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OPTIMALITY CONDITIONS FOR INTERVAL-VALUED VECTOR EQUILIBRIUM PROBLEMS

Ashish Kumar Prasad, Julie Khatri and Izhar Ahmad

In the article, one formulates Fritz John type and Karush–Kuhn–Tucker type necessary conditions for an interval-valued vector equilibrium problem having a locally LU-efficient solution, where convexificators demonstrate the solutions that are regular. Sufficient conditions for a locally weak LU-efficient solution have been entrenched by imposing appropriate assumptions along with generalized convexity. Some applications are presented for a constrained interval-valued vector variational inequality and a constrained interval-valued vector optimization problem.

Keywords: interval-valued vector equilibrium problem, locally LU-efficient solution, optimality, convexificators

Classification: 49J52, 91B50, 90C46

1. INTRODUCTION

The vector problems provide an outstanding base to deal with the problems that include several conflicting objectives. However, decision-makers find it imperative to identify alternative solutions according to multiple criteria and their significance level. Mathematical programming problems may contain uncertainty, which is reflected when coefficients are allowed to vary in closed intervals. Interval-valued optimization problems provide a way to overcome the difficulties in the solution procedure, and there is no need to ignore or gloss over the fact.

Wu [22] formulated Karush–Kuhn–Tucker type optimality conditions for a Wolfe type nonlinear problem, assuming the functions involved in the objective function and constraints are differentiable, and derived duality theorems in weak as well as strong sense. Bhurjee and Panda ([1, 2]) addressed the interrelation between multiobjective interval fractional formulations containing parameters running over some intervals and general optimization problems having fixed real parameters and portrayed the sufficiency criteria along with duality theorems. Jayswal et al. [10] proposed sufficient optimality conditions for functions satisfying generalized invexity, and the same is used to derive weak, strong, and strict duality theorems for Mond–Weir and Wolfe-type dual problems. Ioffe [9] demonstrated a new method to relate problems with and without constraints using the reduction theorem.

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Differentiation of convex functions can be seen as equivalent to the linearization of a function. The perception of subgradients and subdifferentials made it possible to approximate nonsmooth convex functions. The notion of convexificators has been shown to be effective in determining the optimality conditions and duality results in nonsmooth optimization problems. Demyanov [4] was the first who coined the term "convex compact convexificator". The closed nonconvex convexificators for elaborated real-valued continuous functions as well as the approximated Jacobian of continuous vector-valued functions were proposed by Jeyakumar and Luc ([12, 13]). A convexificator is a generalization of certain well-known subdifferential concepts, like the subdifferential of Clarke [3], Mordukhovich and Shao [19], Luu [14], etc. Several researchers have devised optimality conditions for efficiency under convexificators. Recently, Luu ([15, 16]) had introduced Lagrange multipliers rules for efficiency via convexificators. Jayswal et al. [11] set up the sufficiency criteria along with duality results for Mond Weir and Wolfe type duals introduced by Jeyakumar and Luc [12].

In the past few years, the vector equilibrium problems gained a lot of attention. These contain problems like vector variational inequality, vector optimization problems, and other specific cases. Optimality criteria for vector inequality and vector equilibrium problems have been formulated by many researchers, like Gong ([6, 7, 8]), Luu and Hang ([17, 18]), etc. Giannessi et al. [5] established the methodology of vector problems along with the scheme to handle variational inequalities. Morgan and Romaniello [20] proposed Karush–Kuhn–Tucker type conditions for weak vector generalized quasi-variational inequalities by using the scalarization method. Ward and Lee [21] presented the equivalence relations between vector optimization problems and vector variational inequalities.

This article portrays how to implement Fritz John type and Karush–Kuhn–Tucker type conditions to find the locally LU-efficient solution of interval-valued mathematical programming problems using convexificators, which are regular in terms of Ioffe [9]. The article proposes sufficient conditions under appropriate assumptions. Section 2 recalls some basic terminologies and definitions. Section 3 deals with Fritz John necessary conditions for a locally LU-efficient solution of the considered problem, whereas Section 4 constructs Karush–Kuhn–Tucker type conditions for a locally LU-efficient solution with the help of Mangasarian–Fromovitz constraint qualification and stronger Mangasarian–Fromovitz constraint qualification investigated by Luu [14]. Under suitable assumptions on generalized convexity, sufficiency optimality theorems for a locally weak LU-efficient solution have been derived in Section 5, followed by some applications of constrained interval-valued problems in Section 6.

2. PRELIMINARIES

This section begins with the following convention for inequalities and equalities, which are utilized in the later part of the paper. For any $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$, where $\mathbb{R}^n$ symbolizes n-dimensional Euclidean space, we have

(i) $u = v \Leftrightarrow u_i = v_i, \forall i = 1, \ldots, n;$

(ii) $u > v \Leftrightarrow u_i > v_i, \forall i = 1, \ldots, n;$

(*iii*) $u \geqq v \Leftrightarrow u_i \geqq v_i, \forall i = 1, \ldots, n$;

(*iv*) $u \geq v \Leftrightarrow u \geqq v, u \neq v$.

Let $\mathbb{X}^*$ denotes the topological dual space of a real Banach space $\mathbb{X}$. The set of all closed and bounded intervals in $\mathbb{R}$ is denoted by $\mathcal{I}$. For any $\Theta_1 = [\alpha^L, \alpha^U]$ and $\Theta_2 = [\beta^L, \beta^U]$ in $\mathcal{I}$, we define

(*i*) $\Theta_1 + \Theta_2 = [\alpha^L + \beta^L, \alpha^U + \beta^U]$,

(*ii*) $-\Theta_1 = [-\alpha^U, -\alpha^L]$,

(*iii*) $\Theta_1 - \Theta_2 = \{\Theta_1 + (-\Theta_2)\} = [\alpha^L - \beta^U, \alpha^U - \beta^L]$,

(*iv*) $m + \Theta_1 = \{m + \alpha : \alpha \in \Theta_1\} = [m + \alpha^L, m + \alpha^U]$, where m is any real number,

(*v*) $m\Theta_1 = \begin{cases} [m\alpha^L, m\alpha^U], & m \geq 0, \\ [m\alpha^U, m\alpha^L], & m < 0, \end{cases}$ where m is any real number.

If we take $\alpha^L = \alpha^U = \alpha$, then the interval Θ_1 reduces to a real number. A function $\hat{F}$ with domain $\mathbb{X}$ and range $\mathcal{I}$ is known as an interval-valued function. For any member y in $\mathbb{X}$, we define $\hat{F}(y) = [F^L(y), F^U(y)]$, where $F^L(y)$ and $F^U(y)$ are functions defined on $\mathbb{X}$ satisfying $F^L(y) \leqq F^U(y)$. We use the following notation in the rest of the paper: $[F(y)]^L = F^L(y)$ and $[F(y)]^U = F^U(y)$.

We define the interval symbols as per the following scheme:

$$\Theta_1 \leqq_{\mathrm{LU}} \Theta_2 \ \text{ provided } \ \alpha^L \leqq \beta^L \text{ and } \alpha^U \leqq \beta^U,$$

$$\Theta_1 <_{\mathrm{LU}} \Theta_2 \text{ signify } \ \Theta_1 \leqq_{\mathrm{LU}} \Theta_2, \ \Theta_1 \neq \Theta_2.$$

Equivalently, $\Theta_1 <_{\mathrm{LU}} \Theta_2$ if any one of the following three conditions are satisfied:

$\alpha^L < \beta^L, \ \ \alpha^U < \beta^U$;
$\alpha^L \leqq \beta^L, \ \ \alpha^U < \beta^U$;
$\alpha^L < \beta^L, \ \ \alpha^U \leqq \beta^U$.

Consider the following constrained interval-valued mathematical programming problem:

$$\text{(IP)} \quad \text{minimize} \quad \phi(y) = [\phi^L(y), \phi^U(y)]$$
$$\text{subject to } y \in M_1 := \{y \in \mathbb{C} : g_i(y) \leqq 0 \text{ for } i \text{ in } \mathfrak{I}_n \text{ and } h(y) = 0\},$$

where,

(i) the function $\phi : \mathbb{X} \to \mathcal{I}$ is defined so that $\phi(y)$ becomes a bounded and closed interval in $\mathbb{R}$ given by $\phi(y) = [\phi^L(y), \phi^U(y)]$.

(ii) $\mathbb{C} \subseteq \mathbb{X}$ is any closed subset.

(iii) g is a mapping from $\mathbb{X}$ to $\mathbb{R}^n$ and h a mapping from $\mathbb{X}$ to $\mathbb{R}^l$, where g_i ($i \in \mathfrak{I}_n := \{1, \ldots, n\}$) and h_j ($j \in \mathfrak{L}_l := \{1, \ldots, l\}$) are functions from $\mathbb{X}$ to $\mathbb{R} \cup \{-\infty, +\infty\}$.

Definition 2.1. (Ioffe [9]) A point $\bar{y}$ is said to be a regular point for h corresponding to $\mathbb{C}$ if for all $y \in \mathbb{C} \cap B(\bar{y};\delta)$, there exist $\eta > 0$ and $\delta > 0$ satisfying

$$d_Q(y) \leqq \eta\|h(y) - h(\bar{y})\|,$$

where $Q := \{y \in \mathbb{C} : h(y) = h(\bar{y})\}$, $d_Q(y)$ being the distance from y to Q, $B(\bar{y};\delta)$ is an open ball with radius δ and center $\bar{y}$.

Let us now devise necessary optimality conditions for the considered problem, which is based on the reduction theorem due to Ioffe [9].

Proposition 2.2. Let $\bar{y}$ be a regular point for h corresponding to $\mathbb{C}$. Moreover, suppose $\phi^L, \phi^U, h_1, \ldots, h_l$ are locally Lipschitz at $\bar{y}$ and let the mapping $\widetilde{\phi} : X \to \mathbb{R}^2$ be defined by $\widetilde{\phi} = (\phi^L, \phi^U)$. Corresponding to the (isolated) local solution $\bar{y}$ to the problem (IP), there exists $r > 0$ for which the function

$$M_r{}^1(y) := \max\{\widetilde{\phi}(y) - \widetilde{\phi}(\bar{y}), \max_{i\in\mathfrak{I}_n(\bar{y})} g_i(y)\} \; + \; r(\|h(y))\| + d_{\mathbb{C}}(y)),$$

where $\mathfrak{I}_n(\bar{y})$ is the set of all indices i of $\mathfrak{I}_n$ for which $g_i(\bar{y}) = 0$, attains a local minimum at $\bar{y}$.

Conversely, if $M_r{}^1(y)$ attains a strict local minimum at a point $\bar{y}$ for suitable values of r, then $\bar{y}$ also becomes an isolated local solution to the problem (IP).

Now, we recall the notion of a convexificator introduced by Jeyakumar and Luc [12].

Definition 2.3. A function $f : \mathbb{X} \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$) has lower and upper Dini directional derivatives at a point $\bar{y} \in \mathbb{X}$ along the direction $\vartheta \in \mathbb{X}$ provided the limit given by

$$f^-(\bar{y};\vartheta) = \liminf_{t\downarrow 0} \frac{f(\bar{y} + t\vartheta) - f(\bar{y})}{t},$$

$$f^+(\bar{y};\vartheta) = \limsup_{t\downarrow 0} \frac{f(\bar{y} + t\vartheta) - f(\bar{y})}{t},$$

exists. Moreover, if $f^+(\bar{y};\vartheta) = f^-(\bar{y};\vartheta)$, then $f'(\bar{y};\vartheta)$ is symbolized as their common value, which is known as the Dini derivative of f at $\bar{y}$ along the specified direction ϑ.

