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Abstract

We compare several recent approaches to studying right Bousfield localization and algebras over monads. We prove these approaches are equivalent, and we apply this equivalence to obtain several new results regarding right Bousfield localizations (some classical, some new) for spectra, spaces, equivariant spaces, chain complexes, simplicial abelian groups, and the stable module category. En route, we provide conditions so that right Bousfield localization lifts to categories of algebras, so that right Bousfield localization preserves algebras over monads, and so that right Bousfield localization forms a compactly generated model category.

1. Introduction

Bousfield localization is a valuable technique for creating new homotopy theories; it has been extensively studied in the context of model categories, infinity categories, triangulated categories, classical homotopy theory, and group theory. Both left and right Bousfield localization invert morphisms: in the context of model categories both results in a larger class of weak equivalences. Left Bousfield localization begins with a model category $\mathcal{M}$ and a set of maps $\mathcal{C}$, and results in a new model structure $L_{\mathcal{C}}\mathcal{M}$ on the same category, with the same cofibrations, where $\mathcal{C}$ is now contained in the new weak equivalences. Right Bousfield localization (also known as cellularization or colocalization) begins with a model category $\mathcal{M}$ and a set of objects K, and results in a new model structure $R_K\mathcal{M}$, with the same fibrations, where K is now contained in the class of cofibrant objects and where morphisms seen to be weak equivalences by K (via

homotopy function objects) are now weak equivalences. In both cases, hypotheses are needed on $\mathcal{M}$ to guarantee existence of the localization.

Left Bousfield localization is better behaved than right Bousfield localization. For example, $L_\mathcal{C}(\mathcal{M})$ is guaranteed to be cofibrantly generated if $\mathcal{M}$ is, but the same is not true for $R_K\mathcal{M}$. Nevertheless, right Bousfield localization plays an important role in homotopy theory dating back to CW approximation (more generally, A-cellular homotopy theory [11]), n-connected covers and Postnikov pieces [28], and finding point-set models in chain complexes and R-modules for localizing subcategories in the derived category of R and the stable module category of R. A comprehensive list of applications can be found in [39]. Due to the asymmetry between left and right Bousfield localization, more attention has been paid in the literature to left localization, and several different approaches to right localization have emerged (e.g. [22], [2], [12]).

This paper unifies several different approaches [10, 21, 39] to questions related to preservation of algebraic structure under right Bousfield localization. All approaches considered are recalled here, so this paper can be read as a stand-alone paper. The main goal is to prove the following theorem, which contains a converse to the main result of [39]. Here $\mathcal{M}$ is a model category, K is a set of cofibrant objects in $\mathcal{M}$, $R_K\mathcal{M}$ is the right Bousfield localization of $\mathcal{M}$ with respect to K, and T is a monad on $\mathcal{M}$. The category of T-algebras is known as the Eilenberg-Moore category associated to T.

Theorem A. Under Assumption 2.4 suppose further that $\mathsf{Alg}(T; R_K\mathcal{M})$ is semi-admissible over $R_K\mathcal{M}$. Then the following statements are equivalent.

1. The forgetful functor

$$\mathsf{Alg}(T; R_K\mathcal{M}) = R_{T(K)}\mathsf{Alg}(T; \mathcal{M}) \xrightarrow{\ U\ } R_K\mathcal{M}$$

   preserves weak equivalences and cofibrant objects, in which the equality on the left is from Corollary 2.7.
2. R_K preserves T-algebras (Def. 5.2).
3. R_K lifts to the homotopy category of T-algebras (Def. 5.1).
4. The forgetful functor U preserves right Bousfield localization (Def. 5.3).

In Section 2, we provide preliminaries regarding right Bousfield localization and a proof of the equality in (1) above, where $T(K)$ denotes the free T-algebras on the objects K. Often, transferring model structures to categories of T-algebras such as $\mathsf{Alg}(T; R_K\mathcal{M})$ yields only semi-model category structures (see Example 2.8 in [7] for a case where the transfer is provably not a model structure), so we recall semi-model categories in Section 2, and we pay particular attention throughout this paper to highlighting the differences between semi-model categories and model categories throughout. Semi-admissibility in the theorem above refers to $\mathsf{Alg}(T; R_K\mathcal{M})$ having a semi-model structure, which is a strictly weaker condition than having a model structure. In Section 3, we prove Theorem 3.1, which provides conditions under which $\mathsf{Alg}(T; R_K\mathcal{M})$ has a transferred (semi-)model structure from $R_K\mathcal{M}$, such that (1) holds. This result is of independent interest, as it allows for the study of algebras in a colocalized setting. In Section 4, we prove a technical result regarding when $R_K\mathcal{M}$ is compactly generated, a requirement for Theorem 3.1. In Section 5, we recall the definitions required for (2), (3), and (4) above; then we prove Theorem A. Lastly, in Section 6 we provide numerous applications of Theorem A to spectra, (equivariant) topological spaces, chain complexes, and the stable module category.

