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Towards understanding of Napoleon's theorem: Complex numbers are the key

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Abstract. The subject of this article is a generalization—and consequently a better understanding—of a theorem, the authorship of which is often attributed to Napoleon Bonaparte. The article emphasizes the importance of complex numbers when studying the geometry of the plane.

The relationship between the complex numbers and the plane geometry is an interesting topic whose importance does not seem to be adequately emphasized. The concept of complex numbers has an intricate history inextricably tied to solution of algebraic equations. When complex numbers were first introduced they appeared quite mysterious and raised serious questions, even for some prominent mathematicians. However, today a student will be introduced to complex numbers as a natural extension of the real numbers, from the real line to the complex Argand–Gaussian plane equipped with a rectangular coordinate system, as indicated in Fig. 1.

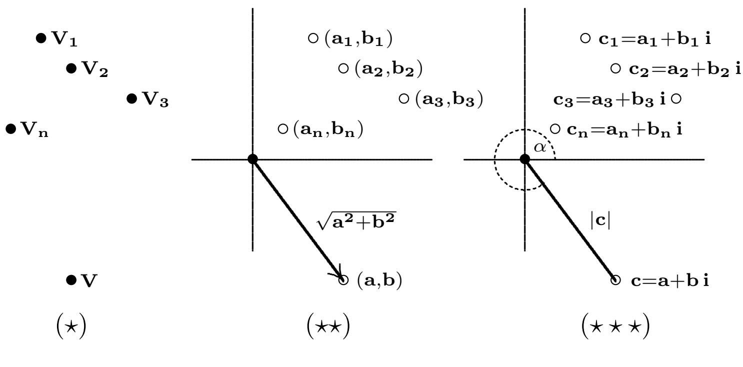

Fig. 1: Bijection between the points of the plane and the complex numbers

The complex plane encourages us to think of complex numbers geometrically. For example, the modulus (the absolute value) of a complex number $c = a + bi$ is equal to $|c| = \sqrt{a^2 + b^2}$ which is the distance of c to the origin 0 and the argument α is the oriented angle well-described in terms of the polar form in every textbook. The geometrical language helps clarify elementary facts such as the description of multiplication in terms of rotations around the origin. The product of two complex numbers is determined by the sum of their arguments and the product of their moduli.

Introduction

The aim of this article is to generalize and understand better a theorem which is often attributed to Napoleon Bonaparte (1769–1821). The first record of the theorem was as a "Gold Medal" competition question at the university of Dublin in October 1820 (see [5]). The result was proposed as a problem in the magazine "The Ladies' Diary" in 1825 with solutions in the following year.

The history of how Napoleon's name is attached to the theorem can be found in [3]). Napoleon was familiar with mathematics, geometry in particular, and knew personally some of the important French mathematicians.

We state the original version of Napoleon's theorem and illustrate the construction in Fig. 2 (see also [2], pp. 209–210).

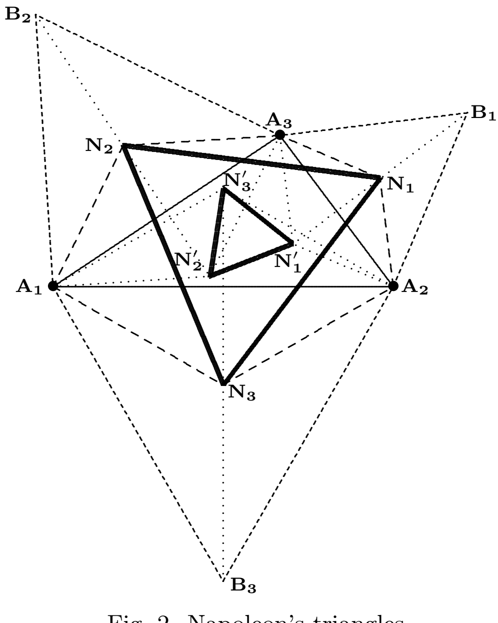

Fig. 2. Napoleon's triangles

Napoleon's theorem. *The centers of the equilateral triangles erected externally on the sides of any triangle, or internally on the sides of any triangle, are the vertices of an equilateral triangle.*

The statement of this theorem is illustrated in Fig. 2: $A_1A_2A_3$ is the given triangle, $A_1A_2B_3, A_2A_3B_1$ and $A_3A_1B_2$ are the equilateral triangles. $N_1N_2N_3$ and $N_1'N_2'N_3'$ are the Napoleon's triangles.

