mathbb{Z}}$ . Furthermore, the **global leaves** of the lamination are just copies of  $\mathbb{C}$ , and more precisely all the global leaves are obtained by translations of the image of the injective and continuous map

$$\nu: \mathbb{C} \rightarrow \mathbb{C}_{\mathbb{Q}}^*, \quad z \mapsto (e^{i(z/j)})_{j \in \mathbb{N}}$$

by elements of  $\widehat{\mathbb{Z}}$ . This algebraic solenoid is thoroughly studied in [1].

**Definition 1.** Let  $N$  be a lattice with dual  $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ . The group

$$(T_N)_{\mathbb{Q}} = \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}_{\mathbb{Q}}^*)$$

is called **proalgebraic torus**.

Note that if  $\text{rank } M = r$ , then we have a natural group isomorphisms

$$N \otimes_{\mathbb{Z}} \mathbb{C}_{\mathbb{Q}}^* \cong (T_N)_{\mathbb{Q}} \cong (\mathbb{C}_{\mathbb{Q}}^*)^r.$$

Hence the proalgebraic torus  $(T_N)_{\mathbb{Q}}$  is also a lamination ( $2r$ -dimensional) with transversal space  $\widehat{\mathbb{Z}}^r$ . In this case, the global leaves are copies of  $\mathbb{C}^r$ , and all of them are obtained via translations by elements of  $\widehat{\mathbb{Z}}^r$ . The maps in 2 used to define the algebraic solenoid extend naturally to the whole complex plane  $\mathbb{C}$ , so we can consider the limit

$$\mathbb{C}_{\mathbb{Q}} = \{(z_j)_{j \in \mathbb{N}} \mid z_j \in \mathbb{C} \text{ for all } j \in \mathbb{N}, p_{n,m}(z_m) = z_n \text{ whenever } n \mid m\}.$$

**Definition 2.** Let  $\sigma$  be a rational strongly convex polyhedral cone with associated semigroup  $S^\sigma$ . The set

$$X_{\mathbb{Q}}^\sigma = \text{Hom}_{sg}(S^\sigma, \mathbb{C}_{\mathbb{Q}})$$

is **the proalgebraic completion** of the affine toric variety  $X^\sigma = \text{Hom}_{sg}(S^\sigma, \mathbb{C})$ .

It is clear that an element  $\gamma$  in  $X_{\mathbb{Q}}^\sigma$  induces a collection  $\{\gamma_j: S^\sigma \rightarrow \mathbb{C}\}_{j \in \mathbb{N}}$  of semigroup homomorphisms satisfying  $(\gamma_m(s))^{m/n} = \gamma_n(s)$  for all  $s \in S^\sigma$  whenever  $n \mid m$ , and there is a bijective correspondence

$$X_{\mathbb{Q}}^\sigma \rightarrow \varprojlim_{q_{n,m}} \text{Hom}_{sg}(S^\sigma, \mathbb{C}), \quad \gamma \mapsto (\gamma_j)_{j \in \mathbb{N}},$$

where

$$q_{n,m}: \text{Hom}_{\mathbb{Z}}(S^\sigma, \mathbb{C}) \rightarrow \text{Hom}_{\mathbb{Z}}(S^\sigma, \mathbb{C}), \quad \gamma \mapsto p_{n,m} \circ \gamma$$

is an algebraic map. Under this correspondence we can endow the proalgebraic completion  $X_{\mathbb{Q}}^\sigma$  with a topology.

More generally, let  $\mathcal{F}$  be a fan and consider the collection of proalgebraic completions

$$\{X_{\mathbb{Q}}^\sigma\}_{\sigma \in \mathcal{F}} = \{\text{Hom}_{sg}(S^\sigma, \mathbb{C}_{\mathbb{Q}})\}_{\sigma \in \mathcal{F}}.$$

As in the classical case, if  $\sigma_1, \sigma_2 \in \mathcal{F}$ , and  $\tau$  is a face of  $\sigma_1 \cap \sigma_2$ , we have natural injective maps

$$(3) \quad X_{\mathbb{Q}}^{\sigma_1} = \text{Hom}_{sg}(S^{\sigma_1}, \mathbb{C}_{\mathbb{Q}}) \hookrightarrow \text{Hom}_{sg}(S^\tau, \mathbb{C}_{\mathbb{Q}}) \hookrightarrow \text{Hom}_{sg}(S^{\sigma_2}, \mathbb{C}_{\mathbb{Q}}) = X_{\mathbb{Q}}^{\sigma_2}.$$

