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The Cayley-Dickson Construction in Homotopy Type Theory

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Abstract

We define in the setting of homotopy type theory an H-space structure on $\mathbb{S}^3$. Hence we obtain a description of the quaternionic Hopf fibration $\mathbb{S}^3 \hookrightarrow \mathbb{S}^7 \twoheadrightarrow \mathbb{S}^4$, using only homotopy invariant tools. A side benefit is that the construction applies to more general ∞-categories than that of spaces.

1. Introduction

Homotopy type theory is the study of a range of homotopy theoretical interpretations of Martin-Löf dependent type theory [10] as well as an exploration in doing homotopy theory inside type theory [17]. This paper concerns the latter aspect, in that we give a purely type theoretic definition of the quaternionic Hopf fibration

$$\mathbb{S}^3 \lhook\joinrel\longrightarrow \mathbb{S}^7 \longrightarrow\joinrel\twoheadrightarrow \mathbb{S}^4.$$

Classically, the 3-sphere can be given the H-space structure given by multiplication of the quaternions of norm 1. For any H-space A, the Hopf construction produces a fibration

$$A \lhook\joinrel\longrightarrow A * A \longrightarrow\joinrel\twoheadrightarrow SA,$$

where $A * A$ denotes the *join* of A with itself, and SA denotes the *suspension* of A. Hence we get the quaternionic Hopf fibration from the H-space structure on $\mathbb{S}^3$ and the Hopf construction.

The Hopf construction has already been developed in homotopy type theory [17, Theorem 8.5.11], so what is needed is to construct the H-space structure on $\mathbb{S}^3$.

When doing homotopy theory in homotopy type theory, we reason directly with (homotopy) types and not with any mediating presentation of these, e.g., as topological spaces or simplicial sets. For example, the spheres are defined as iterated suspensions of the empty type (which represents the (-1)-sphere), rather than subsets of the Euclidean spaces $\mathbb{R}^n$. Therefore, we cannot directly reproduce the classical construction of the H-space structure on $\mathbb{S}^3$, viewed as a subspace of the quaternions.

Instead, our approach in this paper is to find a type-theoretic incarnation of the Cayley-Dickson construction that is used to form the classical algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

It is a natural idea to work instead with the unit spheres inside these algebras and mimic the Cayley-Dickson construction on this level. This almost works, but to complete the construction we need to take one more step back and work with the spheres of *unit imaginaries*.

Our construction of an H-space structure on $\mathbb{S}^3$ uses very little type theoretic machinery, and we expect it can be carried out in any lex ∞-category with homotopy pushouts stable under pullback, for instance any ∞-topos. The full power of an ∞-topos is only needed to go from the H-space structure to the quaternionic Hopf fibration itself.

The type theoretic construction has the benefit of making it simpler to reason about the required higher dimensional filling problems. A second advantage is that it can be formally verified, and our construction has been formalized and checked in the Lean proof assistant [11]. In fact, we developed the results while formalizing them, and the proof assistant was helpful as a tool to develop the mathematics. Our formalization is available as part of the homotopy type theory library for Lean at `https://github.com/leanprover/lean2/`.[1]

The rest of this paper is organized as follows. In Section 2 we recall the classical Cayley-Dickson construction. We then recall the story of the Hopf construction in homotopy type theory in Section 3. In the main section, Section 4, we discuss how to port the Cayley-Dickson construction to type theory in order to construct the H-space structure on $\mathbb{S}^3$. Having performed the construction in homotopy type theory, in Section 5 we discuss the range of models in which the construction can be performed. We conclude in Section 6.

Before we begin, let us address a possible route to our result which we have not taken: It might seem as if the best way to reason about quaternions and related algebraic structures in homotopy type theory would be to construct them in the usual set-theoretic way but such that we could still access, say, the underlying homotopy type of the unit sphere. Indeed, this would be possible to do in cohesive homotopy type theory [14]. However, as of now there is no known interpretation of cohesive homotopy type theory into ordinary homotopy type theory preserving homotopy types, so this would not give a construction in ordinary homotopy type theory. And even if such an interpretation were possible, it might require more machinery to develop than what is used here.

