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HYPERBOLIC SUMMATION INVOLVING THE FUNCTION $\Omega(n)$ AND LCM

Meselem Karras and Mihoub Bouderbala

Abstract. We study the sum $\sum_{abc\leq x} \Omega\left([a,b,c]\right)$, where $\Omega(n)$ denotes the number of distinct prime divisors of $n \in \mathbb{Z}_{\geq 1}$ counted with multiplicity, and $[a,b,c] = \operatorname{lcm}(a,b,c)$. An asymptotic formula is derived for this sum over the hyperbolic region $\left\{(a,b,c) \in \mathbb{Z}^3_{\geq 1},\ abc \leq x\right\}$.

Introduction

Let f be an arithmetic function, and for $r \geq 2$, let

$$[n_1, n_2, \dots n_r] = \operatorname{lcm}(n_1, n_2, \dots n_r)\ .$$

Let $\omega(n)$ denote the number of distinct prime divisors of a positive integer $n \geq 1$, and let $\Omega(n)$ the number of prime divisors of n, counted with multiplicity. The problem of finding an asymptotic formula for sums such as

$$\sum_{n_1 n_2 \dots n_r \leq x} f\left(\gcd(n_1, n_2, \dots n_r)\right) \ \text{ or } \sum_{n_1 n_2 \dots n_r \leq x} f\left([n_1, n_2, \dots n_r]\right)$$

where f is a suitable arithmetic function, to be speced below, has been widely studied in number theory. Previous works by researchers such as Heyman and Tóth [1] include sharp results in the two-variable case. Results for the general case are often restricted to multiplicative arithmetic functions under certain conditions, or to additive arithmetic functions for the first type of sum [2]. For the second type of sum, we first restrict our attention to the case $f = \omega^m$ with $m \geq 1$. This is, however, not our main topic in the present paper. Indeed, the case $f = \Omega^m$ where $m \geq 1$, poses substantial challenges and requires different techniques. In the present paper we are able to make a new and hopefully interesting contribution for the case $f = \Omega$, with $r = 3$.

We begin with two important results which are direct applications of two of Ivic's theorems [4].

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Theorem 1. *Let $r \geq 2$ be fixed integer and N be an arbitrary fixed integer, but for which $N > r$. Then there exist computable constants $a_{r,j}$, $b_{r,j}$, $c_{r,j}$ $(a_{r,j} \neq 0)$ such that*

$$\sum_{n_1 n_2 \ldots n_r \leq x} \omega\left([n_1, n_2, \ldots n_r]\right) = x \sum_{j=1}^{N} \left(a_{r,j} \log\log x + b_{r,j}\right) (\log x)^{r-j} + x \sum_{j=r+1}^{N} c_{r,j} (\log x)^{r-j} + O\big(x (\log x)^{r-N-1}\big). \tag{1}$$

Theorem 2. *Let m, $N \geq 1$ and $r \geq 2$ be fixed integers. Then there exist polynomials $P_{r,m,j}(t)$ $(j = 1, 2, \ldots, N)$ of degree m in t with computable coefficients such that*

$$\sum_{n_1 n_2 \ldots n_r \leq x} \omega^m\left([n_1, n_2, \ldots n_r]\right) = x \sum_{j=1}^{N} P_{r,m,j}(\log\log x) (\log x)^{r-j} + O\big(x (\log x)^{r-N-1} (\log\log x)^m\big). \tag{2}$$

The authors in [1, Theorem 2.9] show the first result in the case $r = 2$ and the same result for the function $\Omega(n)$ (see Theorem 2.10).

Proof. The distinct prime divisors of the integer $[n_1, n_2, \ldots, n_r]$ are the same of the integer $n_1 n_2 \ldots n_r$. Then we obtain

$$\omega\left([n_1, n_2, \ldots, n_r]\right) = \omega\left(n_1 n_2 \ldots n_r\right).$$

Therefore, for any integer $m \geq 1$

$$\begin{aligned} \sum_{n_1 n_2 \ldots n_r = n} \omega^m\left([n_1, n_2, \ldots, n_r]\right) &= \sum_{n_1 n_2 \ldots n_r = n} \omega^m(n) \\ &= \omega^m(n) \sum_{n_1 n_2 \ldots n_r = n} 1 \\ &= \omega^m(n)\, \tau_r(n). \end{aligned}$$