A function f is known as Dini differentiable at a point $\bar{y}$ if and only if its Dini derivatives at point $\bar{y}$ exist along all directions.

Definition 2.4. An extended real function $f : \mathbb{X} \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$) has an upper convexificator $\partial^* f(\bar{y})$ at $\bar{y}$ provided $\partial^* f(\bar{y})$ is weakly* closed subset of $\mathbb{X}^*$ and $f^-(\bar{y};\vartheta) \leqq \sup_{\xi\in\partial^* f(\bar{y})}\langle\xi,\vartheta\rangle, \ \forall\, \vartheta \in \mathbb{X}$.

In the same way, an extended real function $f : \mathbb{X} \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$) has a lower convexificator $\partial_* f(\bar{y})$ at $\bar{y}$ provided $\partial_* f(\bar{y})$ is weakly* closed subset of $\mathbb{X}^*$ and $f^+(\bar{y};\vartheta) \geqq \inf_{\xi\in\partial_* f(\bar{y})}\langle\xi,\vartheta\rangle, \ \forall\, \vartheta \in \mathbb{X}$.

A weakly* closed subset of $\mathbb{X}^*$ is called the convexificator of f and denoted by $\partial f(\bar{y})$, whenever $\partial f(\bar{y}) = \partial^* f(\bar{y}) = \partial_* f(\bar{y})$.

Definition 2.5. An extended real function $f : \mathbb{X} \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$) has an upper semi-regular convexificator $\partial^* f(\bar{y})$ at $\bar{y}$ provides $\partial^* f(\bar{y})$ is weakly* closed subset and $f^+(\bar{y}; \vartheta) \leqq \sup_{\xi \in \partial^* f(\bar{y})} \langle \xi, \vartheta \rangle, \ \forall\, \vartheta \in \mathbb{X}$.

In the same way, an extended real function $f : \mathbb{X} \to \bar{\mathbb{R}}$ ($\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$) has a lower semi-regular convexificator $\partial_* f(\bar{y})$ at $\bar{y}$ provides $\partial_* f(\bar{y})$ is weakly* closed subset and $f^-(\bar{y}; \vartheta) \geqq \inf_{\xi \in \partial_* f(\bar{y})} \langle \xi, \vartheta \rangle, \ \forall\, \vartheta \in \mathbb{X}$.

Moreover, an extended real function f has an upper regular convexificator $\partial^* f(\bar{y})$ at $\bar{y}$ provides $\partial^* f(\bar{y})$ is weakly* closed subset and $f^+(\bar{y}; \vartheta) = \sup_{\xi \in \partial^* f(\bar{y})} \langle \xi, \vartheta \rangle, \ \forall\, \vartheta \in \mathbb{X}$.

In the same way, an extended real function f has a lower regular convexificator $\partial_* f(\bar{y})$ at $\bar{y}$ provides $\partial_* f(\bar{y})$ is weakly* closed subset and $f^-(\bar{y}; \vartheta) = \inf_{\xi \in \partial_* f(\bar{y})} \langle \xi, \vartheta \rangle, \ \forall\, \vartheta \in \mathbb{X}$.

Proposition 2.6. [12] Let us assume that the functions $\phi^L, \phi^U : \mathbb{X} \to \mathbb{R}$ attain upper convexificators $\partial^* \phi^L(\bar{y})$ and $\partial^* \phi^U(\bar{y})$ at point $\bar{y} \in \mathbb{X}$, respectively. If ϕ^L and ϕ^U attain their minimum value at a point $\bar{y}$, then

$$0 \in cl\left(\text{conv } \partial^* \phi^L(\bar{y}) + \text{conv } \partial^* \phi^U(\bar{y})\right), \tag{1}$$

where cl denotes the weak* closure and conv denotes the convex hull.

Example 2.7. Let $\phi : \mathbb{X} \to \mathcal{I}$ be an interval-valued function such that $\phi = [\phi^L, \phi^U]$, where the functions $\phi^L, \phi^U : \mathbb{X} \to \mathbb{R}$ are defined as

$$\phi^L(y) = \begin{cases} 2y^2 + 3, & y \geqq 0 \\ 5y + 3, & y < 0 \end{cases}$$

$$\phi^U(y) = \begin{cases} 6y^3 + 9y + 5, & y \geqq 0 \\ 4y^2 + 7y + 5, & y < 0 \end{cases}$$

A simple calculation will give

$$\phi^{L^+}(0; v) = \phi^{L^-}(0; v) = \begin{cases} 0, & v \geqq 0 \\ 5v, & v < 0 \end{cases}$$

$$\phi^{U^+}(0; v) = \phi^{U^-}(0; v) = \begin{cases} 9v, & v \geqq 0 \\ 7v, & v < 0 \end{cases}$$

The sets {0,5} and {9,7} are upper semi-regular convexificators of ϕ^L and ϕ^U, respectively, at $\bar{y} = 0$.

Definition 2.8. (Clarke [3]) The Clarke directional derivative of $f : \mathbb{X} \to \mathbb{R}$ at point $\bar{y}$ along the direction ϑ is defined by

$$f^\circ(\bar{y}; \vartheta) := \limsup_{\substack{y \to \bar{y} \\ t \downarrow 0}} \frac{f(y + t\vartheta) - f(y)}{t}.$$

The Clarke subdifferential of f at $\bar{y}$ can be expressed mathematically as

$$\partial f_\circ(\bar{y}) := \left\{ \xi \in \mathbb{X}^* : \langle \xi, \vartheta \rangle \leqq f^\circ(\bar{y}; \vartheta), \ \forall\, \vartheta \in \mathbb{X} \right\}.$$

Remark 2.9. If f is a strictly differentiable function, then the Clarke subdifferential of f becomes the strict derivative. If f is locally Lipschitz, then the Clarke subdifferential transforms to the convexificator of f at point $\bar{y}$. A locally Lipschitz function f is said to be regular at point $\bar{y}$, if there exist $f'(\bar{y};\vartheta)$ for each $\vartheta \in \mathbb{X}$ having a value the same as that of $f^{\circ}(\bar{y};\vartheta)$. For a function f of this type, Clarke subdifferential $\partial f_{\circ}(\bar{y})$ is an upper regular convexificator, and the convexificator mapping ∂f is locally bounded at a point $\bar{y}$. Furthermore, if dim $\mathbb{X} < \infty$, then the mapping ∂f becomes upper semicontinuous at a point $\bar{y}$.
For a set $\mathbb{C}$, the Clarke tangent cone $\mathcal{T}_{\mathbb{C}}(\bar{y})$ at $\bar{y} \in \mathbb{C}$ can be defined by

$$\begin{aligned}\mathcal{T}_{\mathbb{C}}(\bar{y}) := \big\{\vartheta \in \mathbb{X} : \forall\, y_n \in \mathbb{C},\ y_n \to \bar{y},\ \forall\, t_n \downarrow 0,\ \exists\, \vartheta_n \to \vartheta \\ \text{with } y_n + t_n\vartheta_n \in \mathbb{C},\ \forall\, n\big\},\end{aligned}$$

whereas for a set $\mathbb{C}$, the Clarke normal cone $\mathcal{N}_{\mathbb{C}}(\bar{y})$ at $\bar{y}$ can be defined by

$$\mathcal{N}_{\mathbb{C}}(\bar{y}) \ := \big\{\xi \in \mathbb{X}^* : \langle \xi, \vartheta\rangle \ \leqq 0,\ \forall\, \vartheta \in \mathcal{T}_{\mathbb{C}}(\bar{y})\big\}.$$

Now, we construct the interval-valued vector equilibrium problem (IEP), the interval-valued vector variational inequality (IVI), and the interval-valued vector optimization problem (IOP). The interval-valued vector equilibrium problem (IEP) is an important topic in nonlinear analysis. It is a generalized form of the scalar interval-valued problem (IP) and provides a unified mathematical framework that includes the (IVI) and (IOP) as special cases.

The interval-valued vector equilibrium problem (IEP):
Let $M \subset \mathbb{X}$ be a nonempty subset, and Φ_k be a function from $\mathbb{X} \times \mathbb{X} \to \mathcal{I}$, $\Phi_k = [\Phi_k^L, \Phi_k^U]$ being an interval-valued function $\forall\, k \in \mathfrak{J}_m := \{1, \ldots, m\}$. Let the mapping $\Phi_k^L, \Phi_k^U : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$. In short, we write $\Phi^L = (\Phi_1^L, \ldots, \Phi_m^L)$ and $\Phi^U = (\Phi_1^U, \ldots, \Phi_m^U)$. Suppose $\mathbb{P} \subseteq \mathbb{R}^m$ is a closed, pointed, and convex cone. Let us take the interval-valued vector equilibrium problem (IEP): find $\bar{y} \in M$ satisfying

$$\Phi^L(\bar{y}, z),\ \Phi^U(\bar{y}, z) \notin -\mathbb{P} \setminus \{0\},\ \forall\, z \in M. \tag{2}$$

A solution $\bar{y}$ of interval-valued equilibrium problem (IEP) is called a locally LU-efficient if and only if there exists $\delta > 0$ which satisfies the condition (2) for every $z \in M \cap \mathbb{B}(\bar{y};\delta)$. If int$\mathbb{P} \neq \emptyset$, a point $\bar{y}$ of interval-valued equilibrium problem (IEP) is called a locally weak LU-efficient iff there exists $\delta > 0$ so that for all $z \in M \ \cap\ \mathbb{B}(\bar{y};\delta)$, we have

$$\Phi^L(\bar{y}, z),\ \Phi^U(\bar{y}, z) \notin -\text{int}\mathbb{P}.$$

Now, let us denote $\Phi_{\bar{y}}(z) := \Phi(\bar{y}, z),\ \Phi_{k,\bar{y}}(z) := \Phi_k(\bar{y}, z),\ \forall\, k \in \mathfrak{J}_m := \{1, \ldots, m\}$ and suppose $\Phi_{\bar{y}}(\bar{y}) = 0$. If $\mathbb{P} = \mathbb{R}_+^m$, a solution $\bar{y} \in M$ of the interval-valued vector equilibrium problem (IEP) is said to be a locally LU-efficient (locally weak LU-efficient) solution iff $\exists\, \delta > 0$ so that there does not exist any $z \in M \ \cap\ \mathbb{B}(\bar{y};\delta)$ which satisfies

$$\Phi_{k,\bar{y}}(z) \leqq_{\text{LU}} 0,\ \forall\, k \in \mathfrak{J}_m,$$

$$\Phi_{s,\bar{y}}(z) <_{\text{LU}} 0 \text{ for at least one index } s \in \mathfrak{J}_m,$$

$$\big(\Phi_{k,\bar{y}}(z) <_{\text{LU}} 0 \ \ \forall\, k \in \mathfrak{J}_m\big).$$