2. The classical Cayley-Dickson construction

Classically, the 1-, 3- and 7-dimensional spheres are subspaces of $\mathbb{R}^2$, $\mathbb{R}^4$ and $\mathbb{R}^8$, respectively. Each of these vector spaces can be given the structure of a normed division algebra, and we get the complex numbers $\mathbb{C}$, Hamilton's quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$ of Graves and Cayley.

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[1]See the files `imaginaroid.hlean` and `quaternionic_hopf.hlean` in the `hott/homotopy/` subdirectory.

Since, in each of these algebras, the product preserves norm, the unit sphere is a subgroup of the multiplicative group.

Cayley's construction of the octonions was later generalized by Dickson [7], who gave a uniform procedure for generating each of these algebras from the previous one. The process can be continued indefinitely, giving for instance the 16-dimensional sedenion-algebra after the octonions.

Here we describe one variant of the Cayley-Dickson construction, following the presentation in [2]. For this purpose, let an *algebra* be a vector space A over $\mathbb{R}$ together with a bilinear multiplication, which need not be associative, and a unit element 1. A $*$-*algebra* is an algebra equipped with a linear involution $*$ (called the *conjugation*) satisfying $1^* = 1$ and $(ab)^* = b^*a^*$.

If A is a $*$-algebra, then $A' := A \oplus A$ is again a $*$-algebra using the definitions

$$(a,b)(c,d) := (ac - db^*, a^*d + cb), \quad 1 := (1,0), \quad (a,b)^* := (a^*, -b). \tag{1}$$

If A is *nicely normed* in the sense that (i) for all a, we have $a + a^* \in \mathbb{R}$ (i.e., the subspace spanned by 1), and (ii) $aa^* = a^*a > 0$ for nonzero a, then so is A'. In the nicely normed case, we get a norm by defining

$$\|a\| = aa^*,$$

and we have inverses given by $a^{-1} = a^*/\|a\|$. By applying this construction repeatedly, starting with $\mathbb{R}$, we obtain the following sequence of algebras, each one having slightly fewer good properties than the preceding one:

- $\mathbb{R}$ is a *real* (i.e., $a^* = a$) commutative associative nicely normed $*$-algebra,
- $\mathbb{C}$ is a commutative associative nicely normed $*$-algebra,
- $\mathbb{H}$ is an associative nicely normed $*$-algebra,
- $\mathbb{O}$ is an *alternative* (i.e., any subalgebra generated by two elements is associative) nicely normed $*$-algebra,
- the sedenions and the following algebras are nicely normed $*$-algebras, which are neither commutative, nor alternative.

Being alternative, the first four are normed division algebras, as a, b, a^*, b^* are in the subalgebra generated by $a - a^*$ and $b - b^*$, so we get

$$\|ab\|^2 = (ab)(ab)^* = (ab)(b^*a^*) = a(bb^*)a^* = \|a\|^2\|b\|^2.$$

However, starting with the sedenions, this fails and we get nontrivial zero divisors. In fact, the zero divisors of norm one in the sedenions form a group homeomorphic to the exceptional Lie group G_2.

To sum up the story as it relates to us, we first form the four normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ by applying the Cayley-Dickson construction starting with $\mathbb{R}$, and then we carve out the unit spheres and get spaces with multiplication $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$.

In homotopy type theory, we cannot use this strategy directly. Before we discuss our alternative construction, let us recall some basics regarding H-spaces in homotopy type theory.

3. H-spaces and the Hopf construction

First, let us briefly recall that the (homotopy) pushout $A \sqcup^C B$ of a span

$$A \xleftarrow{\ f\ } C \xrightarrow{\ g\ } B$$

can be modeled in homotopy type theory as a higher inductive type with *point constructors* $\mathsf{inl} : A \to A \sqcup^C B$ and $\mathsf{inr} : B \to A \sqcup^C B$ and a *path constructor* $\mathsf{glue} : \Pi_{(c:C)} \mathsf{inl}(f(c)) = \mathsf{inr}(g(c))$. The *suspension SA* of a type A is the pushout of the span $1 \leftarrow A \to 1$, which is equivalently described as the higher inductive type with point constructors N and S (corresponding to left and right injections) and path constructor $\mathsf{merid} : A \to (\mathsf{N} = \mathsf{S})$, generating the meridians in the suspension. The *join $A * B$* of two types A and B is the pushout of the span $A \leftarrow A \times B \to B$ given by the projections, which is equivalently described as the higher inductive type with point constructors $\mathsf{inl} : A \to A * B$, $\mathsf{inr} : B \to A * B$ and path constructor $\mathsf{glue} : \Pi_{(a:A)} \Pi_{(b:B)} \mathsf{inl}(a) = \mathsf{inr}(b)$.