Note that, for an integer $r \geq 2$,

$$\tau_r(n) = \sum_{n_1 n_2 \ldots n_r = n} 1$$

is the Piltz divisor function. Thus,

$$\begin{aligned} \sum_{n_1 n_2 \ldots n_r \leq x} \omega^m\left([n_1, n_2, \ldots, n_r]\right) &= \sum_{n \leq x} \sum_{n_1 n_2 \ldots n_r = n} \omega^m(n) \\ &= \sum_{n \leq x} \omega^m(n)\, \tau_r(n). \end{aligned}$$

So, our results are Theorems 1 and 2 in [4]. □

Theorem 3. *We have*

$$\sum_{abc\leq x}\Omega\left([a,b,c]\right)=\frac{3}{2}x\left(\log x\right)^{2}\log\log x+3(b-1)x^{2}\log x+\frac{(C_2-3b)}{2}x\left(\log x\right)^{2}+O\left(x\log x\log\log x\right),$$

where $b=A+\sum_{p}\frac{1}{p\left(p-1\right)}$ *such that* $A=\gamma+\sum_{p}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)$ *and*

$$C_2=\sum_{p}\frac{1}{p^3-1}\approx 0.1941\,.$$

The proof of the theorem is based on the following lemmas:

Lemma 1. *Let* f *be an arithmetic function. Then*

$$\sum_{abc\leq x}f\left([a,b,c]\right)=3\sum_{an\leq x}f\left(a\right)\tau\left(n\right)-3x\sum_{n\leq x}\frac{1}{n}\sum_{ab=n}f\Big(\operatorname{gcd}\left(a,b\right)\Big)+\sum_{abc\leq x}f\left(\operatorname{gcd}\left(a,b,c\right)\right)+O\Big(\sum_{ab\leq x}f\Big(\operatorname{gcd}\left(a,b\right)\Big)\Big),\tag{3}$$

where $\tau\left(n\right)$ *is the number of positive divisors of* n*.*

Proof. Using the inclusion-exclusion principle, we have

$$\begin{aligned}\sum_{abc=n}f\left([a,b,c]\right)&=\sum_{abc=n}f\left(a\right)+\sum_{abc=n}f\left(b\right)+\sum_{abc=n}f\left(c\right)-\sum_{abc=n}f\Big(\operatorname{gcd}\left(a,b\right)\Big)\\&\quad-\sum_{abc=n}f\Big(\operatorname{gcd}\left(a,c\right)\Big)-\sum_{abc=n}f\Big(\operatorname{gcd}\left(b,c\right)\Big)+\sum_{abc=n}f\Big(\operatorname{gcd}\left(a,b,c\right)\Big)\\&=3\sum_{abc=n}f\left(a\right)-3\sum_{abc=n}f\Big(\operatorname{gcd}\left(a,b\right)\Big)\\&\quad+\sum_{abc=n}f\Big(\operatorname{gcd}\left(a,b,c\right)\Big)\,.\end{aligned}$$

Thus, we obtain:

$$\begin{aligned}\sum_{abc\leq x}f\left([a,b,c]\right)&=3\sum_{abc\leq x}f\left(a\right)-3\sum_{abc\leq x}f\Big(\operatorname{gcd}\left(a,b\right)\Big)+\sum_{abc\leq x}f\Big(\operatorname{gcd}\left(a,b,c\right)\Big)\\&=3\sum_{an\leq x}f\left(a\right)\tau\left(n\right)-3\sum_{ab\leq x}f\Big(\operatorname{gcd}\left(a,b\right)\Big)\sum_{c\leq\frac{x}{ab}}1\\&\quad+\sum_{abc\leq x}f\Big(\operatorname{gcd}\left(a,b,c\right)\Big)\,.\end{aligned}$$

Furthermore, since:

$$\sum_{ab\leq x} f\big(\gcd(a,b)\big)\sum_{c\leq \frac{x}{ab}} 1 = x\sum_{ab\leq x}\frac{f\big(\gcd(a,b)\big)}{ab} + O\Big(\sum_{ab\leq x} f\big(\gcd(a,b)\big)\Big)$$
$$= x\sum_{n\leq x}\frac{1}{n}\sum_{ab=n} f\big(\gcd(a,b)\big) + O\Big(\sum_{ab\leq x} f\big(\gcd(a,b)\big)\Big).$$