These maps allow to define an equivalence relation on

$$\bigsqcup_{\sigma \in \mathcal{F}} \text{Hom}_{sg}(S^\sigma, \mathbb{C}_{\mathbb{Q}})$$

as follows:  $(\gamma_1: S^{\sigma_1} \rightarrow \mathbb{C}_{\mathbb{Q}}) \sim (\gamma_2: S^{\sigma_2} \rightarrow \mathbb{C}_{\mathbb{Q}})$  if and only if there exists a face  $\tau$  of  $\sigma_1 \cap \sigma_2$  and a homomorphism  $\gamma: S^\tau \rightarrow \mathbb{C}_{\mathbb{Q}}$  such that  $\gamma|_{S^{\sigma_1}} = \gamma_1, \gamma|_{S^{\sigma_2}} = \gamma_2$ .

**Definition 3.** Let  $N$  be a lattice and let  $\mathcal{F}$  be a fan in  $N \otimes_{\mathbb{Z}} \mathbb{R}$ . We define **the proalgebraic completion**  $X_{\mathbb{Q}}^{\mathcal{F}}$  of the toric variety  $X^{\mathcal{F}}$  as the quotient space

$$X_{\mathbb{Q}}^{\mathcal{F}} := \bigsqcup_{\sigma \in \mathcal{F}} \text{Hom}_{sg}(S^\sigma, \mathbb{C}_{\mathbb{Q}}) / \sim.$$

As in the affine case, the proalgebraic completion  $X_{\mathbb{Q}}^{\mathcal{F}}$  turns out to be a projective limit. More precisely we have the following (see [1] Section 4):

**Theorem 2.** *The proalgebraic completion  $X_{\mathbb{Q}}^{\mathcal{F}}$  of a toric variety  $X^{\mathcal{F}}$  is homomorphic to the projective limit over all its toric covers.*

With the previous result it is clear that there is a canonical inclusion

$$(T_N)_{\mathbb{Q}} \subset X_{\mathbb{Q}}^{\mathcal{F}}$$

induced by the inclusion  $T_N \subset X^{\mathcal{F}}$ . Moreover, the action described in (1) induces the action

$$(4) \quad (T_N)_{\mathbb{Q}} \times X_{\mathbb{Q}}^{\mathcal{F}} \rightarrow X_{\mathbb{Q}}^{\mathcal{F}}, \quad \tilde{\gamma} \cdot [\gamma] = [\tilde{\gamma}|_{S^{\sigma}} \cdot \gamma],$$

where  $\gamma: S^{\sigma} \rightarrow \mathbb{C}_{\mathbb{Q}}$ ,  $\tilde{\gamma}: M \rightarrow \mathbb{C}_{\mathbb{Q}}^*$ .

**Remark 1.** If  $X$  is a toric variety defined by a fan  $\mathcal{F}$  in  $N \otimes_{\mathbb{Z}} \mathbb{R}$ , for every  $m \in \mathbb{N}$  we take  $X_m = X$ , and then for every  $n \mid m$  we have a map  $N \rightarrow N$  which is multiplication by the integer  $\frac{m}{n}$ . Since this map takes cones of  $\mathcal{F}$  into cones of  $\mathcal{F}$ , it defines a map of toric varieties  $X_m \rightarrow X_n$ , and thus a projective system where all the maps involved are algebraic. Moreover, these maps are affine, so we get a scheme structure on  $X_{\mathbb{Q}}$ .

**Remark 2.** Note that if  $X$  is singular, it is not a manifold, and perhaps a more suitable name for the proalgebraic completion would be solenoidal variety, instead of solenoidal manifold.

#### 4. THE ORBIT-CONE CORRESPONDENCE

In order to study the  $(T_N)_{\mathbb{Q}}$ -orbits of the proalgebraic completion  $X_{\mathbb{Q}}^{\mathcal{F}}$  we have the following definition which is the proalgebraic version of the special points in  $X^{\mathcal{F}}$ .

**Definition 4.** Let  $\sigma$  be a rational strongly convex polyhedral cone in  $N \otimes_{\mathbb{Z}} \mathbb{R}$ . We define the **special point** (or distinguished point) in  $X_{\mathbb{Q}}^{\sigma} = \text{Hom}_{sg}(S^{\sigma}, \mathbb{C}_{\mathbb{Q}}^*)$  as the semigroup homomorphism

$$\gamma_{\mathbb{Q}}^{\sigma}(m) = \begin{cases} (1_j)_j & \text{if } m \in S^{\sigma} \cap \sigma^{\perp} = \sigma^{\perp} \cap M, \\ (0_j)_j & \text{otherwise,} \end{cases}$$

for  $m \in S^{\sigma}$ .