In [17], an H-space[2] in homotopy type theory is defined to consist of a pointed type (A, e) with a multiplication $\mu : A \times A \to A$ and equalities $\lambda_a : \mu(e, a) = a$ and $\rho_a : \mu(a, e) = a$ for all $a : A$. However, for the Hopf construction it is useful to require that the left and right *translation maps*, $\mu(a, -)$ and $\mu(-, a)$, are equivalences for all $a : A$. This is automatic if A is connected, but also holds, e.g., if the induced multiplication makes $\pi_0(A)$ into a group.

Definition 3.1. An *H-space* is a pointed type (A, e) with a multiplication $\mu : A \times A \to A$, and homotopies $\mu(e, -) \sim \mathsf{id}_A$ and $\mu(-, e) \sim \mathsf{id}_A$.[3]

In section 8.5.2 of [17], there is a description of the Hopf construction, which takes a connected H-space A, and produces a type family H over SA by letting the fibers over N and S be A, and giving the equivalence $\mu(a, -)$ for the meridian $\mathsf{merid}(a)$. The total space is then shown to be the join $A * A$ of A with itself. The projection map $A * A \to SA$ can be taken to send the left component to N, the right component to S, and for $a, b : A$ the glue path between $\mathsf{inl}\, a$ and $\mathsf{inr}\, b$ to the meridian $\mathsf{merid}(\mu(a, b))$.[4] With the described map $A * A \to SA$, we have a commuting triangle

$$
\begin{array}{ccc}
\Sigma_{(x:SA)} H(x) & \xrightarrow{\;\simeq\;} & A * A \\
& \searrow \qquad \swarrow & \\
& SA. &
\end{array}
$$

Note that the only point in the Hopf construction where the connectedness of A is used, is to conclude that $\mu(a, -)$ and $\mu(-, a)$ are equivalences for each $a : A$. Hence the Hopf construction also works if we make this requirement directly, so that it becomes applicable in a slightly more general setting including the H-space $\mathbb{S}^0$.

Lemma 3.2 (The Hopf construction)**.** *Let A be an H-space for which the translation maps $\mu(a, -)$ and $\mu(-, a)$ are equivalences, for each $a : A$. Then there is a fibration $H : SA \to U$ such that*

$$
H(\mathsf{N}) = H(\mathsf{S}) = A, \qquad and \qquad \left(\Sigma_{(x:SA)} H(x)\right) = A * A.
$$

This is Lemma 8.5.7 of [17], and it follows from the proof given there, that we get a fibration sequence $A \to A * A \to SA$ where we may take the first map to be one of the inclusions.

We also recall that the join operation on types is associative (Lemma 8.5.9 of [17]), and that the suspension SA of A is the join $\mathbb{S}^0 * A$ (Lemma 8.5.10 of [17]). In particular, it follows that

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[2]Serre [13] introduced H-spaces in honor of Hopf.

[3]Sometimes, the multiplication on an H-space is required to give a pointed map $A \times A \to A$. We could achieve that up to homotopy by additionally requiring an equality $\mathsf{coh} : \lambda_e = \rho_e$, but we do not need this hypothesis in the present work.

[4]With the precise equivalence $(\Sigma_{(x:SA)} H(x)) \simeq (A * A)$ from [17] this will be mirrored.

$\mathbb{S}^{2n+1} \simeq \mathbb{S}^n * \mathbb{S}^n$, for any $n : \mathbb{N}$. To give the four Hopf fibrations in homotopy type theory, it thus suffices to give the H-space structures on the spheres $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$.