The interval-valued vector variational inequality (IVI):
Let $\mathbb{T} : \mathbb{X} \to \mathcal{I}$ be an interval-valued function, where $\mathbb{T} = [\mathbb{T}^L, \mathbb{T}^U]$. Let $\mathfrak{L}_l(\mathbb{X}; \mathbb{R}^m)$ denotes the space of all the continuous linear mappings from $\mathbb{X} \to \mathbb{R}^m$ and $\mathbb{T}^L,\ \mathbb{T}^U : \mathbb{X} \to \mathfrak{L}_l(\mathbb{X}; \mathbb{R}^m)$ be any function. Consider the interval-valued vector variational inequality (IVI), which is the particular case of the interval-valued vector equilibrium problem (IEP):

$$\text{(IVI) Find } y \in M \text{ so that } \mathbb{T}^L(y)(z-y),\ \mathbb{T}^U(y)(z-y) \notin -\mathbb{P} \setminus \{0\},\ \forall\, z \in M. \tag{3}$$

A solution $\bar{y}$ of an interval-valued vector variational inequality (IVI) is called locally LU-efficient if and only if there exists $\delta > 0$ which satisfies the condition (3) for each $z \in M \cap \mathbb{B}(\bar{y}; \delta)$. If $\text{int}\mathbb{P} \neq \emptyset$, then a solution $\bar{y}$ of the interval-valued vector variational inequality (IVI) is said to be locally weak LU-efficient iff there exists $\delta > 0$ so that

$$\mathbb{T}^L(\bar{y})(z-\bar{y}),\ \mathbb{T}^U(\bar{y})(z-\bar{y}) \notin -\text{int}\mathbb{P},\ \forall\, z \in M \cap \mathbb{B}(\bar{y}; \delta).$$

Example 2.10. (Interval Cournot–Nash Duopoly as an IVI) Consider a market with two firms choosing production levels $y = (y_1, y_2) \in \mathbb{R}^2$. The inverse demand function is given by

$$p(Q) = a - bQ, \qquad Q = y_1 + y_2,$$

where the demand parameters are uncertain within intervals

$$a \in [a^L, a^U], \qquad b \in [b^L, b^U],$$

with $0 < a^L \leq a^U,\ 0 < b^L \leq b^U$. Each firm i faces linear production cost

$$C_i(y_i) = c_i y_i, \qquad c_i \in [c_i^L, c_i^U], \quad i = 1, 2.$$

The feasible set is given by

$$M = \{\, y \in \mathbb{R}^2 :\ 0 \leq y_i \leq \bar{y}_i,\ i = 1, 2 \,\},$$

which is nonempty, closed, and convex. Let $P = \mathbb{R}^2_+$ be the ordering cone.
For interval operator with fixed parameters, the standard Cournot mapping is

$$\hat{F}_i(y) = c_i - a + b\,(2y_i + y_j), \qquad i \neq j,\ i, j \in \{1, 2\}.$$

That is,

$$F_i^L(y) = c_i^L - a^U + b^L(2y_i + y_j), \qquad F_i^U(y) = c_i^U - a^L + b^U(2y_i + y_j).$$

We define the interval operator by $\mathbb{T}(y) = [\mathbb{T}^L(y), \mathbb{T}^U(y)]$, with linear forms

$$\mathbb{T}^L(y)(v) = \langle F^L(y), v \rangle, \qquad \mathbb{T}^U(y)(v) = \langle F^U(y), v \rangle, \quad v \in \mathbb{R}^2.$$

The problem involving interval variational inequality can be stated as follows:
Find $\bar{y} \in M$ such that for all $z \in M$,

$$\mathbb{T}^L(\bar{y})(z-\bar{y}),\ \mathbb{T}^U(\bar{y})(z-\bar{y})\ \notin\ -P \setminus \{0\}.$$

Equivalently, there is no $z \in M$ with

$$\mathbb{T}(\bar{y})(z-\bar{y}) \leq_{LU} 0.$$

This example demonstrates how an interval variational inequality problem can be used to formulate the Cournot–Nash duopoly with interval-valued parameters.

Note: In case $\mathbb{P} = \mathbb{R}^m_+$ the definition of locally LU-efficient solution (locally weak LU-efficient solution if $\mathbb{P} = \mathbb{R}^m_{++}$) is of the form: There is no any $z \in M \cap \mathbb{B}(\bar{y};\delta)$ such that

$$\mathbb{T}_k(\bar{y})(z-\bar{y}) \leqq_{\mathrm{LU}} 0, \ \forall\ k \in \mathfrak{J}_m,$$
$$\mathbb{T}_s(\bar{y})(z-\bar{y}) <_{\mathrm{LU}} 0 \text{ for at least one index } s \in \mathfrak{J}_m,$$
$$\big(\mathbb{T}_k(\bar{y})(z-\bar{y}) <_{\mathrm{LU}} 0 \ \ \forall\ k \in \mathfrak{J}_m\big),$$

where $\mathbb{T}_k(\bar{y}) = [\mathbb{T}^L_k(\bar{y}), \mathbb{T}^U_k(\bar{y})],\ \mathbb{T}^L_k(\bar{y}), \mathbb{T}^U_k(\bar{y}) : \mathbb{X} \to \mathbb{R}\ k \in \mathfrak{J}_m,\ \mathbb{R}^m_{++} = \mathrm{int}\mathbb{R}^m_+$.

The interval-valued vector optimization problem (IOP):
The problem (IEP) transforms to the interval-valued vector optimization problem (IOP), if we define $\Phi^L(y,z) = \phi^L(z) - \phi^L(y)$ and $\Phi^U(y,z) = \phi^U(z) - \phi^U(y)$ $(y, z \in M)$, then

$$\min\{[\phi^L(y), \phi^U(y)] :\ y \in M\}.$$

We call $\bar{y}$ to be a locally LU-efficient solution for $\mathbb{P} = \mathbb{R}^m_+$ (a locally weak LU-efficient solution for $\mathbb{P} = \mathbb{R}^m_{++}$) if $\nexists\ y \in M\ \cap\ \mathbb{B}(\bar{y};\delta)$ satisfying

$$\phi_k(y) \leqq_{\mathrm{LU}}\ \ \phi_k(\bar{y}),\ \forall\ k \in \mathfrak{J}_m,$$
$$\phi_s(y) <_{\mathrm{LU}}\ \ \phi_s(\bar{y}) \text{ for at least one } s \in \mathfrak{J}_m,$$
$$\big(\phi_k(y) <_{\mathrm{LU}}\ \ \phi_k(\bar{y})\ \forall\ k \in \mathfrak{J}_m\big).$$

Next, we consider the problem (IEP) together with the feasible solution set $M = M_1$, which we denote as the constrained interval-valued vector equilibrium problem (CIEP). Likewise, adding the feasible set $M = M_1$ to the problems (IVI) and (IOP), we obtain the constrained interval-valued vector variational inequality (CIVI) and the constrained interval-valued vector optimization problem (CIOP), respectively, which will be studied in the following sections.

3. FRITZ JOHN TYPE NECESSARY CONDITIONS

In this section, we derive the necessary optimality conditions for (CIEP). To obtain the necessary conditions for a locally LU-efficient solution at the point $\bar{y}$ of (CIEP), we introduce the following assumptions.

Assumption 3.1.

(i) For $s \in \mathfrak{J}_m$, the functions $\Phi^L_{s,\bar{y}}(.), \Phi^U_{s,\bar{y}}(.)$, and $h_1, \ldots, h_l$ are locally Lipschitz at a point $\bar{y}$. Moreover, $\Phi^L_{k,\bar{y}}(.), \Phi^U_{k,\bar{y}}(.)$ $(s \neq k \in \mathfrak{J}_m), g_i,\ i \in \mathfrak{J}_n(\bar{y})$ are continuous, and $\mathbb{C}$ is convex.

(ii) $\Phi^L_{k,\bar{y}}(.)$, $\Phi^U_{k,\bar{y}}(.)$ and g_i admit an upper convexificator $\partial^*\Phi^L_{k,\bar{y}}(\bar{y})$, $\partial^*\Phi^U_{k,\bar{y}}(\bar{y})$ $(s \neq k \in \mathfrak{I}_m)$ and $\partial^* g_i(\bar{y})$, i in $\mathfrak{I}_n(\bar{y})$ at a point $\bar{y}$.

(iii) The functions $|h_j|$, $j \in \mathfrak{L}_l$ are regular at a point $\bar{y}$ in the sense of Clarke [3].

Theorem 3.2. Let $\bar{y}$ be a locally LU-efficient solution to (CIEP) and regular point corresponding to $\mathbb{C}$ satisfying the conditions $\Phi^L_{\bar{y}}(\bar{y}) = 0$ and $\Phi^U_{\bar{y}}(\bar{y}) = 0$. Further, if it satisfies the conditions specified in Assumption 3.1, then there exist $\bar{\alpha}^L_k$, $\bar{\alpha}^U_k \geqq 0$ (for all k in $\mathfrak{I}_m$), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\bar{\tau}_j \geqq 0$ (for all j in $\mathfrak{L}_l$), $\sum_{j\in\mathfrak{L}_l} \bar{\tau}_j = 1$, $\bar{r} > 0$ satisfying $\sum_{k\in\mathfrak{I}_m}\left(\bar{\alpha}^L_k + \bar{\alpha}^U_k\right) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i = 1$ along with

$$0 \in \text{cl}\Bigg\{ \sum_{k\in\mathfrak{I}_m} \Big(\bar{\alpha}^L_k \text{ conv } \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}^U_k \text{ conv } \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y})$$
$$+\bar{r}\Bigg(\sum_{j\in\mathfrak{L}_l} \bar{\tau}_j \text{ conv } \partial^*(|h_j(\bar{y})|) + \partial_\circ d_{\mathbb{C}}(\bar{y})\Bigg)\Bigg\}, \tag{4}$$

where $\partial_\circ d_{\mathbb{C}}(\bar{y})$ is the subdifferential of $d_{\mathbb{C}}(y)$ at $\bar{y}$ as defined by Clarke [3].