So, considering this last formula, we obtain the desired result. □

Lemma 2. *For $x\geq 2$, we have*

$$\sum_{an\leq x}\Omega(a)\,\tau(n) = \frac{x}{2}(\log x)^2\log\log x + (b-1)x^2\log x - \frac{b}{2}x(\log x)^2$$
$$+ O\left(x\log x\log\log x\right), \tag{4}$$

where $b = A+\sum_p \frac{1}{p(p-1)}$ *such that* $A = \gamma+\sum_p\Big(\log\Big(1-\frac{1}{p}\Big)+\frac{1}{p}\Big)\approx 0.2614972\ldots.$

Proof. By the well-known estimate formula,

$$\sum_{n\leq x}\tau(n) = x(\log x + C) + O\left(x^{\theta+\varepsilon}\right),$$

where $C = 2\gamma - 1$ and $\frac{1}{4}<\theta<\frac{1}{2}$, we have

$$\sum_{an\leq x}\Omega(a)\,\tau(n) = \sum_{a\leq x}\Omega(a)\sum_{n\leq\frac{x}{a}}\tau(n)$$
$$= \sum_{a\leq x}\Omega(a)\left(\frac{x}{a}\left(\log\frac{x}{a}+C\right)+O\Big(\Big(\frac{x}{a}\Big)^{\theta+\varepsilon}\Big)\right)$$
$$= x(\log x + C)\sum_{n\leq x}\frac{\Omega(n)}{n} - x\sum_{n\leq x}\frac{\Omega(n)\log n}{n} + O\Big(x^{\theta+\varepsilon}\sum_{n\leq x}\frac{\Omega(n)}{n^{\theta+\varepsilon}}\Big).$$

Estimate of the following sums $\sum_{n\leq x}\frac{\Omega(n)}{n}$, $\sum_{n\leq x}\frac{\Omega(n)\log n}{n}$ and $\sum_{n\leq x}\frac{\Omega(n)}{n^{\theta+\varepsilon}}$.

We know that,

$$\sum_{n\leq x}\Omega(n) = x\log\log x + bx + O\Big(\frac{x}{\log x}\Big),$$

where b is given by (4). We use a partial summation, we get

$$\sum_{n\leq x}\frac{\Omega(n)}{n} = (\log x)\log\log x + (b-1)\,x + O(\log\log x).$$

Again by partial summation, we have

$$\sum_{n\leq x}\frac{\Omega(n)\log n}{n} = \log x\sum_{n\leq x}\frac{\Omega(n)}{n} - \int_2^x\frac{1}{t}\Big(\sum_{n\leq t}\frac{\Omega(n)}{n}\Big)dt$$
$$= \frac{1}{2}(\log x)^2\log\log x + \frac{b}{2}(\log x)^2 + O\big((\log x)\log\log x\big),$$

and we have

$$\sum_{n\leq x}\frac{\Omega\left(n\right)}{n^{\theta+\varepsilon}}=O\left(x^{1-\theta-\varepsilon}\log\log x\right)\,.$$

□

Remark 4. We can use the Theorem 3 from [4] with parameters $k=3$, $m=0$ and $r=1$.

Lemma 3. *We have*

$$\sum_{n\leq x}\frac{1}{n}\sum_{ab=n}\Omega\left(\gcd\left(a,b\right)\right)=\frac{C_{\Omega}}{2}\log^{2}\left(x\right)+\left(C_{\Omega}+1\right)\log x+D_{\Omega}+O\left(x^{-1/2}\right)\,, \tag{5}$$

where C_{Ω} and D_{Ω} are two positive constants.