For $\mathbb{S}^0$, i.e., the group $\mathbb{Z}/2\mathbb{Z}$, this is trivial, and in the case of the circle $\mathbb{S}^1$, this has already been formalized and appears in [17]. In the next section we shall see how to construct the H-space structure on $\mathbb{S}^3$.

4. Spheroids and imaginaroids

We saw in Section 2 the classical Cayley-Dickson construction on the level of $*$-algebras. We would obtain nothing of interest by imitating this directly in homotopy type theory, as any real vector space is contractible and thus equivalent to the one-point type 1.

A first idea, which turns out to not quite work, is to give an analog of the Cayley-Dickson construction on the level of the unit spheres inside the $*$-algebras, as what we are ultimately after is the H-space structure on these unit spheres. Thus we propose:

Definition 4.1. A *Cayley-Dickson spheroid*[5] consists of an H-space S (we write 1 for the base point, and concatenation denotes multiplication) with additional operations

$$x \mapsto x^* \qquad\qquad (conjugation)$$
$$x \mapsto -x \qquad\qquad (negation)$$

satisfying the further laws

$$
\begin{aligned}
1^* &= 1 & (-x)^* &= -x^* \\
-(-x) = x &= x^{**} & x(-y) &= -xy \\
(xy)^* &= y^* x^* & x^* x &= 1.
\end{aligned}
$$

Lemma 4.2. *For any two points x and y of a Cayley-Dickson spheroid, we have $xx^* = 1$ and $(-x)y = -xy$.*

Proof. For the first, simply note that $xx^* = x^{**}x^* = 1$. For the second, we have:

$$
\begin{aligned}
(-x)y &= ((-x)y)^{**} & \cdots &= (-y^* x^*)^* \\
&= (y^*(-x)^*)^* & &= -(y^* x^*)^* \\
&= (y^*(-x^*))^* & &= -(xy)^{**} \\
&= \cdots & &= -xy. \qquad \square
\end{aligned}
$$

The hope is now that if S is an *associative* Cayley-Dickson spheroid, then we can give the join $S * S$ the structure of a Cayley-Dickson spheroid. This turns out not quite to work, but it is instructive to see where we get stuck.

We wish to define the multiplication xy for $x, y : S * S$ by induction on x and y. To do the induction on x we must define elements $(\mathsf{inl}\,a)y$, $(\mathsf{inl}\,b)y$ and paths $(\mathsf{glue}\,a\,b)_*y : (\mathsf{inl}\,a)y = (\mathsf{inr}\,b)y$ for $a, b : S$. This is of course the same as giving the two bent arrows such that the outer square commutes in following diagram, where the inner square is the pushout square defining $S * S$

---

[5]We use the term "spheroid" to emphasize that S is to be thought of as a unit sphere, but we do not require S to be an actual sphere.

pulled back along the projection $S * S \to 1$ corresponding to the variable y. The dotted arrow is the desired multiplication map:

$$
\begin{array}{ccc}
S \times S \times (S * S) & \longrightarrow & S \times (S * S) \\
\downarrow & & \downarrow \\
S \times (S * S) & \longrightarrow & (S * S) \times (S * S) \\
& & \searrow \quad \dashrightarrow \\
& & S * S
\end{array}
$$

In each case we do an induction on y, giving the following point constructor problems, which we solve using equation (1):

$$(\mathsf{inl}\, a)(\mathsf{inl}\, c) := \mathsf{inl}(ac) \quad (\mathsf{inl}\, a)(\mathsf{inr}\, d) := \mathsf{inr}(a^* d)$$
$$(\mathsf{inr}\, b)(\mathsf{inl}\, c) := \mathsf{inr}(cb) \quad (\mathsf{inr}\, b)(\mathsf{inr}\, d) := \mathsf{inl}(-db^*)$$

We must define four dependent paths corresponding to the interaction of a point constructor with a path constructor, and these we all fill with glue (or its inverse). There results a dependent path problem in an identity type family, which we can think of as the problem of filling the square on the left, also depicted on the right as a *diamond*:

$$
\begin{array}{ccc}
\mathsf{inl}(ac) \xlongequal{\;\mathsf{glue}\;} \mathsf{inr}(cb) \\
{\scriptstyle\mathsf{glue}}\| \qquad\qquad \|{\scriptstyle\mathsf{glue}^{-1}} \\
\mathsf{inr}(a^* d) \xlongequal[\;\mathsf{glue}^{-1}\;]{} \mathsf{inl}(-db^*)
\end{array}
\qquad
\begin{array}{c}
cb \\
-db^* \diamond ac \\
a^* d
\end{array}
\tag{2}
$$