There are many proofs of this theorem, such as [1], the aforementioned [2], and [4]–[10]. In this article we shall show that Napoleon's theorem is a special case of a much more general result which we formulate in the following theorem and illustrate in the first and the second drawings in Fig. 3.

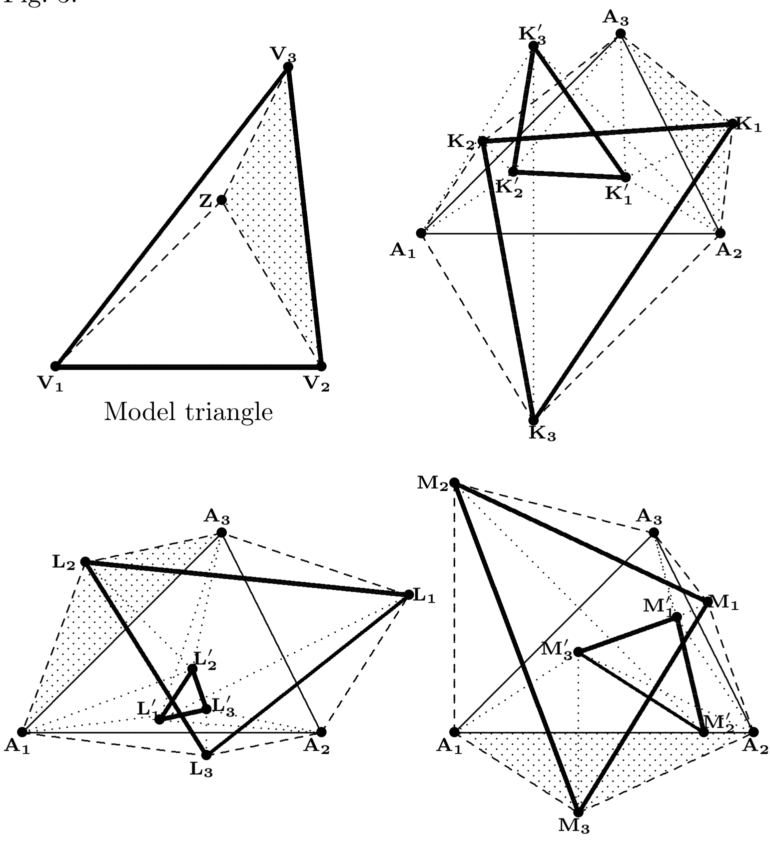

Fig. 3: Model and the six similar triangles

Theorem. *Given two arbitrary triangles $V_1V_2V_3$ and $A_1A_2A_3$ together with a point Z inside the triangle $V_1V_2V_3$,[1)] construct the points K_1, K_2, K_3 and K_1', K_2', K_3' so that the following triangles are similar:*

$$A_1A_2K_3 \sim V_2V_1Z,\ A_2A_3K_1 \sim V_3V_2Z,\ A_3A_1K_2 \sim V_1V_3Z$$

and

$$A_1A_2K_3' \sim V_2V_1Z,\ A_2A_3K_1' \sim V_3V_2Z,\ A_3A_1K_2' \sim V_1V_3Z.$$

Then the triangles $K_1K_2K_3$ and $K_1'K_2'K_3'$ are similar to the triangle $V_1V_2V_3$.

The reader should have no difficulty to check that the following statement follows from the Theorem.

Consequence. *If the triangle $V_1V_2V_3$ is equilateral and the point Z is its center, we get the Napoleon theorem.*

The third and fourth drawings of Fig. 3 illustrate the situations when the triangles V_2V_1Z, V_3V_2Z and V_1V_3Z are drawn on the sides A_1A_2, A_2A_3 and A_3A_1 of the triangle $A_1A_2A_3$ in a different order.