This paper can be viewed as the dual of [7], which unified approaches for left Bousfield localization, but the methods are far from formally dual. In particular, more care must be taken

regarding the properties of $R_K\mathcal{M}$ (e.g. Section 4). Additionally, there are instances where the asymmetry works to make right Bousfield localization *easier* than left Bousfield localization. For example, Theorem 2.1 (and Corollary 2.7) proves that whenever one of the (semi-)model structures $\mathsf{Alg}(T; R_K\mathcal{M})$ or $R_{T(K)}\mathsf{Alg}(T; \mathcal{M})$ exists, then both exist and coincide. The corresponding results are false for left Bousfield localization (see Remark 5.7 in [7]). The key reason is an adjunction argument that gives control over the $T(K)$-colocal equivalences, whereas for left Bousfield localization we only had information about local objects.

Theorem A has already been applied in several papers, including [10, 26, 39]. Furthermore, just as the dual theorem [7, Theorem A] was applied in [4, 5, 6, 36], the first author plans to apply Theorem A in future joint work with Michael Batanin, continuing the program of McClure and Smith regarding operad-algebra structures on total spaces of cosimplicial spaces [27].

2. Lifting Right Bousfield Localization to Eilenberg-Moore Categories

In this paper, we will be transferring model structures to categories of T-algebras, for various monads T. In practice, there is often not a full model structure on T-algebras, but rather only a semi-model structure (see Example 2.8 in [7]). We begin with the definition of a semi-model category [5, 2], inspired by the definition from [32], where it is called a J-semi model category. A weaker notion of semi-model category has appeared in [16], analogous to what Spitzweck called an (I, J)-semi model category. The definition provided here is the most structure we can transfer to T-algebras, and all the results from [16] remain true, since Fresse requires less structure on $\mathcal{D}$.

Definition 2.1. A *semi-model structure* on a category $\mathcal{D}$ consists of classes of weak equivalences W, fibrations F, and cofibrations Q satisfying the following axioms:

M1 Fibrations are closed under pullback.

M2 The class W is closed under the two-out-of-three property.

M3 W, F, and Q are all closed under retracts.

M4   i Cofibrations have the left lifting property with respect to trivial fibrations.

     ii Trivial cofibrations whose domain is cofibrant have the left lifting property with respect to fibrations.

M5   i Every map in $\mathcal{D}$ can be functorially factored into a cofibration followed by a trivial fibration.

     ii Every map whose domain is cofibrant can be functorially factored into a trivial cofibration followed by a fibration.

  If, in addition, $\mathcal{D}$ is bicomplete, then we call $\mathcal{D}$ a *semi-model category*.

In practice, most often there is an adjunction $F : \mathcal{M} \rightleftarrows \mathcal{D} : U$ where $\mathcal{M}$ is a cofibrantly generated model category, $\mathcal{D}$ is bicomplete, the right adjoint U preserves colimits over non-empty ordinals, and U preserves and reflects fibrations and trivial fibrations. We say $\mathcal{D}$ is a *semi-model category over* $\mathcal{M}$ if in addition to the axioms above, trivial cofibrations in $\mathcal{D}$ whose domain is cofibrant in $\mathcal{M}$ have the left lifting property with respect to fibrations, if every map in $\mathcal{D}$ whose domain is cofibrant in $\mathcal{M}$ can be functorially factored into a trivial cofibration followed by a fibration, and if the initial object in $\mathcal{D}$ is cofibrant in $\mathcal{M}$. In our examples $\mathcal{D}$ will be the category of T-algebras and (T, U) will be the free-forgetful adjunction.

Let I' be a class of morphisms in $\mathcal{D}$. Denote by I'-inj the class of maps that have the right lifting property with respect to maps in I'. We say $\mathcal{D}$ is *cofibrantly generated* if there are sets of morphisms I' and J' in $\mathcal{D}$ such that the following conditions are satisfied.

1. I'-inj is the class of trivial fibrations.
2. J'-inj is the class of fibrations in $\mathcal{D}$.
3. The domains of I' are small relative to I'-cell.
4. The domains of J' are small relative to maps in J'-cell whose domain is sent by U to a cofibrant object in $\mathcal{M}$.

Every model category is a semi-model category. A semi-model category in which all objects are cofibrant is a model category. In practice, anything that can be done in a model category can be done in a semi-model category, if one first cofibrantly replaces everything in sight. Several examples of the uses of semi-model categories can be found in [18], [2], [32], [13], [35], [34], [37], [39], and [20] among other places.