These diamond shapes will play an important role in the construction. We can define these diamond types as certain square types sitting in a join, $A * B$, for any $a, a' : A$ and $b, b' : B$:

$$
\begin{array}{ccc}
a \xlongequal{\;\mathsf{glue}\;} b \\
{\scriptstyle\mathsf{glue}}\| \qquad\qquad \|{\scriptstyle\mathsf{glue}^{-1}} \\
b' \xlongequal[\;\mathsf{glue}^{-1}\;]{} a'
\end{array}
\qquad
\begin{array}{c}
b \\
a' \diamond a \\
b'
\end{array}
\tag{3}
$$

The geometric intuition behind the shape is that we picture the join $A * B$ as A lying on a horizontal line, B on a vertical line, and glue-paths connecting every point in A to every point in B.

Definition 4.3. Given a diamond problem corresponding to $a, a' : A$ and $b, b' : B$ as in (3), if we have either a path $p : a =_A a'$ or a path $q : b =_B b'$, then we can solve it (i.e., fill the square on the left).

Construction. By path induction on p resp. q followed by easy 2-dimensional box filling. $\qquad\square$

Definition 4.4. Given types A_1, A_2, B_1, B_2 and functions $f : A_1 \to A_2$ and $g : B_1$ to B_2, if we have a solution to the diamond problem in $A_1 * B_1$ given by $a, a' : A_1$, $b, b' : B_1$, then we apply the induced function $f * g : A_1 * B_1 \to A_2 * B_2$ to obtain a solution to the diamond problem in $A_2 * B_2$ given by $f\, a, f\, a' : A_2$, $g\, b, g\, b' : B_2$:

$$
\begin{array}{c}
b \\
a' \diamond a \\
b'
\end{array}
\qquad \mapsto \qquad
\begin{array}{c}
g\, b \\
f\, a' \diamond f\, a \\
g\, b'
\end{array}
$$

Construction. This is an instance of applying a function to a square.   □

Coming back to (2) and fixing $a, b, c, d : S$, consider the functions $f, g : S \to S$:

$$f(x) := -acx, \qquad g(y) := cyb$$

(we are leaving out the parentheses since we are assuming the multiplication is associative).

Lemma 4.5. *If the multiplication is associative, then we have* $f(-1) = ac$, $f(c^*a^*db^*) = -db^*$, $g(1) = cb$, *and* $g(c^*a^*db^*) = a^*d$.

Proof. For example,

$$
\begin{aligned}
ac(-c^*a^*db^*) &= -acc^*a^*db^* & \cdots &= -(aa^*)db^* \\
&= -a(cc^*)a^*db^* & &= -1db^* \\
&= -a1a^*db^* & &= -db^*.
\end{aligned}
$$ □

Thus, it suffices to solve the diamond problem,

with $x = c^*a^*db^*$. Naively, we might hope to solve this problem for every $x : S$. However, considering the case where S is the unit 0-sphere $\{\pm 1\}$ in $\mathbb{R}$, it seems necessary to make a case distinction on x to do so. This motivates the following revised strategy.

4.1   Cayley-Dickson imaginaries   Instead of just axiomatizing the unit sphere, we shall make use of the fact that all the unit spheres in the Cayley-Dickson algebras are suspensions of the unit sphere of imaginaries (the unit 0-sphere in $\mathbb{R}$ is of course the suspension of the -1-sphere, i.e., the empty type, which corresponds to the fact the $\mathbb{R}$ is a *real* algebra with no imaginaries).