Proof. We apply a partial summation to the estimate (2.16) in [1], we get (5). □

Lemma 4. *We have*

$$\sum_{abc\leq x}\Omega\big(\gcd\left(a,b,c\right)\big)=\frac{C_{2}}{2}x\log^{2}\left(x\right)+O\left(x\log x\right)\,, \tag{6}$$

where $C_{2}=\sum_{p}\frac{1}{p^{3}-1}\approx 0.1941.$

We note that formula (6) has been proved in the general case in [2, Theorem 2.3]. Here we give the explicit form of the polynomial $P_{\Omega,2}\left(x\right)$ of degree 2. First, by Proposition 5.1 in [5], we get

$$\sum_{abc=n}\Omega\big(\gcd\left(a,b,c\right)\big)=\sum_{d^{3}m=n}\left(\mu*\Omega\right)\left(d\right)\tau_{3}\left(m\right)\,,$$

then

$$\begin{aligned}\sum_{abc\leq x}\Omega\big(\gcd\left(a,b,c\right)\big)&=\sum_{d^{3}m\leq x}\left(\mu*\Omega\right)\left(d\right)\tau_{3}\left(m\right)\\&=\sum_{d\leq x^{1/3}}\left(\mu*\Omega\right)\left(d\right)\sum_{m\leq\left(\frac{x}{d}\right)^{1/3}}\tau_{3}\left(m\right)\,.\end{aligned}$$

We use the estimate

$$\sum_{m\leq x}\tau_{3}\left(m\right)=\frac{x\log^{2}x}{2}+O\left(x\log x\right)\,,$$

see, e.g., Nathanson [6, Th 7.6]. According to this estimate

$$\begin{aligned}\sum_{abc\leq x}\Omega\big(\gcd(a,b,c)\big) &= \sum_{d\leq x^{1/3}}(\mu*\Omega)(d)\left(\frac{x}{2d^3}\log^2\left(\frac{x}{d^3}\right)+O\left(\frac{x}{d^3}\log\left(\frac{x}{d^3}\right)\right)\right)\\ &= \frac{x\log^2 x}{2}\sum_{d\leq x^{1/3}}\frac{(\mu*\Omega)(d)}{d^3}+\Big(-3x\log x+\frac{9}{2}x\Big)\\ &\quad\times\sum_{d\leq x^{1/3}}\frac{(\mu*\Omega)(d)\log d}{d^3}\\ &\quad+O\bigg(x\log x\sum_{d\leq x^{1/3}}\frac{(\mu*\Omega)(d)}{d^3}\bigg). \qquad (7)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(\mu*\Omega)(n)}{n^s} &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\times\sum_{n=1}^{\infty}\frac{\Omega(n)}{n^s}\\ &= \frac{1}{\zeta(s)}\times\zeta(s)\sum_{p}\frac{1}{p^s-1}\\ &= \sum_{p}\frac{1}{p^s-1},\quad \operatorname{Re}(s)>1\,.\end{aligned}$$

then

$$\begin{aligned}\sum_{d\leq x^{1/3}}\frac{(\mu*\Omega)(d)}{d^3} &= \sum_{d=1}^{\infty}\frac{(\mu*\Omega)(d)}{d^3}+O\Big(\frac{1}{x^{2/3}}\Big)\\ &= \sum_{p}\frac{1}{p^3-1}+O\Big(\frac{1}{x^{2/3}}\Big)=C_2+O\Big(\frac{1}{x^{2/3}}\Big)\,. \qquad (8)\end{aligned}$$

Using this last estimate with a partial sum, we find

$$\sum_{d\leq x^{1/3}}\frac{(\mu*\Omega)(d)\log d}{d^3}=O(1) \qquad (9)$$

According (8) and (9) in (7), therefore we obtain (6).

Proof of Theorem 3. By Lemma 1, we have

$$\begin{aligned}\sum_{abc\leq x}\Omega([a,b,c]) &= 3\sum_{an\leq x}\Omega(a)\tau(n)-3x\sum_{n\leq x}\frac{1}{n}\sum_{ab=n}\Omega\big(\gcd(a,b)\big)\\ &\quad+\sum_{abc\leq x}\Omega\big(\gcd(a,b,c)\big)+O\bigg(\sum_{ab\leq x}\Omega\big(\gcd(a,b)\big)\bigg),\end{aligned}$$

and by Lemmas 2, 3 and 4 we get

$$\sum_{abc\leq x} \Omega\left([a,b,c]\right) = \frac{3}{2}x\left(\log x\right)^2 \log\log x + 3(b-1)x^2\log x + \frac{(C_2-3b)}{2}x\left(\log x\right)^2 + O\big(x\log x\log\log x\big)\,,$$

which concludes the proof. □

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