Proof. Since $\bar{y}$ is locally LU-efficient solution to (CIEP), and $\Phi^L_{\bar{y}}(\bar{y}) = 0$ and $\Phi^U_{\bar{y}}(\bar{y}) = 0$, the solution $\bar{y}$ is also locally LU-efficient to the interval-valued vector optimization problem (MP):

$$\begin{aligned} &\text{minimize} \quad [\Phi^L_{\bar{y}}(y), \Phi^U_{\bar{y}}(y)] \\ &\text{subject to} \;\; y \in M_1 := \{y \in \mathbb{C} : g_i(y) \leqq 0, \text{for } i \text{ in } \mathfrak{I}_n, h(y) = 0\}. \end{aligned}$$

So, the solution $\bar{y}$ is locally LU-efficient for the following vector programming problem:

$$\min\Bigg\{ \Phi^L_{s,\bar{y}}(y), \Phi^U_{s,\bar{y}}(y) : \Phi^L_{k,\bar{y}}(y) \leqq 0, \Phi^U_{k,\bar{y}}(y) \leqq 0, \;\; s \neq k \in \mathfrak{I}_m,$$
$$g_i(y) \leqq 0, \;\; i \in \mathfrak{I}_n, \;\; h(y) = 0, \;\; y \in \mathbb{C}\Bigg\}.$$

Using Proposition 2.2, we get $\bar{r} > 0$ where $\bar{y}$ is a local minimum of $M_{\bar{r}}(y)$ over $\mathbb{X}$:

$$M_{\bar{r}}(y) := \max\Bigg\{ \Phi^L_{s,\bar{y}}(y), \Phi^U_{s,\bar{y}}(y), \max_{s\neq k\in\mathfrak{I}_m} \Phi^L_{k,\bar{y}}(y), \max_{s\neq k\in\mathfrak{I}_m} \Phi^U_{k,\bar{y}}(y),$$
$$\max_{i\in\mathfrak{I}_n(\bar{y})} g_i(y)\Bigg\} \; + \; \bar{r}\Big(\|h(y))\| + d_{\mathbb{C}}(y)\Big), \tag{5}$$

where norm can be defined by $\|h(y))\| = \max_{1\leqq j\leqq l}|h_j(y)|$. Therefore, using Proposition 2.6, $\partial^* M_r(\bar{y})$ can be taken as the upper convexificator of M_r at a point $\bar{y}$, and hence

$$0 \in \text{cl conv } \partial^* M_{\bar{r}}(\bar{y}). \tag{6}$$

Since $d_{\mathbb{C}}(y)$ is a Lipschitz function on $\mathbb{X}$, therefore we can take the Clarke subdifferential $\partial_\circ d_{\mathbb{C}}(\bar{y})$ as a convexificator for $d_{\mathbb{C}}(y)$ at a point $\bar{y}$. Let the function $\max_{i\in\mathfrak{I}_n(\bar{y})} \; g_i(y)$,

$\max_{s\neq k\in\mathfrak{I}_m}\Phi^L_{k,\bar{y}}(y)$ and $\max_{s\neq k\in\mathfrak{I}_m}\Phi^U_{k,\bar{y}}(y)$ admit an upper convexificator at a point $\bar{y}$ given by

$$\bigcup_{i\in\mathfrak{I}_n(\bar{y})}\partial^* g_i(\bar{y})\ ,\ \bigcup_{s\neq k\in\mathfrak{I}_m}\partial^*\Phi^L_{k,\bar{y}}(\bar{y}) \text{ and } \bigcup_{s\neq k\in\mathfrak{I}_m}\partial^*\Phi^U_{k,\bar{y}}(\bar{y}),$$

respectively. Therefore, the function

$$\max\left\{\Phi^L_{s,\bar{y}}(y),\Phi^U_{s,\bar{y}}(y),\max_{s\neq k\in\mathfrak{I}_m}\Phi^L_{k,\bar{y}}(y),\max_{s\neq k\in\mathfrak{I}_m}\Phi^U_{k,\bar{y}}(y),\max_{i\in\mathfrak{I}_n(\bar{y})}g_i(y)\right\},$$

has an upper convexificator at a point $\bar{y}$ specified as

$$\bigcup_{k\in\mathfrak{I}_m}\left(\partial^*\Phi^L_{k,\bar{y}}(\bar{y})\ \cup\partial^*\Phi^U_{k,\bar{y}}(\bar{y})\right)\bigcup\left(\bigcup_{i\in\mathfrak{I}_n(\bar{y})}\partial^* g_i(\bar{y})\right).$$

Since $|h_j|,\ \forall\ j\in\mathfrak{L}_l$ are regular at a point $\bar{y}$, so $\max_{1\leqq j\leqq l}|h_j(y)|$ is also regular at $\bar{y}$. Therefore, $\|\ h(y)\|$ has a regular upper convexificator at a point $\bar{y}$ given by $\bigcup_{j\in\mathfrak{L}_l}\partial^*(|h_j(\bar{y})|)$. By Rule (4.2) given by Jeyakumar and Luc [12], the function defined by

$$\max\left\{\Phi^L_{s,\bar{y}}(y),\Phi^U_{s,\bar{y}}(y),\max_{s\neq k\in\mathfrak{I}_m}\Phi^L_{k,\bar{y}}(y),\max_{s\neq k\in\mathfrak{I}_m}\Phi^U_{k,\bar{y}}(y),\max_{i\in\mathfrak{I}_n(\bar{y})}g_i(y)\right\}+\bar{r}\|\ h(y)\|,$$

has an upper convexificator at a point $\bar{y}$ given by

$$\bigcup_{k\in\mathfrak{I}_m}\left(\partial^*\Phi^L_{k,\bar{y}}(\bar{y})\ \cup\partial^*\Phi^U_{k,\bar{y}}(\bar{y})\right)\bigcup\left(\bigcup_{i\in\mathfrak{I}_n(\bar{y})}\partial^* g_i(\bar{y})+\bar{r}\bigcup_{j\in\mathfrak{L}_l}\partial^*(|h_j(\bar{y})|)\right).$$

The convexity of $\mathbb{C}$ implies the convexity of $d_{\mathbb{C}}(y)$ which in turn is regular, and the Clarke subdifferential $\partial_\circ d_{\mathbb{C}}(\bar{y})$ is the upper regular convexificator of $d_{\mathbb{C}}(y)$ at $\bar{y}$. It has to be noted that the set $\partial_\circ d_{\mathbb{C}}(\bar{y})$ is convex and weakly* compact. Again using Rule (4.2), given Jeyakumar and Luc [12], the function $M_{\bar{r}}(y)$ is an upper convexificator at a point $\bar{y}$ as

$$\bigcup_{k\in\mathfrak{I}_m}\left(\partial^*\Phi^L_{k,\bar{y}}(\bar{y})\ \cup\partial^*\Phi^U_{k,\bar{y}}(\bar{y})\right)\bigcup\left(\bigcup_{i\in\mathfrak{I}_n(\bar{y})}\partial^* g_i(\bar{y})+\bar{r}\left(\bigcup_{j\in\mathfrak{L}_l}\partial^*(|h_j(\bar{y})|)+\partial_\circ d_{\mathbb{C}}(\bar{y})\right)\right).$$

Therefore, from (6) we conclude that

$$\begin{aligned}0\in\ &\text{cl conv}\left\{\bigcup_{k\in\mathfrak{I}_m}\left(\partial^*\Phi^L_{k,\bar{y}}(\bar{y})\ \cup\partial^*\Phi^U_{k,\bar{y}}(\bar{y})\right)\bigcup\left(\bigcup_{i\in\mathfrak{I}_n(\bar{y})}\partial^* g_i(\bar{y})\right.\right.\\ &\qquad\left.\left.+\bar{r}\Big(\bigcup_{j\in\mathfrak{L}_l}\partial^*(|h_j(\bar{y})|)+\partial_\circ d_{\mathbb{C}}(\bar{y})\Big)\right)\right\}\\ =\ &\text{cl}\left\{\text{conv}\left(\bigcup_{k\in\mathfrak{I}_m}\left(\partial^*\Phi^L_{k,\bar{y}}(\bar{y})\ \cup\partial^*\Phi^U_{k,\bar{y}}(\bar{y})\right)\bigcup\left(\bigcup_{i\in\mathfrak{I}_n(\bar{y})}\partial^* g_i(\bar{y})\right)\right)\right.\end{aligned}$$

$$+\bar{r}\Big(\operatorname{conv}\big(\bigcup_{j\in\mathfrak{L}_l}\partial^*(|h_j(\bar{y})|)\big)+\partial_\circ d_{\mathbb{C}}(\bar{y})\Big)\Big\}.$$

This guarantees the existence of $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\bar{\tau}_j \geqq 0$ (for all j in $\mathfrak{L}_l$) with $\sum_{j\in\mathfrak{L}_l}\bar{\tau}_j = 1$, so that $\sum_{k\in\mathfrak{J}_m}(\bar{\alpha}_k^L+\bar{\alpha}_k^U)+\sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i = 1$, along with

$$0 \in \operatorname{cl}\Big\{\Big(\sum_{k\in\mathfrak{J}_m}\big(\bar{\alpha}_k^L\operatorname{conv}\partial^*\Phi_{k,\bar{y}}^L(\bar{y})\ +\bar{\alpha}_k^U\operatorname{conv}\partial^*\Phi_{k,\bar{y}}^U(\bar{y})\big)+\sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i\operatorname{conv}\partial^*g_i(\bar{y})\Big)$$

$$+\bar{r}\Big(\sum_{j\in\mathfrak{L}_l}\bar{\tau}_j\operatorname{conv}\partial^*(|h_j(\bar{y})|)+\partial_\circ d_{\mathbb{C}}(\bar{y})\Big)\Big\}.$$

The proof is complete. □

Proposition 3.3. (Luu [14]) For all $j \in \mathfrak{L}_l$, suppose h_j are endowed with a convexificator $\partial^*h_j(y)$ at a point y around $\bar{y}$ and the Lipschitz functions h_j satisfy Assumption 3.1 and ∂^*h_j is upper semicontinuous at a point $\bar{y}$. Then, $\partial^*h_j(\bar{y})\cup(-\partial^*h_j(\bar{y}))$ is the convexificator of $|h_j|$ at a point $\bar{y}$, and

$$\operatorname{cl}\operatorname{conv}\partial^*(|h_j(\bar{y})|)\subseteq\operatorname{cl}\operatorname{conv}\Big(\partial^*h_j(\bar{y})\cup(-\partial^*h_j(\bar{y}))\Big). \tag{7}$$

Remark 3.4. The sets $\partial^*h_j(\bar{y})\cup(-\partial^*h_j(\bar{y}))$ and $\operatorname{cl}\operatorname{conv}\Big(\partial^*h_j(\bar{y})\cup(-\partial^*h_j(\bar{y}))\Big)$ may not be upper regular convexificators of $|h_j|$ at a point $\bar{y}$, and

$$\operatorname{cl}\operatorname{conv}\partial^*(|h_j(\bar{y})|)\subsetneqq\operatorname{cl}\operatorname{conv}\Big(\partial^*h_j(\bar{y})\cup(-\partial^*h_j(\bar{y}))\Big).$$

Example 3.5. Let the function $h:\mathbb{R}\to\mathbb{R}$ be defined by

$$h(y)=\begin{cases}-(y+1)^3, & y\geqq 0\\ -(y-1)^4, & y<0\end{cases}$$

Then,

$$|h(y)|=\begin{cases}(y+1)^3, & y\geqq 0\\ (y-1)^4, & y<0\end{cases}$$

It can be seen that the Lipschitz function $|h|$ is regular in terms of Clarke at $\bar{y}=0$. Furthermore, $\partial h(0)=[-3,4]$, $\partial(|h(0)|)=[-4,3]$ and $\partial h(0)\cup(-\partial h(0))=[-4,4]$. Therefore, the set $[-4,3]$ is an upper regular convexificator of the function $|h|$ at the point $\bar{y}=0$ and $\partial(|h(0)|)\subsetneqq\partial h(0)\cup\partial|h(0)|$.

If the convexificator maps ∂^*h_j $(j\in\mathfrak{L}_l)$ ae upper semicontinuous at a point $\bar{y}$, then we get the following Fritz John type necessary conditions of a locally LU-efficient solution at point $\bar{y}$ of (CIEP).