First we note that both conjugation and negation on a Cayley-Dickson sphere are determined by the negation acting on the imaginaries. In fact, we can make the following general constructions:

Definition 4.6. Suppose A is a type with a negation operation. Then we can define a conjugation and a negation on the suspension SA of A:

$$
\begin{aligned}
\mathsf{N}^* &:= \mathsf{N} & -\mathsf{N} &:= \mathsf{S} \\
\mathsf{S}^* &:= \mathsf{S} & -\mathsf{S} &:= \mathsf{N}
\end{aligned}
$$

$$\mathsf{ap}\,(\lambda x.x^*)\,(\mathsf{merid}\,a) := \mathsf{merid}(-a) \quad \mathsf{ap}\,(\lambda x.-x)\,(\mathsf{merid}\,a) := \mathsf{merid}(-a)^{-1}$$

We give SA the base point N, which we also write as 1. If the negation on A is involutive, then so is the conjugation and negation on SA.

Definition 4.7. A *Cayley-Dickson imaginaroid* consists of a type A with an involutive negation, together with a binary multiplication operation on the suspension SA, such that SA becomes an H-space satisfying the *imaginaroid laws*
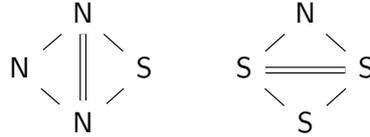
$$
\begin{aligned}
x(-y) &= -xy \\
xx^* &= 1 \\
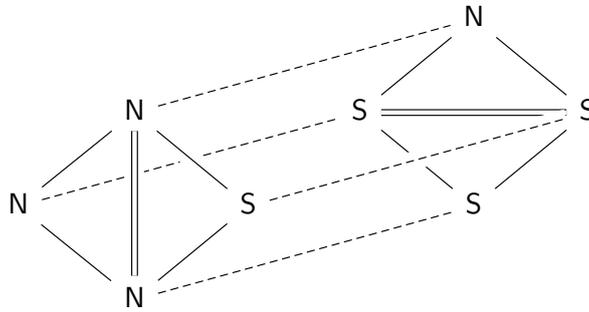(xy)^* &= y^*x^*
\end{aligned}
$$

for $x, y : SA$.

Note that if A is a Cayley-Dickson imaginaroid, then SA becomes a Cayley-Dickson spheroid.

Definition 4.8. Let A be a Cayley-Dickson imaginaroid where the multiplication on SA is associative. Then $A' := SA * SA$ can be given the structure of an H-space.

Construction. We can define the multiplication on $SA * SA$ as in the previous section, leading to the diamond problem (4). This we now solve by induction on $x : SA$. The diamonds for the poles are easily filled using Definition 4.3:

$$
\begin{array}{ccccccc}
 & \mathsf{N} & & & & \mathsf{N} & \\
\mathsf{N} & \| & \mathsf{S} & \quad & \mathsf{S} & = & \mathsf{S} \\
 & \mathsf{N} & & & & \mathsf{S} &
\end{array}
$$

These solutions must now be connected by filling, for every $a : A$, the following hollow cube connecting the diamonds:

$$
\begin{array}{ccc}
 & & \mathsf{N} \\
\mathsf{N} & \quad \mathsf{S} = \mathsf{S} & \\
\mathsf{N} & \quad \mathsf{S} \quad \mathsf{S} & \\
 & \mathsf{N} &
\end{array}
$$

Here, the two dashed paths $\mathsf{N} = \mathsf{N}$ and $\mathsf{S} = \mathsf{S}$ are identities, while the other two are each the meridian, $\mathsf{merid}\, a : \mathsf{N} = \mathsf{S}$. Generalizing a bit, we see that we can fill any cube in a symmetric join, $B * B$, with $p : x =_B y$, of this form:

$$
\begin{array}{ccc}
 & & x \\
x & \quad y = y & \\
x & \quad y \quad y & \\
 & x &
\end{array}
$$
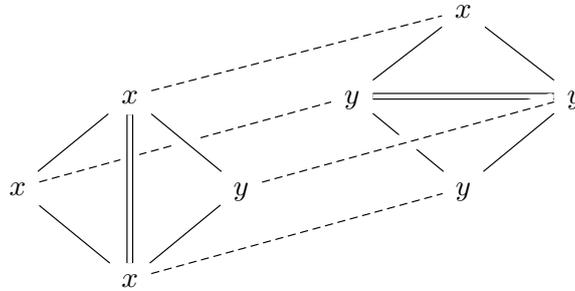
Indeed, this follows by path induction on p followed by trivial manipulations.