Complex numbers and similarity

To a large degree, the study of triangles is the study of the classes of similar triangles. Rather than study a specific triangle, we consider the sets of all triples of points that are "similar".

We limit ourselves to "oriented" triples of points. This is the way we shall proceed: Triangles ABC and UVW are similar if the side lengths AB, BC and CA are, for certain positive t, t-multiples of the corresponding sides UV, VW and WU. Consequently, the interior angles of triangles ABC and UVW are the same. In this article when we speak of a triple of points we shall always assume that the three points do not lie on the same straight line (otherwise the corresponding triangle would be degenerate). We can thus talk about their orientation, i.e. how their vertices line up on their circumscribed circle. By the positive orientation we shall understand the counterclockwise orientation. The similarity of two triangles will require in this article equal orientation as illustrated in Figs. 4 and 5. We may speak about "direct" similarity: The triangle $C_1C_2C_3$

[1)] The point Z can be located anywhere (including at the vertices of the triangle $V_1V_2V_3$), but no new interesting situations appear, see [12].

and $C_1'C_2'C_3'$ in Fig. 4 are not directly similar. The easily comparable positions $U_1U_2U_3$ and $V_1V_2V_3$ acquired by simple rotations clarify it all.

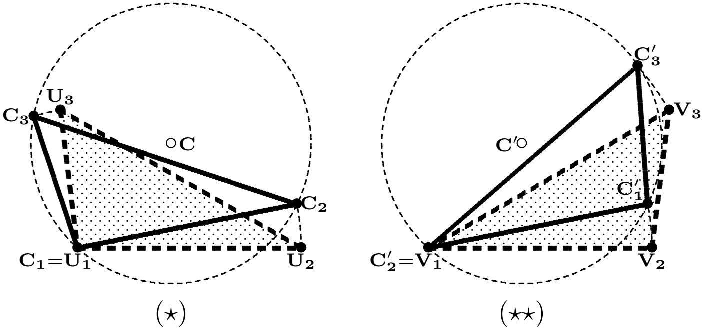

Fig. 4: Orientation of triangles: $C_1C_2C_3 \not\sim C_1'C_2'C_3'$

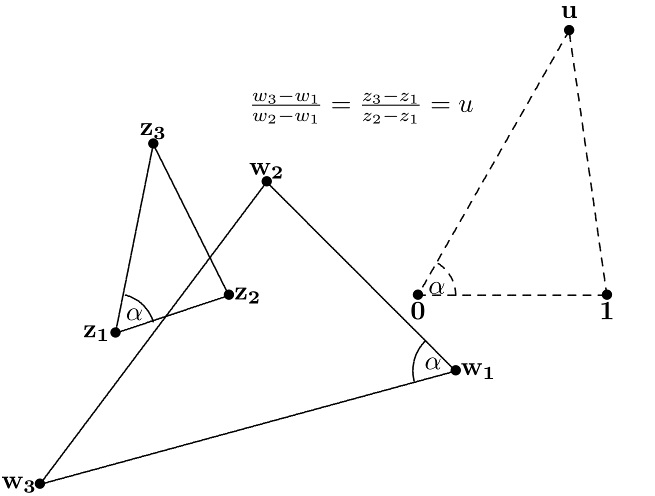

Fig. 5: Triangles $z_1z_2z_3$, $w_1w_2w_3$ and $01u$ are similar

Our modus operandi will be a complex plane, i.e. its points U, V, W, X, Y, Z, … will be identified with the complex numbers u, v, w, x, y, z, … The field of complex numbers will be denoted by $\mathbb{C}$.

The initial statement and one of the motivations of our approach is the following lemma. As already indicated, by similarity we mean direct similarity.

Lemma. *Triangles $Z_1Z_2Z_3$, i.e. $z_1z_2z_3$, and $W_1W_2W_3$, i.e. $w_1w_2w_3$, are similar if and only if*

$$\frac{w_3 - w_1}{w_2 - w_1} = \frac{z_3 - z_1}{z_2 - z_1}.$$ [2]

Proof. The proof is quite elementary. Realizing that the modulus $|u|$ of the number

$$u = \frac{w_3 - w_1}{w_2 - w_1} = \frac{z_3 - z_1}{z_2 - z_1}$$

is the ratio of the sizes of the sides W_1W_3 and W_1W_2, as well as of the sizes of the sides Z_1Z_3 and Z_1Z_2, and the argument of u is the angle α between these adjacent sides, Fig. 5 illustrates this situation very clearly.