Theorem 3.6. Suppose the solution $\bar{y}$ is locally LU-efficient to (CIEP) and $\bar{y}$ is a regular point corresponding to $\mathbb{C}$ with $\Phi^L_{\bar{y}}(\bar{y}) = 0$ and $\Phi^U_{\bar{y}}(\bar{y}) = 0$, and it satisfies the Assumption 3.1. Additionally, assume that h_j (for each j in $\mathfrak{L}_l$) admits a convexificator $\partial^* h_j(y)$ at a point y near $\bar{y}$ and the convexificator map $\partial^* h_j$ is upper semicontinuous at a point $\bar{y}$. Then, there exist $\bar{\alpha}^L_k,\ \bar{\alpha}^U_k \geqq 0$ (for all k in $\mathfrak{I}_m$), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\rho_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) satisfying $\sum_{k\in\mathfrak{I}_m}\left(\bar{\alpha}^L_k + \bar{\alpha}^U_k\right) + \sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i = 1$ along with

$$0 \in \operatorname{cl}\Bigg\{ \sum_{k\in\mathfrak{I}_m}\Big(\bar{\alpha}^L_k \operatorname{conv} \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}^U_k \operatorname{conv} \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \operatorname{conv} \partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \operatorname{conv} \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Bigg\}. \tag{8}$$

Proof. In the light of the assumptions of Theorem 3.2, there exist $\bar{\alpha}^L_k,\ \bar{\alpha}^U_k \geqq 0$, for all k in $\mathfrak{I}_m$, $\bar{\beta}_i \geqq 0$ for all i in $\mathfrak{I}_n(\bar{y})$, $\bar{\tau}_j \geqq 0$ for all j in $\mathfrak{L}_l$, $\sum_{j\in\mathfrak{L}_l}\bar{\tau}_j = 1$, $\bar{r} > 0$ satisfies $\sum_{k\in\mathfrak{I}_m}(\bar{\alpha}^L_k + \bar{\alpha}^U_k) + \sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i = 1$, along with

$$0 \in \operatorname{cl}\Bigg\{ \sum_{k\in\mathfrak{I}_m}\Big(\bar{\alpha}^L_k \operatorname{conv} \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}^U_k \operatorname{conv} \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \operatorname{conv} \partial^* g_i(\bar{y}) + \bar{r}\Big(\sum_{j\in\mathfrak{L}_l} \bar{\tau}_j \operatorname{conv} \partial^*(|h_j(\bar{y})|) + \partial_\circ d_{\mathbb{C}}(\bar{y})\Big)\Bigg\}. \tag{9}$$

Since all the hypotheses of Proposition 3.3 are satisfied, therefore inclusion (9) together with (7) gives

$$0 \in \operatorname{cl}\Bigg\{ \sum_{k\in\mathfrak{I}_m}\Big(\bar{\alpha}^L_k \operatorname{conv} \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}^U_k \operatorname{conv} \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \operatorname{conv} \partial^* g_i(\bar{y}) + \bar{r}\Big(\sum_{j\in\mathfrak{L}_l} \bar{\tau}_j \operatorname{cl} \operatorname{conv} \partial^* h_j(\bar{y}) \cup (-\partial^* h_j(\bar{y})) + \partial_\circ d_{\mathbb{C}}(\bar{y})\Big)\Bigg\}. \tag{10}$$

Using the condition $\operatorname{cl}A + \operatorname{cl}B \subseteq \operatorname{cl}(A + B)$, we obtain

$$\begin{aligned} 0 \in\ & \operatorname{cl}\Bigg\{ \sum_{k\in\mathfrak{I}_m}\Big(\bar{\alpha}^L_k \operatorname{conv} \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}^U_k \operatorname{conv} \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \operatorname{conv} \partial^* g_i(\bar{y}) \\ & + \bar{r} \operatorname{cl}\Big(\sum_{j\in\mathfrak{L}_l} \bar{\tau}_j \operatorname{conv} \partial^* h_j(\bar{y}) \cup (-\partial^* h_j(\bar{y})) + \partial_\circ d_{\mathbb{C}}(\bar{y})\Big)\Bigg\} \\ \subseteq\ & \operatorname{cl}\Bigg\{ \sum_{k\in\mathfrak{I}_m}\Big(\bar{\alpha}^L_k \operatorname{conv} \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}^U_k \operatorname{conv} \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \operatorname{conv} \partial^* g_i(\bar{y}) \\ & + \bar{r}\Big(\sum_{j\in\mathfrak{L}_l} \bar{\tau}_j \operatorname{conv} \partial^* h_j(\bar{y}) \cup (-\partial^* h_j(\bar{y})) + \partial_\circ d_{\mathbb{C}}(\bar{y})\Big)\Bigg\} \end{aligned}$$

which shows that $\bar{p}_j \geqq 0$ and $\bar{q}_j \geqq 0$ exist with the condition $\bar{p}_j + \bar{q}_j = 1$ for all j in $\mathfrak{L}_l$ such that

$$0 \in \operatorname{cl}\Big\{ \sum_{k\in\mathfrak{I}_m} \Big(\bar{\alpha}_k^L \text{ conv } \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y})$$

$$+\bar{r}\Big(\sum_{j\in\mathfrak{L}_l} \bar{\tau}_j(\bar{p}_j - \bar{q}_j) \text{ conv } \partial^* h_j(\bar{y}) + \partial_\circ d_{\mathbb{C}}(\bar{y})\Big)\Big\}.$$

Putting $\bar{\rho}_j = \bar{r}\bar{\tau}_j(\bar{p}_j - \bar{q}_j)$, where $\rho_j \in \mathbb{R}$ and using inclusion $\bar{r}\ \partial_\circ d_{\mathbb{C}}(\bar{y}) \subseteq \mathbb{N}_{\mathbb{C}}(\bar{y})$, we get

$$0 \in \operatorname{cl}\Big\{ \sum_{k\in\mathfrak{I}_m} \Big(\bar{\alpha}_k^L \text{ conv } \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y})$$

$$+ \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \text{ conv } \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Big\}.$$

This completes the proof. □

4. KARUSH–KUHN–TUCKER TYPE NECESSARY CONDITIONS

To derive the Karush–Kuhn–Tucker type necessary conditions under locally LU-efficient solutions for problem (CIEP), we use the following Mangasarian–Fromovitz constraint qualification (MFCQ), as stated in Luu [14]: There exists $\omega_0 \in \mathbb{T}_{\mathbb{C}}(\bar{y})$ and numbers $a_i > 0$ $(i \in \mathfrak{I}_n(\bar{y}))$ so that

(i) $\langle \zeta_i, \omega_0\rangle \leqq -a_i$ $(\forall\ \zeta_i \in \partial^* g_i(\bar{y}),\ \forall i \in \mathfrak{I}_n(\bar{y})$;

(ii) $\langle \mu_j, \omega_0\rangle = 0$ $(\forall\ \mu_j \in \partial^* h_j(\bar{y}),\ \forall j \in \mathfrak{L}_l)$.

Theorem 4.1. Suppose $\bar{y}$ is a locally LU-efficient solution to (CIEP) and the Mangasarian–Fromovitz constraint qualification (MFCQ) hold. Under the presumption of Theorem 3.6, there exist $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{I}_m$, not all zero simultaneously), $\bar{\beta}_i \geqq 0$ (for each i in $\mathfrak{I}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) in such a way

$$0 \in \operatorname{cl}\Big\{ \sum_{k\in\mathfrak{I}_m} \Big(\bar{\alpha}_k^L \text{ conv } \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y})$$

$$+ \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \text{ conv } \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Big\}. \tag{11}$$

Proof. Since the hypotheses of Theorem 3.6 are satisfied, we claim that there exist $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{I}_m$), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) satisfying $\sum_{k\in\mathfrak{I}_m}(\bar{\alpha}_k^L + \bar{\alpha}_k^U) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i = 1$, along with

$$0 \in \operatorname{cl}\Big\{ \sum_{k\in\mathfrak{I}_m} \Big(\bar{\alpha}_k^L \text{ conv } \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y})$$

$$+\sum_{j\in\mathfrak{L}_l}\bar{\rho}_j \text{ conv } \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Big\}.$$

If $\bar{\alpha}_k^L = 0,\ \bar{\alpha}_k^U = 0$ (for all k in $\mathfrak{I}_m$), then $\sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i = 1$. Consequently, there exist $\zeta_i^{(\nu)} \in \text{conv } \partial^* g_i(\bar{y})$ (for all $i \in \mathfrak{I}_n(y)$), $\mu_j^{(\nu)} \in \text{conv } \partial^* h_j(\bar{y})$ (for all $j \in \mathfrak{L}_l$) and $\xi^{(\nu)} \in \mathbb{N}_{\mathbb{C}}(\bar{y})$ such that

$$0 = \lim_{\nu\to\infty}\Big[\sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i\zeta_i^{(\nu)} + \sum_{j\in\mathfrak{I}_m}\bar{\rho}_j\mu_j^{(\nu)} + \xi^{(\nu)}\Big]. \tag{12}$$

which imply that

$$0 = \lim_{\nu\to\infty}\Big[\sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i\langle\zeta_i^{(\nu)},\omega_0\rangle + \sum_{j\in\mathfrak{I}_m}\bar{\rho}_j\langle\mu_j^{(\nu)},\omega_0\rangle + \langle\xi^{(\nu)},\omega_0\rangle\Big]. \tag{13}$$

Also, since $\sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i = 1$, using (MFCQ), we get

$$\begin{aligned}
&\lim_{\nu\to\infty}\Big[\sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i\langle\zeta_i^{(\nu)},\omega_0\rangle + \sum_{j\in\mathfrak{I}_m}\bar{\rho}_j\langle\mu_j^{(\nu)},\omega_0\rangle + \langle\xi^{(\nu)},\omega_0\rangle\Big]\\
&\leqq \lim_{\nu\to\infty}\Big[\sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i\langle\zeta_i^{(\nu)},\omega_0\rangle + \sum_{j\in\mathfrak{I}_m}\bar{\rho}_j\langle\mu_j^{(\nu)},\omega_0\rangle\Big]\\
&\leqq -\sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i a_i < 0.
\end{aligned}$$

Hence, it contradicts equation (13). □

To determine the nonzero components of the Lagrange multipliers relative to the objective function, we use the following stronger Mangasarian–Fromovitz–type constraint qualification (SMFCQ), as stated in Luu [14]: There exist $s \in \mathfrak{I}_m$, $\omega_0 \in \mathbb{T}_{\mathbb{C}}(\bar{y})$ and numbers $a_i > 0$ ($i \in \mathfrak{I}_n(\bar{y})$), $b_k > 0$ ($s \neq k \in \mathfrak{I}_m$) satisfying the following assumptions

(i) $\langle\zeta_i,\omega_0\rangle \leqq -a_i$ ($\forall\ \zeta_i \in \partial^* g_i(\bar{y}),\ \forall i \in \mathfrak{I}_n(\bar{y})$); $\langle\varkappa_k^L,\omega_0\rangle \leqq -b_k^L$ ($\forall\ \varkappa_k^L \in \partial^*\Phi_{k,\bar{y}}^L(\bar{y})$); $\langle\varkappa_k^U,\omega_0\rangle \leqq -b_k^U$ ($\forall\ \varkappa_k^U \in \partial^*\Phi_{k,\bar{y}}^U(\bar{y}),\ \forall s \neq k \in \mathfrak{I}_m$);

(ii) $\langle\mu_j,\omega_0\rangle = 0$ (for all $\mu_j \in \partial^* h_j(\bar{y})$), where j runs over $\mathfrak{L}_l$.

Remark 4.2.

(*a*) (SMFCQ) $\implies$ (MFCQ).