This multiplication has the virtue that the H-space laws $1x = x1 = x$ are very easy to prove; indeed, for point constructors they follow from the H-space laws on SA, and since these point constructors land in the two different sides of the join, we can glue them together trivially on path constructors. $\qquad\square$

Let us finish this section by stating the result of combining the Hopf construction (Lemma 3.2) and the H-space structure on $\mathbb{S}^3$, which we obtain from Definition 4.8 using the obvious imaginaroid structure on $\mathbb{S}^0$ and the associativity of the H-space structure on $\mathbb{S}^1 = S\mathbb{S}^0$:

Theorem 4.9. *There is a fibration sequence*

$$\mathbb{S}^3 \to \mathbb{S}^7 \to \mathbb{S}^4$$

of pointed maps.

Corollary 4.10. *There is an element of infinite order in $\pi_7(\mathbb{S}^4)$.*

Proof. Consider the long exact sequence of homotopy groups [17, Theorem 8.4.6] corresponding to the above fibration sequence. In particular, we get the exactness of

$$\pi_7(\mathbb{S}^3) \to \pi_7(\mathbb{S}^7) \to \pi_7(\mathbb{S}^4).$$

The inclusion of the fiber, $\mathbb{S}^3 \hookrightarrow \mathbb{S}^3 * \mathbb{S}^3 = \mathbb{S}^7$, is nullhomotopic, so the first map is zero. Since $\pi_7(\mathbb{S}^7) = \mathbb{Z}$, we get an exact sequence

$$0 \to \mathbb{Z} \to \pi_7(\mathbb{S}^4),$$

which gives the desired element of infinite order.   □

5. Semantics

One expects that anything that is done in homotopy type theory, can also be done in most $(\infty, 1)$-toposes. However, general $(\infty, 1)$-topos semantics of homotopy type theory is currently still conjectural. Nonetheless, there is semantics for homotopy type theory in the usual $(\infty, 1)$-topos of ∞-groupoids (in terms of simplicial sets [8], and in cubical sets [4, 6]), and in certain presheaf $(\infty, 1)$-toposes [15, 16].

On the other hand, given a particular construction in homotopy type theory, one can investigate what semantics is needed to perform the construction in other 'homotopy theories', for instance in $(\infty, 1)$-categories presented by (Quillen) model categories. An example of this kind is given by [12], who translated the formalized proof of the Blakers-Massey theorem to obtain a new, purely homotopy theoretic proof in the category of spaces. In this section, we describe what seems to be needed to perform (i) the construction of the H-space structure on $\mathbb{S}^3$ (Section 4), and (ii) the Hopf construction (Lemma 3.2).

The Hopf construction requires some form of univalence, for instance an object classifier as in an $(\infty, 1)$-topos. For any H-space A we always have a map $A * A \to SA$, but in general the homotopy fiber may fail to be A (consider, e.g., $\mathbb{S}^0$ in the category of sets equipped with the trivial model structure).

Observe that for the construction of the H-space structure on $\mathbb{S}^3$, we have used only a small fragment of homotopy type theory. We have used dependent sums, identity types, and homotopy pushouts. And for the latter, we only use the elimination principle into small types (so a universe is not needed). A priori we also use the inductive families of squares and cubes (of paths in a type), but these can be equivalently defined in terms of identity types, see the next subsection 5.1.

In general, to model dependent sums and identity types in a Quillen model category, some extra coherence is needed [1, 3]. However, to reproduce a particular type theoretic construction, this extra coherence may not be needed. Since a Quillen model category has homotopy pushouts, an empty space and a unit space, it also has the n-spheres. The construction corresponding to Definition 4.8 only uses finite homotopy colimits and their universal properties, but in arbitrary contexts. Therefore, we expect that the construction of the H-space structure on $\mathbb{S}^3$ is possible in any Quillen model category presenting a finitely homotopy cocomplete ∞-category in which the homotopy pushouts are stable under pullback.

5.1 The cubical machinery In the formalization we use the cubical methods of [9], which consists in using inductively defined families of square, cubes, squareovers, etc. These are available in any model category (up to pullback stability), because there are alternative definitions just in terms of identity types and dependent sums.