In fact, Lemma follows from simple calculations using the concepts of translation and rotation of the complex plane. A detailed proof may be presented in the form of the following elucidating statement:

For each ordered triple $z_1z_2z_3$ there exists exactly one triple of the form $01u$ with $z_1z_2z_3 \sim 01u$. Moreover, if $z_1z_2z_3$ is positively oriented, then $\mathrm{Im}(u) > 0$.

Proof. Applying $t(z) = z - z_1$ shows that $z_1z_2z_3 \sim 0z_2'z_3'$ with $z_2' = z_2 - z_1$ and $z_3' = z_3 - z_1$. Then applying $r(z) = (z_2')^{-1}z$ shows that $0z_2'z_3' \sim 01u$ with $u = (z_2')^{-1}z_3'$. Thus $z_1z_2z_3 \sim 01u$. In addition, if $z_1z_2z_3$ is positively oriented, then so is $01u$ and then u lies in the upper half-plane. Finally, it follows from these calculations that

$$u = \frac{z_3 - z_1}{z_2 - z_1}$$

and so $01u$ is the only such triple.

Proof of the main theorem is based on a repeated application of Lemma. Theorem claims that the triangle $K_1K_2K_3$ is similar to the triangle $V_1V_2V_3$. The construction of the vertices K_1, K_2 and K_3 implies that

$$\frac{a_2 - a_1}{k_3 - a_1} = \frac{v_1 - v_2}{z - v_2}, \quad \frac{a_3 - a_2}{k_1 - a_2} = \frac{v_2 - v_3}{z - v_3}$$

and

$$\frac{a_1 - a_3}{k_2 - a_3} = \frac{v_3 - v_1}{z - v_1},$$

[2] This fact will be exploited and elaborated in a separate publication [13].

and hence

$$k_3 = a_1 + \frac{(a_2 - a_1)(z - v_2)}{v_1 - v_2}, \quad k_1 = a_2 + \frac{(a_3 - a_2)(z - v_3)}{v_2 - v_3}$$

and

$$k_2 = a_3 + \frac{(a_1 - a_3)(z - v_1)}{v_3 - v_1}.$$

From here, $k_3 - k_1$ equals

$$\frac{(a_1-a_2)(v_1-v_2)(v_2-v_3)+(a_2-a_1)(z-v_2)(v_2-v_3)-(a_3-a_2)(z-v_3)(v_1-v_2)}{(v_1-v_2)(v_2-v_3)},$$

and therefore

$$k_3-k_1 = \frac{a_1(v_1 - z)(v_2 - v_3) + a_2(v_2 - z)(v_3 - v_1) + a_3(v_3 - z)(v_1 - v_2)}{(v_1 - v_2)(v_2 - v_3)}.$$

In a similar manner, one can prove that $k_2 - k_1$ equals

$$\frac{(a_3-a_2)(v_3-v_1)(v_2-v_3)+(a_1-a_3)(z-v_1)(v_2-v_3)-(a_3-a_2)(z-v_3)(v_3-v_1)}{(v_2-v_3)(v_3-v_1)},$$

and therefore

$$k_2-k_1 = \frac{a_1(v_1 - z)(v_2 - v_3) + a_2(v_2 - z)(v_3 - v_1) + a_3(v_3 - z)(v_1 - v_2)}{(v_2 - v_3)(v_1 - v_3)}.$$

One can see from here that

$$\frac{k_3 - k_1}{k_2 - k_1} = \frac{(v_2 - v_3)(v_1 - v_3)}{(v_1 - v_2)(v_2 - v_3)} = \frac{v_3 - v_1}{v_2 - v_1},$$

as required.

For the Czech version of this article we refer to [11].

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Slovníček k článku

aforementioned	dříve zmíněný
center of a triangle	těžiště
circumscribed circle	opsaná kružnice
modus operandi	(pracovní) postup
similar triangles	podobné trojúhelníky