(*b*) If (SMFCQ) holds for an element $s \in \mathfrak{I}_m$, then $\bar{\alpha}_s^L,\ \bar{\alpha}_s^U > 0$ and $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for $k \neq s$ in $\mathfrak{I}_m$), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) with

$$0 \in \text{cl}\Big\{\sum_{k\in\mathfrak{I}_m}\Big(\bar{\alpha}_k^L \text{ conv } \partial^*\Phi_{k,\bar{y}}^L(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\Phi_{k,\bar{y}}^U(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y})$$
$$+\sum_{j\in\mathfrak{L}_l}\bar{\rho}_j \text{ conv } \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Big\}.$$

Theorem 4.3. Let $\bar{y}$ be a locally LU-efficient solution to (CIEP). Suppose all the assumptions of Theorem 3.6 and the constraint qualification (SMFCQ) hold. Then, there exist $\bar{\alpha}_s^L,\ \bar{\alpha}_s^U > 0$ (for s in $\mathfrak{I}_m$), $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all $s \neq k$ in $\mathfrak{I}_m$), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) such that

$$0 \in \mathrm{cl}\Big\{ \sum_{k\in\mathfrak{I}_m} \Big(\bar{\alpha}_k^L \text{ conv } \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \text{ conv } \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Big\}. \tag{14}$$

P r o o f. In view of Remark 4.2, we see that for all s in $\mathfrak{I}_m$, there exist $\alpha_s^{(s)L},\ \alpha_s^{(s)U} > 0$, $\alpha_k^{(s)L},\ \alpha_k^{(s)U} \geqq 0$ (for all $s \neq k \in \mathfrak{I}_m$), $\beta_i^{(s)} \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\rho_j^{(s)} \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) satisfying

$$0 \in \mathrm{cl}\Big\{ \sum_{k\in\mathfrak{I}_m} \Big(\alpha_k^{(s)L} \text{ conv } \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \alpha_k^{(s)U} \text{ conv } \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \beta_i^{(s)} \text{ conv } \partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l} \rho_j^{(s)} \text{ conv } \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Big\}. \tag{15}$$

Using the condition $\mathrm{cl}A + \mathrm{cl}B \subseteq \mathrm{cl}(A+B)$ and putting $s = 1$ to m in (15) and finally summing all equations, we get

$$\begin{aligned}0 \in \sum_{s\in\mathfrak{I}_m} \mathrm{cl}\Big\{ &\sum_{k\in\mathfrak{I}_m} \Big(\alpha_k^{(s)L} \text{ conv } \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \alpha_k^{(s)U} \text{ conv } \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) \\ &+ \sum_{i\in\mathfrak{I}_n(\bar{y})} \beta_i^{(s)} \text{ conv } \partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l} \rho_j^{(s)} \text{ conv } \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Big\}, \\ \subseteq \mathrm{cl}\Big\{ &\sum_{k\in\mathfrak{I}_m} \Big(\bar{\alpha}_k^L \text{ conv } \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) \\ &+ \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \text{ conv } \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Big\},\end{aligned}$$

where, $\bar{\alpha}_k^L = \alpha_s^{(s)L} + \sum_{k\neq s\in\mathfrak{I}_m} \alpha_k^{(s)L} > 0,\ \bar{\alpha}_k^U = \alpha_s^{(s)U} + \sum_{k\neq s\in\mathfrak{I}_m} \alpha_k^{(s)U} > 0$ (for all k in $\mathfrak{I}_m$), $\bar{\beta}_i = \sum_{s\in\mathfrak{I}_m} \beta_i^{(s)} \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\bar{\rho}_j = \sum_{s\in\mathfrak{I}_m} {\rho_j}^{(s)} \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$). □

Corollary 4.4. Suppose that $\mathbb{X}$ is finite-dimensional. Let $\bar{y}$ be a locally LU-efficient solution to (CIEP). Assume that $\bar{y}$ is a regular point for h corresponding to $\mathbb{C}$ and the constraint qualification (MFCQ) hold along with Assumption 3.1. Then, there exist

$\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$, at least one is nonzero), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) such that

$$0 \in \text{cl}\Bigg\{ \sum_{k\in\mathfrak{J}_m} \Big(\bar{\alpha}_k^L \text{ conv } \partial^*\Phi_{k,\bar{y}}^L(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\Phi_{k,\bar{y}}^U(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y})$$

$$+ \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j\ \partial_\circ h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Bigg\},$$

where $\partial_\circ h_j(\bar{y})$ is Clarke subdifferential of h_j at $\bar{y}$.

5. SUFFICIENCY FOR WEAK LU-EFFICIENT SOLUTIONS

Definition 5.1. (Luu [14]) A function ϕ defined on $\mathbb{X}$ having an upper convexificator $\partial^*\phi(\bar{y})$ is known as asymptotic pseudoconvex at a point $\bar{y}$ regarding $\mathbb{C}$ if and only if for some $y_\nu^* \in \text{conv } \partial^*\phi(\bar{y})$, the following condition is satisfied:

$$\lim_{\nu\to\infty} \langle y_\nu^*, y-\bar{y}\rangle \geqq 0 \implies \phi(y) \geqq \phi(\bar{y}),\ \forall\, y \in \mathbb{C}.$$

Definition 5.2. (Luu [14]) A function ϕ defined on $\mathbb{X}$ having an upper convexificator $\partial^*\phi(\bar{y})$ is known as asymptotic quasiconvex at a point $\bar{y}$ regarding $\mathbb{C}$ if and only if for some $y_\nu^* \in \text{conv } \partial^*\phi(\bar{y})$, the following condition is satisfied:

$$\phi(y) \leqq \phi(\bar{y}) \implies \lim_{\nu\to\infty} \langle y_\nu^*, y-\bar{y}\rangle \leqq 0,\ \forall\, y \in \mathbb{C}.$$

A function ϕ is known as asymptotic quasiconcave at a point $\bar{y}$ regarding to $\mathbb{C}$ if and only if $-\phi$ is asymptotic quasiconvex at a point $\bar{y}$ regarding to $\mathbb{C}$.

Definition 5.3. (Luu [14]) An asymptotic quasilinear function ϕ at a point $\bar{y}$ relative to $\mathbb{C}$ is characterized by the fact that it is both asymptotic quasiconcave and asymptotic quasiconvex at a point $\bar{y}$ regarding $\mathbb{C}$.

Theorem 5.4. A solution $\bar{y} \in M_1$ is weak LU-efficient to the problem (CIEP), if it satisfies Assumption 3.1 for which $\Phi_{\bar{y}}^L(\bar{y}) = 0$ and $\Phi_{\bar{y}}^U(\bar{y}) = 0$. Furthermore, let us assume that there exist $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$, at least one being nonzero), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\bar{\tau}_j \geqq 0$ (for all j in $\mathfrak{L}_l$), $\sum_{j\in\mathfrak{L}_l} \bar{\tau}_j = 1$, such as $M_{\bar{r}}(y)$, is asymptotic pseudoconvex at a point $\bar{y}$ regarding to M_1 for $\bar{r} > 0$ and

$$0 \in \text{cl}\Bigg\{ \sum_{k\in\mathfrak{J}_m} \Big(\bar{\alpha}_k^L \text{ conv } \partial^*\Phi_{k,\bar{y}}^L(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\Phi_{k,\bar{y}}^U(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y})$$

$$+\bar{r}\Bigg(\sum_{j\in\mathfrak{L}_l} \bar{\tau}_j \text{ conv } \partial^*(|h_j(\bar{y})|) + \partial_\circ d_{\mathbb{C}}(\bar{y})\Bigg)\Bigg\}. \tag{16}$$

Proof. As we have $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$, at least one being nonzero), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), satisfying $\sum_{k\in\mathfrak{J}_m}(\bar{\alpha}_k^L + \bar{\alpha}_k^U) + \sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i = 1$. Therefore, using (16), we get

$$0 \in \mathrm{cl}\Big\{\mathrm{conv}\Big(\bigcup_{k\in\mathfrak{J}_m}\Big(\partial^*\Phi^L_{k,\bar{y}}(\bar{y})\ \cup \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big)\bigcup\Big(\bigcup_{i\in\mathfrak{I}_n(\bar{y})}\partial^* g_i(\bar{y})\Big)\Big)$$
$$+\bar{r}\Big(\mathrm{conv}\Big(\bigcup_{j\in\mathfrak{L}_l}\partial^*(|h_j(\bar{y})|)\Big) + \partial_\circ d_{\mathbb{C}}(\bar{y})\Big)\Big\}$$
$$= \ \mathrm{cl\ conv}\Big\{\bigcup_{k\in\mathfrak{J}_m}\Big(\partial^*\Phi^L_{k,\bar{y}}(\bar{y})\ \cup \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big)\bigcup\Big(\bigcup_{i\in\mathfrak{I}_n(\bar{y})}\partial^* g_i(\bar{y})$$
$$+\bar{r}\Big(\bigcup_{j\in\mathfrak{L}_l}\partial^*(|h_j(\bar{y})|) + \partial_\circ d_{\mathbb{C}}(\bar{y})\Big)\Big)\Big\}.$$

Due to the fact that $\mathbb{C}$ and $d_{\mathbb{C}}$ are convex, and the function $d_{\mathbb{C}}$ is regular at a point $\bar{y}$, the Clarke subdiffferential $\partial_\circ d_{\mathbb{C}}(\bar{y})$ can be taken as the upper regular convexificator of $d_{\mathbb{C}}$ at a point $\bar{y}$. Therefore, the set of upper convexificators of $M_{\bar{r}}$ at a point $\bar{y}$ is specified by

$$\bigcup_{k\in\mathfrak{J}_m}\Big(\partial^*\Phi^L_{k,\bar{y}}(\bar{y})\ \cup \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big)\bigcup\Big(\bigcup_{i\in\mathfrak{I}_n(\bar{y})}\partial^* g_i(\bar{y}) + \bar{r}\Big(\bigcup_{j\in\mathfrak{L}_l}\partial^*(|h_j(\bar{y})|) + \partial_\circ d_{\mathbb{C}}(\bar{y})\Big)\Big).$$

Consequently, one can have

$$0 \in \mathrm{cl\ conv}\partial^* M_{\bar{r}}(\bar{y}),$$

and, hence, one can get a sequence $\{y_n^*\} \subseteq\ \mathrm{conv}\partial^* M_{\bar{r}}(\bar{y})$ so that

$$\lim_{\nu\to\infty}\ y_\nu^* = 0.$$

Therefore, we can conclude that

$$\lim_{\nu\to\infty}\ \langle y_\nu^*, y-\bar{y}\rangle = 0,\ \forall y\in\mathbb{C}. \tag{17}$$

Since $M_{\bar{r}}$ is asymptotic pseudoconvex at a point $\bar{y}$, (17) yield

$$M_{\bar{r}}(y) \geqq M_{\bar{r}}(\bar{y});\ \forall\ x\in M.$$

Using Proposition 2.2, we can conclude that solution $\bar{y}$ becomes a weak minimum of the vector optimization problem (MP). Also, $\Phi^L_{\bar{y}}(\bar{y}) = 0$ and $\Phi^U_{\bar{y}}(\bar{y}) = 0$, therefore, we can conclude that solution $\bar{y}$ is weak LU-efficient to the problem (CIEP). □