As we shall see in the following Lemma, the  $(T_N)_{\mathbb{Q}}$ -orbits in  $X_{\mathbb{Q}}^{\mathcal{F}}$  corresponding to the special points are proalgebraic torus themselves.

**Lemma 1.** *Let  $\sigma$  be a rational strongly convex polyhedral cone in  $N \otimes_{\mathbb{Z}} \mathbb{R}$  and consider the lattice  $N(\sigma) = N/N_{\sigma}$ , where  $N_{\sigma}$  is the sublattice generated by the elements in  $\sigma \cap N$ . Then there is a bijection*

$$(T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\sigma} \simeq (T_{N(\sigma)})_{\mathbb{Q}} = \text{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}_{\mathbb{Q}}^*).$$

**Proof.** We define the map

$$(5) \quad (T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\sigma} \rightarrow (T_{N(\sigma)})_{\mathbb{Q}}, \quad \gamma \cdot \gamma_{\mathbb{Q}}^{\sigma} \mapsto (\gamma \cdot \gamma_{\mathbb{Q}}^{\sigma})|_{\sigma^{\perp} \cap M},$$

which is clearly injective by definition of  $\gamma_{\mathbb{Q}}^{\sigma}$ . To see that this correspondence is surjective, consider the short exact sequence

$$0 \rightarrow N_{\sigma} \rightarrow N \rightarrow N(\sigma) \rightarrow 0.$$

If we tensor with  $\mathbb{C}_{\mathbb{Q}}^*$  then we obtain a surjective homomorphism

$$N \otimes_{\mathbb{Z}} \mathbb{C}_{\mathbb{Q}}^* \rightarrow N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}_{\mathbb{Q}}^* \rightarrow 0,$$

and hence a transitive action of  $N \otimes_{\mathbb{Z}} \mathbb{C}_{\mathbb{Q}}^*$  on  $N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}_{\mathbb{Q}}^*$ .

Under the natural isomorphisms

$$N \otimes_{\mathbb{Z}} \mathbb{C}_{\mathbb{Q}}^* \cong \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}_{\mathbb{Q}}^*) = (T_N)_{\mathbb{Q}}$$

and

$$N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}_{\mathbb{Q}}^* \cong \text{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}_{\mathbb{Q}}^*) = (T_{N(\sigma)})_{\mathbb{Q}},$$

the action of  $N \otimes_{\mathbb{Z}} \mathbb{C}_{\mathbb{Q}}^*$  on  $N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}_{\mathbb{Q}}^*$  corresponds with the action

$$(T_N)_{\mathbb{Q}} \times (T_{N(\sigma)})_{\mathbb{Q}} \rightarrow (T_{N(\sigma)})_{\mathbb{Q}}$$

given by  $\gamma \cdot \alpha = \gamma|_{\sigma^{\perp} \cap M} \alpha$ .

Note that by restricting the special point  $\gamma_{\mathbb{Q}}^{\sigma}$  to the group  $\sigma^{\perp} \cap M$  we get an element in  $(T_{N(\sigma)})_{\mathbb{Q}}$ . Since the previous action is transitive, the map (5) is surjective.  $\square$

**Remark 3.** The group  $\text{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}_{\mathbb{Q}}^*)$  from the previous lemma can be described as the following set of semigroup homomorphisms:

$$\{\gamma: S^{\sigma} \rightarrow \mathbb{C}_{\mathbb{Q}} \mid \gamma(m) \neq (0)_j \Leftrightarrow m \in \sigma^{\perp} \cap M\}.$$

Indeed, any element of this set can be restricted to obtain an element in

$$\text{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}_{\mathbb{Q}}^*).$$

Conversely, extending by zero, any element on the group  $\text{Hom}_{\mathbb{Z}}(\sigma^{\perp} \cap M, \mathbb{C}_{\mathbb{Q}}^*)$  can be seen as an element in the set above.