Consider for instance the type of squares in a type A. These are parameterized by the top-left corner $a_{00} : A$. The dependent sum type

$$B := \Sigma_{(a_{02}:A)} \Sigma_{(a_{20}:A)} \Sigma_{(a_{22}:A)} (a_{00} = a_{02}) \times (a_{20} = a_{22}) \times (a_{00} = a_{20}) \times (a_{02} = a_{22})$$

describes the type of boundaries of squares in A with top-left corner a_{00}. There is an obvious element $r := (a_{00}, a_{00}, a_{00}, 1_{a_{00}}, 1_{a_{00}}, 1_{a_{00}}, 1_{a_{00}})$ representing the trivial boundary. Now the type of squares with boundary $b : B$ can be represented simply as the identity type $(b = r)$. The representation of cubes and squareovers proceeds in a similar manner.

We are grateful to Christian Sattler for this observation, which derives from considerations of the Reedy fibrant replacement of the constant diagram over the semi-cubical indexing category.

6. Conclusion

One might also wonder whether our construction applies to other H-spaces in the usual homotopy category besides the spheres $\mathbb{S}^0$, $\mathbb{S}^1$, and $\mathbb{S}^3$, in other words, what are the associative imaginaroids in ordinary homotopy theory?

We are grateful to Mark Grant and Qiaochu Yuan for the following observations (in response to a question on MathOverflow [5]). If a space X is a suspension, then it is automatically a co-H-space, and [18] proved that the only finite complexes which are both H-spaces and co-H-spaces are the spheres $\mathbb{S}^0$, $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$. Beyond the finite complexes, note that the rationalization $\mathbb{S}_{\mathbb{Q}}^{2n+1}$ of any odd-dimensional sphere is an associative H-space that is also a suspension, but in this case we already know that the join $\mathbb{S}_{\mathbb{Q}}^{2n+1} * \mathbb{S}_{\mathbb{Q}}^{2n+1} \simeq \mathbb{S}_{\mathbb{Q}}^{4n+3}$ is again an H-space. It remains to be seen whether there are non-trivial applications in other homotopy theories.

The classical Cayley-Dickson construction gives more than just the H-space structure on $\mathbb{S}^3$, namely it presents $\mathbb{S}^3$ as the topological group $Sp(1)$ (which is also $SU(2)$). Topological groups can be represented in homotopy type theory via their classifying types, but we do not know how to define a delooping of $\mathbb{S}^3$ in homotopy type theory (classically this would be the infinite-dimensional quaternionic projective space $\mathbb{H}P^\infty$).

One of the other fascinating aspects of the classical Cayley-Dickson construction is of course that it can be iterated. Our construction as it stands does not allow for iteration, and of course we can not expect it to be indefinitely applicable as we need the associativity condition. However, it is conceivable that for a strengthened notion of imaginaroid A (including some coherence conditions on the algebraic structure), we could have that $A * SA$ is again an imaginaroid. We state this as:

Conjecture 6.1. *Suppose A is a Cayley-Dickson imaginaroid where the multiplication on SA is associative and some further (for now unspecified) coherence conditions hold. Then $A' := A * SA$ can also be given the structure of a Cayley-Dickson imaginaroid, which is associative if A is furthermore commutative.*

We get of course a negation on A' in a canonical way using the negations on A and SA. Using associativity of join and the fact that $\mathbb{S}^0 * X = SX$ for any X, we get $SA' = \mathbb{S}^0 * (A * SA) = (\mathbb{S}^0 * A) * SA = SA * SA$. Thus the multiplication on SA comes from the previous construction.

The hard part is then to verify the algebraic laws, which is where we expect that coherence conditions on the algebraic structure for A will come in.

Proving this conjecture and that the further conditions hold for the case of $\mathbb{S}^1$ would be one way to obtain the H-space structure on $\mathbb{S}^7$ in homotopy type theory, but we leave this to future work.

Another byproduct of the classical Cayley-Dickson construction is that we find the exceptional Lie group G_2 as the zero divisors in the sedenions. Unfortunately, there seems to be no hope for our current approach to yield such fruits.

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