Theorem 5.5. If the solution $\bar{y}\in M_1$ is weak LU-efficient to the problem (CIEP), it holds the conditions

(i) $\Phi^L_{\bar{y}}(\bar{y}) = 0$, $\Phi^U_{\bar{y}}(\bar{y}) = 0$,

(ii) there exist $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$, at least one being nonzero), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) such that

$$0 \in \operatorname{cl}\Bigg\{ \sum_{k\in\mathfrak{J}_m} \Big(\bar{\alpha}_k^L \text{ conv } \partial^*\Phi_{k,\bar{y}}^L(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\Phi_{k,\bar{y}}^U(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \text{ conv } \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Bigg\}, \tag{18}$$

(iii) $\partial^*\Phi_{k,\bar{y}}^L(\bar{y})$ and $\partial^*\Phi_{k,\bar{y}}^U(\bar{y})$ $(k \in \mathfrak{J}_m)$ are upper regular at $\bar{y}$ for at most one of the upper convexificators, the function $\bar{\alpha}^L\Phi_{\bar{y}}^L(.) := \sum_{k\in\mathfrak{J}_m} \bar{\alpha}_k^L\Phi_{k,\bar{y}}^L(.)$ and $\bar{\alpha}^U\Phi_{\bar{y}}^U(.) := \sum_{k\in\mathfrak{J}_m} \bar{\alpha}_k^U\Phi_{k,\bar{y}}^U(.)$ are asymptotic pseudoconvex at point $\bar{y}$ regarding M_1, g_i are asymptotic quasiconvex at point $\bar{y}$ regarding M_1 (for all i in $\mathfrak{I}_n(\bar{y})$), h_j are asymptotic quasilinear at point $\bar{y}$ regarding M_1 (for all j in $\mathfrak{L}_l$), $\mathbb{C}$ is convex.

P r o o f. From condition (18), we may conclude that $\varkappa_k^{(\nu)L} \in \text{conv } \partial^*\Phi_{k,\bar{y}}^L(\bar{y})$, $\varkappa_k^{(\nu)U} \in \text{conv } \partial^*\Phi_{k,\bar{y}}^U(\bar{y})$, $\zeta_i^{(\nu)} \in \text{conv } \partial^* g_i(\bar{y})$, $\mu_j^{(\nu)} \in \text{conv } \partial^* h_j(\bar{y})$, $\xi^{(\nu)} \in \mathbb{N}_{\mathbb{C}}(\bar{y})$ such that

$$0 = \lim_{\nu\to\infty} \Bigg[\sum_{k\in\mathfrak{J}_m} \Big(\bar{\alpha}_k^L \varkappa_k^{(\nu)L} + \bar{\alpha}_k^U \varkappa_k^{(\nu)U}\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i\zeta_i^{(\nu)} + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j\mu_j^{(\nu)} + \xi^{(\nu)}\Bigg],$$

which implies that

$$\lim_{\nu\to\infty} \Bigg[\sum_{k\in\mathfrak{J}_m} \Big(\bar{\alpha}_k^L \langle\varkappa_k^{(\nu)L}, y-\bar{y}\rangle + \bar{\alpha}_k^U \langle\varkappa_k^{(\nu)U}, y-\bar{y}\rangle\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i\langle\zeta_i^{(\nu)}, y-\bar{y}\rangle + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j\langle\mu_j^{(\nu)}, y-\bar{y}\rangle + \langle\xi^{(\nu)}, y-\bar{y}\rangle\Bigg] = 0,\ \forall y \in M_1. \tag{19}$$

For all $y \in M_1$, we get $g_i(y) \leqq 0 = g_i(\bar{y}),\ \forall\ i \in \mathfrak{I}_n(\bar{y})$. Therefore, asymptotic quasiconvexity of g_i at point $\bar{y}$, gives

$$\lim_{\nu\to\infty} \langle\zeta_i^{(\nu)}, y-\bar{y}\rangle \leqq 0,\ \forall y \in M_1. \tag{20}$$

Also, we have $h_j(y) = 0 = h_j(\bar{y})$ $(\forall y \in M_1)$, so using asymptotic quasilinearity, we obtain

$$\lim_{\nu\to\infty} \langle\mu_j^{(\nu)}, y-\bar{y}\rangle = 0;\ \forall y \in M_1. \tag{21}$$

Due to the convexity of $\mathbb{C}$, $y-\bar{y} \in \mathbb{T}_{\mathbb{C}}(\bar{y})\ \forall\ y \in \mathbb{C}$, we have

$$\lim_{\nu\to\infty} \langle\xi^{(\nu)}, y-\bar{y}\rangle \leqq 0;\ \forall y \in M_1. \tag{22}$$

Summing up equations (19)–(22), we have

$$\lim_{\nu\to\infty} \Bigg\langle \sum_{k\in\mathfrak{J}_m} \bar{\alpha}_k^L \varkappa_k^{(\nu)L}, y-\bar{y} + \sum_{k\in\mathfrak{J}_m} \bar{\alpha}_k^U \varkappa_k^{(\nu)U}, y-\bar{y}\Bigg\rangle \geqq 0.$$

As $\partial^*\Phi^L_{k,\bar{y}}(\bar{y})$ and $\partial^*\Phi^U_{k,\bar{y}}(\bar{y})$ $(k \in \mathfrak{I}_m)$ are upper regular at point $\bar{y}$ for at most one of the upper convexificators; therefore, the function $\sum_{k\in\mathfrak{I}_m} \bar{\alpha}^L_k \Phi^L_{k,\bar{y}}(.)$ and $\sum_{k\in\mathfrak{I}_m} \bar{\alpha}^U_k \Phi^U_{k,\bar{y}}(.)$ has an upper convexificator $\sum_{k\in\mathfrak{I}_m} \bar{\alpha}^L_k \partial^*\Phi^L_{k,\bar{y}}(\bar{y})$ and $\sum_{k\in\mathfrak{I}_m} \bar{\alpha}^U_k \partial^*\Phi^U_{k,\bar{y}}(\bar{y})$ respectively at a point $\bar{y}$, using asymptotic pseudoconvexity of $\bar{\alpha}^L\Phi^L_{\bar{y}}(.)$ and $\bar{\alpha}^U\Phi^U_{\bar{y}}(.)$ $\forall y \in M_1$, we obtain

$$\bar{\alpha}^L\Phi^L_{\bar{y}}(y) \geqq \bar{\alpha}^L\Phi^L_{\bar{y}}(\bar{y}) = 0,$$
$$\bar{\alpha}^U\Phi^U_{\bar{y}}(y) \geqq \bar{\alpha}^U\Phi^U_{\bar{y}}(\bar{y}) = 0.$$

Therefore, $\bar{y}$ is the minima of the functions $\bar{\alpha}^L\Phi^L_{\bar{y}}(.)$ and $\bar{\alpha}^U\Phi^U_{\bar{y}}(.)$ over M_1. Hence, the solution $\bar{y}$ is weak LU-efficient to the problem (CIEP). □

Theorem 5.6. A point $\bar{y} \in M_1$ is a weak LU-efficient solution to (CIEP) provided it fulfills the following criteria:

(i) $\Phi^L_{\bar{y}}(\bar{y}) = 0$, $\Phi^U_{\bar{y}}(\bar{y}) = 0$,

(ii) there exist $\bar{\alpha}^L_s$, $\bar{\alpha}^U_s > 0$, $\bar{\alpha}^L_k$, $\bar{\alpha}^U_k \geqq 0$ (for all $s \neq k$ in $\mathfrak{I}_m$), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) such that

$$0 \in \operatorname{cl}\Bigg\{ \sum_{k\in\mathfrak{I}_m} \Big(\bar{\alpha}^L_k \operatorname{conv} \partial^*\Phi^L_{k,\bar{y}}(\bar{y}) + \bar{\alpha}^U_k \operatorname{conv} \partial^*\Phi^U_{k,\bar{y}}(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})} \bar{\beta}_i \operatorname{conv} \partial^* g_i(\bar{y})$$
$$+ \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \operatorname{conv} \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Bigg\}.$$

(iii) The functions $\Phi^L_{s,\bar{y}}(.)$ and $\Phi^U_{s,\bar{y}}(.)$ are asymptotic pseudoconvex at point $\bar{y}$ regarding M_1, $\Phi^L_{k,\bar{y}}(.)$, $\Phi^U_{k,\bar{y}}(.)$ and g_i are asymptotic quasiconvex at point $\bar{y}$ regarding M_1 ($\forall\ k \in \mathfrak{I}_m, k \neq s, \forall\ i \in \mathfrak{I}_n(\bar{y})$), h_j are asymptotic quasilinear at point $\bar{y}$ regarding M_1 ($\forall\ j \in \mathfrak{L}_l$), $\mathbb{C}$ is convex.

6. APPLICATIONS

In this section, we formulate optimality conditions of constrained interval-valued vector variational inequality (CIVI) and constrained interval-valued vector optimization problem (CIOP).

We state the following assumptions, which are used to formulate the optimality conditions for locally LU-efficient solutions at the point $\bar{y}$ of problem (CIVI).

Assumption 6.1.

(i) The functions $h_1, \ldots, h_l$ are locally Lipschitz at the point $\bar{y}$. Moreover, h_j (j varies over $\mathfrak{L}_l$) have convexificators defined by $\partial^* h_j(y)$ at a point y near $\bar{y}$, the convexificator map $\partial^* h_j$ is upper semicontinuous at a point $\bar{y}$, g_i (i varies over $\mathfrak{I}_n(\bar{y})$) are continuous functions, and $\mathbb{C}$ is convex.

(ii) $\partial^* g_i(\bar{y})$ represent an upper regular convexificator of g_i at a point $\bar{y}$.

(iii) The functions $|h_j|$, j in $\mathfrak{L}_l$ are regular at the point $\bar{y}$ in the sense of Clarke [3].

Fritz John type necessary conditions for (CIVI) can be stated as follows.

Theorem 6.2. Let the solution $\bar{y}$ be a locally LU-efficient to the problem (CIVI). Suppose that $\bar{y}$ is regular point for h, corresponding to $\mathbb{C}$, and satisfies Assumption 6.1. Then there exist $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) satisfying $\sum_{k\in\mathfrak{J}_m}\left(\bar{\alpha}_k^L + \bar{\alpha}_k^U\right) + \sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i = 1$, and

$$0 \in \operatorname{cl}\Bigg\{ \sum_{k\in\mathfrak{J}_m}\Big(\bar{\alpha}_k^L \mathbb{T}_k^L(\bar{y}) + \bar{\alpha}_k^U \mathbb{T}_k^L(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i \operatorname{conv}\partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l}\bar{\rho}_j \operatorname{conv}\partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Bigg\}. \tag{23}$$

P r o o f. Since $\mathbb{T}^L(\bar{y})(.)$ and $\mathbb{T}^U(\bar{y})(.)$ are continuous linear mappings, strictly differentiable, and locally Lipschitz. Therefore, $\{\mathbb{T}_k^L(\bar{y})\}$ and $\{\mathbb{T}_k^U(\bar{y})\}$ are upper convexificators of $\mathbb{T}^L(\bar{y})(.)$ and $\mathbb{T}^U(\bar{y})(.)$ ($\forall\ k \in \mathfrak{J}_m$), respectively. Putting $\Phi^L(y,z) = \mathbb{T}^L(y)(z-y)$ and $\Phi^U(y,z) = \mathbb{T}^U(y)(z-y)$, we get $\Phi_{\bar{y}}^L(\bar{y}) = 0$ and $\Phi_{\bar{y}}^U(\bar{y}) = 0$. Since all the hypotheses of Theorem 3.6 are fulfilled (since Assumption 6.1 is satisfied), then there exist $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), and $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) so that inclusion (23) holds and $\sum_{k\in\mathfrak{J}_m}\left(\bar{\alpha}_k^L + \bar{\alpha}_k^U\right) + \sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i = 1$. □

Now, we state Karush–Kuhn–Tucker type necessary conditions for locally LU-efficient solutions to (CIVI).