**Theorem 3** (The Orbit-Cone correspondence). *Let  $\mathcal{F}$  be a fan in  $N \otimes_{\mathbb{Z}} \mathbb{R}$  and let  $X_{\mathbb{Q}}^{\mathcal{F}}$  be the proalgebraic completion of the normal toric variety  $X^{\mathcal{F}}$ . Then*

(a) *There is a bijective correspondence*

$$\begin{aligned} \{\text{Cones } \sigma \in \mathcal{F}\} &\longleftrightarrow \{(T_N)_{\mathbb{Q}}\text{-orbits in } X_{\mathbb{Q}}^{\mathcal{F}}\} \\ \sigma &\longleftrightarrow (T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\sigma}. \end{aligned}$$

(b) *If  $\dim N \otimes_{\mathbb{Z}} \mathbb{R} = r$ , then for every cone  $\sigma \in \mathcal{F}$ , the orbit  $(T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\sigma}$  is a lamination of dimension  $2(r - \dim \sigma)$ .*

(c) *For every  $\sigma \in \mathcal{F}$  the completion  $X_{\mathbb{Q}}^{\sigma}$  is the union of orbits*

$$X_{\mathbb{Q}}^{\sigma} = \bigcup_{\tau \text{ face of } \sigma} (T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\tau}.$$

**Proof.** For surjectivity in (a), let  $O$  be a  $(T_N)_{\mathbb{Q}}$ -orbit of  $X_{\mathbb{Q}}^{\mathcal{F}}$ . Since

$$X_{\mathbb{Q}}^{\mathcal{F}} = \bigcup_{\sigma \in \mathcal{F}} X_{\mathbb{Q}}^{\sigma},$$

if  $\sigma_1, \sigma_2 \in \mathcal{F}$ , the elements of  $X_{\mathbb{Q}}^{\sigma_1}$  that are related to the elements of  $X_{\mathbb{Q}}^{\sigma_2}$  are precisely the elements of  $X_{\mathbb{Q}}^{\sigma_1 \cap \sigma_2}$ , i.e.,

$$X_{\mathbb{Q}}^{\sigma_1} \cap X_{\mathbb{Q}}^{\sigma_2} = X_{\mathbb{Q}}^{\sigma_1 \cap \sigma_2}.$$

Moreover each  $X_{\mathbb{Q}}^{\sigma}$  is invariant under the action (4) of the torus  $(T_N)_{\mathbb{Q}}$ . Hence there exists a unique minimal cone  $\sigma \in \mathcal{F}$  such that  $O \subseteq X_{\mathbb{Q}}^{\sigma}$ . We shall prove that  $O = (T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\sigma}$ . Indeed, let  $\gamma \in O$  and consider the set

$$S = \{m \in S^{\sigma} \mid \gamma(m) \neq (0_j)_j\}.$$

For every  $j \in \mathbb{N}$ , let  $\pi_j: \mathbb{C}_{\mathbb{Q}} \rightarrow \mathbb{C}$  be the canonical projection which is in particular a semigroup homomorphism, and note that

$$S = \{m \in S^{\sigma} \mid \gamma(m) \neq (0_j)_j\} = \{m \in S^{\sigma} \mid (\pi_j \circ \gamma)(m) \neq 0\}$$

for every  $j \in \mathbb{N}$ . It follows that  $S = \Gamma \cap M$  for some face  $\Gamma$  of  $\sigma^{\vee}$ , and consequently there exists a face  $\tau$  of  $\sigma$  such that  $\Gamma = \sigma^{\vee} \cap \tau^{\perp}$ .

Now the equality

$$S = \Gamma \cap M = \sigma^{\vee} \cap \tau^{\perp} \cap M$$

implies  $\gamma \in X_{\mathbb{Q}}^{\tau}$ , and then  $\tau = \sigma$  because of the minimality of  $\sigma$ . By Remark 3,

$$S = \sigma^{\perp} \cap M$$

implies that  $\gamma \in (T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\sigma}$  and therefore  $O = (T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\sigma}$ . This proves surjectivity.

For injectivity, let  $\sigma_1 \neq \sigma_2$  be cones in  $\mathcal{F}$ . If  $\dim \sigma_1 \neq \dim \sigma_2$ , it follows from Lemma 1 that the orbits are not equal. If  $\dim \sigma_1 = \dim \sigma_2$ , an easy computation shows that

$$(T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\sigma_1}, (T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\sigma_2} \not\subseteq X_{\mathbb{Q}}^{\sigma_1 \cap \sigma_2},$$

and hence  $(T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\sigma_1} \neq (T_N)_{\mathbb{Q}} \cdot \gamma_{\mathbb{Q}}^{\sigma_2}$ .

Part (b) follows from Lemma 1 and the fact that any proalgebraic torus is a lamination; part (c) follows directly from part (a).  $\square$

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