Theorem 6.3. Let the solution $\bar{y}$ be locally LU-efficient to (CIVI). Suppose that the constraint qualification (MFCQ) and the assumptions of Theorem 6.2 hold. Then there exist $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$, at least one being nonzero), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{I}_n(\bar{y})$), $\rho_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) such that

$$0 \in \operatorname{cl}\Bigg\{ \sum_{k\in\mathfrak{J}_m}\Big(\bar{\alpha}_k^L \mathbb{T}_k^L(\bar{y}) + \bar{\alpha}_k^U \mathbb{T}_k^U(\bar{y})\Big) + \sum_{i\in\mathfrak{I}_n(\bar{y})}\bar{\beta}_i \operatorname{conv}\partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l}\bar{\rho}_j \operatorname{conv}\partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Bigg\}. \tag{24}$$

P r o o f. Since hypotheses that were used to prove Theorem 4.1 are satisfied, therefore using Theorem 4.1 and Theorem 6.2, we get the Karush–Kuhn–Tucker type necessary condition (24) for the problem (CIVI). □

Followed by Theorem 5.5, we obtain the following sufficient optimality condition for weak LU-efficient solution to (CIVI).

Theorem 6.4. The solution $\bar{y} \in M_1$ becomes weak LU-efficient to the problem (CIVI) provided

(i) there exist $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$, at least one being nonzero), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{J}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) such that

$$0 \in \operatorname{cl}\Big\{ \sum_{k\in\mathfrak{J}_m} \Big(\bar{\alpha}_k^L \mathbb{T}_k^L(\bar{y}) + \bar{\alpha}_k^U \mathbb{T}_k^U(\bar{y})\Big) + \sum_{i\in\mathfrak{J}_n(\bar{y})} \bar{\beta}_i \operatorname{conv} \partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \operatorname{conv} \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Big\},$$

(ii) g_i are asymptotic quasiconvex at point $\bar{y}$ corresponding to M_1 for all i in $\mathfrak{J}_n(\bar{y})$, h_j are asymptotic quasilinear at point $\bar{y}$ corresponding to M_1 for all j in $\mathfrak{L}_l$, and $\mathbb{C}$ is convex.

Proof. Since $\mathbb{T}^L(\bar{y})(.)$ and $\mathbb{T}^U(\bar{y})(.)$ are continuous linear mappings, strictly differentiable, and locally Lipschitz. Therefore, $\{\mathbb{T}_k^L(\bar{y})\}$ and $\{\mathbb{T}_k^U(\bar{y})\}$ are upper convexificators of $\mathbb{T}^L(\bar{y})(.)$ and $\mathbb{T}^U(\bar{y})(.)$ (for all k in $\mathfrak{J}_m$), respectively. $\bar{\alpha}^L\mathbb{T}^L(\bar{y})(.) := \sum_{k\in\mathfrak{J}_m} \bar{\alpha}_k^L \mathbb{T}_k^L(\bar{y})(.)$ and $\bar{\alpha}^U\mathbb{T}^U(\bar{y})(.) := \sum_{k\in\mathfrak{J}_m} \bar{\alpha}_k^U \mathbb{T}_k^U(\bar{y})(.)$ are asymptotic pseudoconvex functions at point $\bar{y}$ corresponding to M_1. Thus, all the assumptions of Theorem 5.5 are satisfied. Hence, applying Theorem 5.5 to problem (CIVI), we get the result. □

To derive the optimality conditions for a locally LU-efficient solution at $\bar{y}$ to the problem (CIOP), we discuss the following assumptions.

Assumption 6.5.

(i) For $s \in \mathfrak{J}_m$, the functions $\phi_s^L,\ \phi_s^U$ and $h_1,\dots,h_l$ are locally Lipschitz at a point $\bar{y}$. Moreover, h_j (for all j in $\mathfrak{L}_l$) have convexificators $\partial^* h_j(y)$ at a point y near $\bar{y}$, the convexificator map $\partial^* h_j$ is upper semicontinuous at a point $\bar{y}$. The functions $\phi_k^L,\ \phi_k^U$ $(s \neq k \in \mathfrak{J}_m)$, g_i (for every i in $\mathfrak{J}_n(\bar{y})$) are continuous, and $\mathbb{C}$ is taken as convex.

(ii) $\partial^*\phi_k^L(\bar{y}),\ \partial^*\phi_k^U(\bar{y})$ $(s \neq k \in \mathfrak{J}_m)$, $\partial^* g_i(\bar{y})$ (for all i in $\mathfrak{J}_n(\bar{y})$) are upper regular convexificators of $\phi_k^L,\ \phi_k^U$ and g_i at a point $\bar{y}$, respectively.

(iii) The functions $|\Phi_j^L|$ and $|\Phi_j^U|$ for all j in $\mathfrak{L}_l$ are regular at the point $\bar{y}$ in the sense of Clarke [3].

Fritz John type necessary conditions to the problem (CIOP) can be stated as follows:

Theorem 6.6. Let the solution $\bar{y}$ be a locally LU-efficient to (CIOP) and $\bar{y}$ be a regular point for h, corresponding to $\mathbb{C}$. Moreover, if Assumption 6.5 are satisfied, then there exist $\bar{\alpha}_k^L,\ \bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{J}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) satisfying $\sum_{k\in\mathfrak{J}_m}\left(\bar{\alpha}_k^L + \bar{\alpha}_k^U\right) + \sum_{i\in\mathfrak{J}_n(\bar{y})} \bar{\beta}_i = 1$, and

$$0 \in \operatorname{cl}\Big\{ \sum_{k\in\mathfrak{J}_m} \Big(\bar{\alpha}_k^L \operatorname{conv} \partial^*\phi_k^L(\bar{y}) + \bar{\alpha}_k^U \operatorname{conv} \partial^*\phi_k^U(\bar{y})\Big) + \sum_{i\in\mathfrak{J}_n(\bar{y})} \bar{\beta}_i \operatorname{conv} \partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \operatorname{conv} \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Big\}.$$

P r o o f. Observe that under Assumption 6.5, all the hypotheses of Theorem 3.6 hold to the poblem (CIOP) with $\Phi^L(y,z) = \phi^L(z) - \phi^L(y)$ and $\Phi^U(y,z) = \phi^U(z) - \phi^U(y)$ $(y, z \in M)$.$\Phi^L(y,z) = \phi^L(z) - \phi^L(y)$ and $\Phi^U(y,z) = \phi^U(z) - \phi^U(y)$ $(y, z \in M)$. Thus, applying Theorem 3.6 to (CIOP), we get the result. □

Next is the Karush–Kuhn–Tucker type necessary conditions of (CIOP).

Theorem 6.7. Let the solution $\bar{y}$ be a locally LU-efficient to (CIOP). Suppose that the constraint qualification (MFCQ) and the assumptions of Theorem 6.6 hold. Then there exist $\bar{\alpha}_k^L$, $\bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$, at least one being nonzero), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{J}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) such that

$$0 \in \operatorname{cl}\Bigg\{ \sum_{k\in\mathfrak{J}_m} \Big(\bar{\alpha}_k^L \text{ conv } \partial^*\phi_k^L(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\phi_k^U(\bar{y})\Big) + \sum_{i\in\mathfrak{J}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \text{ conv } \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Bigg\}. \tag{25}$$

P r o o f. Since the hypotheses that were used to prove Theorem 4.1 are satisfied, therefore, using Theorem 6.2 and Theorem 4.1, we get the desired result. □

Theorem 6.8. The solution $\bar{y} \in M_1$ becomes weak LU-efficient to the problem (CIOP) provided

(i) there exist $\bar{\alpha}_k^L$, $\bar{\alpha}_k^U \geqq 0$ (for all k in $\mathfrak{J}_m$, at least one being nonzero), $\bar{\beta}_i \geqq 0$ (for all i in $\mathfrak{J}_n(\bar{y})$), $\bar{\rho}_j \in \mathbb{R}$ (for all j in $\mathfrak{L}_l$) such that

$$0 \in \operatorname{cl}\Bigg\{ \sum_{k\in\mathfrak{J}_m} \Big(\bar{\alpha}_k^L \text{ conv } \partial^*\phi_k^L(\bar{y}) + \bar{\alpha}_k^U \text{ conv } \partial^*\phi_k^U(\bar{y})\Big) + \sum_{i\in\mathfrak{J}_n(\bar{y})} \bar{\beta}_i \text{ conv } \partial^* g_i(\bar{y}) + \sum_{j\in\mathfrak{L}_l} \bar{\rho}_j \text{ conv } \partial^* h_j(\bar{y}) + \mathbb{N}_{\mathbb{C}}(\bar{y})\Bigg\},$$

(ii) $\partial^*\phi_k^L(\bar{y})$ and $\partial^*\phi_k^U(\bar{y})$ $(k \in \mathfrak{J}_m)$ are upper regular at point $\bar{y}$ for at most one of the upper convexificators, the function $\bar{\alpha}^L\phi^L := \sum_{k\in\mathfrak{J}_m} \bar{\alpha}_k^L \phi_k^L$ and $\bar{\alpha}^U\phi^U := \sum_{k\in\mathfrak{J}_m} \bar{\alpha}_k^U \phi_k^U$ are asymptotic pseudoconvex at point $\bar{y}$ regarding M_1, g_i are asymptotic quasiconvex at point $\bar{y}$ regarding M_1 (for all i in $\mathfrak{J}_n(\bar{y})$), h_j are asymptotic quasilinear at point $\bar{y}$ regarding M_1 (for all j in $\mathfrak{L}_l$), and $\mathbb{C}$ is convex.

P r o o f. Using Theorem 5.5 and the result $\Phi^L(y,z) = \phi^L(z) - \phi^L(y)$, $\Phi^U(y,z) = \phi^U(z) - \phi^U(y)$ $(y, z \in M)$, we can obtain the result. □

7. CONCLUSIONS

In this article, we have derived Fritz John type necessary conditions with convexificators for the interval-valued programming problem (CIEP), where solutions are considered regular in the sense of Ioffe [9]. The necessary Karush–Kuhn–Tucke–type conditions for locally LU-efficient solutions of the (CIEP) are discussed by applying the Mangasarian–Fromovitz–type constraint qualification (MFCQ). The stronger Mangasarian–Fromovitz–type constraint qualification (SMFCQ) is required to identify the component of the Lagrange multipliers contributed by the objective function. Under appropriate assumptions, combined with generalized convexity, we have established sufficiency criteria for equilibrium problems. Furthermore, we have presented optimality conditions for locally LU-efficient solutions of the interval-valued variational inequality problem (CIVI) and the interval-valued optimization problem (CIOP) under well-suited assumptions.

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