Recall that a model category structure is completely determined by the classes of weak equivalences and of fibrations [22] (Prop. 7.2.7). The same is true for semi-model categories, since a map is a cofibration if and only if it satisfies the left lifting property with respect to all trivial fibrations [2] (Lemma 1.7). A left Quillen functor between semi-model categories is a left adjoint functor F whose right adjoint preserves fibrations and trivial fibrations. This implies that F preserves cofibrations and trivial cofibrations between cofibrant objects [16] (12.1.8).

Definition 2.2. Suppose $\mathcal{M}$ is a model category, T is a monad on $\mathcal{M}$, and $\mathsf{Alg}(T;\mathcal{M})$ is the category of T-algebras in $\mathcal{M}$. We say that $\mathsf{Alg}(T;\mathcal{M})$ is *(semi-)admissible over* $\mathcal{M}$ if it admits a (semi-)model category structure over $\mathcal{M}$ in which a map f is a weak equivalence (resp., fibration) if $Uf \in \mathcal{M}$ is a weak equivalence (resp., fibration), where $U : \mathsf{Alg}(T;\mathcal{M}) \longrightarrow \mathcal{M}$ is the forgetful functor.

The homotopy function complex in a (semi-)model category $\mathcal{M}$ is denoted $\mathrm{map}_{\mathcal{M}}$. Specifically, we will use the right homotopy function complex [22] (Def. 17.2.1). For semi-model categories, we use the homotopy function complexes of [32] (Prop. 3) and [2] (Def. 3.63). To construct the simplicial set $\mathrm{map}_{\mathcal{M}}(X, Y)$, we cofibrantly replace both X and Y, then take a simplicial resolution of the cofibrant replacement of Y.

Definition 2.3. Suppose $\mathcal{M}$ is a model category, and $K \subseteq \mathcal{M}$ is a set of cofibrant objects.

1. A *K-colocal equivalence* is a map $f : A \longrightarrow B \in \mathcal{M}$ such that the induced map

$$\mathrm{map}_{\mathcal{M}}(X, A) \xrightarrow{\mathrm{map}_{\mathcal{M}}(X,f)} \mathrm{map}_{\mathcal{M}}(X, B)$$

   is a weak equivalence of simplicial sets for all $X \in K$.
2. A *K-colocal object* is a cofibrant object Y in $\mathcal{M}$ such that the induced map

$$\mathrm{map}_{\mathcal{M}}(Y, A) \xrightarrow{\mathrm{map}_{\mathcal{M}}(Y,f)} \mathrm{map}_{\mathcal{M}}(Y, B)$$

   is a weak equivalence of simplicial sets for all K-colocal equivalences $f : A \longrightarrow B$.
3. Define a new category $R_K\mathcal{M}$ as being the same as $\mathcal{M}$ as a category, together with the following distinguished classes of maps. A map f in $R_K\mathcal{M}$ is called a:
   - *weak equivalence* if it is a K-colocal equivalence.
   - *fibration* if it is a fibration in $\mathcal{M}$.
4. If $R_K\mathcal{M}$ is a model category with these weak equivalences and fibrations, then it is called the *right Bousfield localization of $\mathcal{M}$ with respect to K* [22] (Def. 3.3.1(2) and Theorem 5.1.1). In this case, K-colocal objects are precisely the cofibrant objects in $R_K\mathcal{M}$.

Assumption 2.4. Suppose $\mathcal{M}$ is a model category, $K \subseteq \mathcal{M}$ is a set of cofibrant objects such that the right Bousfield localization $R_K\mathcal{M}$ exists, and T is a monad on $\mathcal{M}$ such that $\mathsf{Alg}(T;\mathcal{M})$ is semi-admissible over $\mathcal{M}$.

Lemma 2.5. *Under Assumption 2.4 a map f in $\mathsf{Alg}(T;\mathcal{M})$ is a $T(K)$-colocal equivalence if and only if $Uf \in \mathcal{M}$ is a K-colocal equivalence.*

Proof. Suppose $f : A \longrightarrow B$ is a map of T-algebras in $\mathcal{M}$, and $f_\bullet : A_\bullet \longrightarrow B_\bullet$ is a simplicial resolution of $f \in \mathsf{Alg}(T;\mathcal{M})$ ([22] Def. 16.1.2(2) and Prop. 16.1.22(2) for model categories, [2] Def. 3.63, Thm. 3.12 for semi-model categories). Note that to construct simplicial resolutions in the Reedy semi-model category, one must first cofibrantly replace the constant diagram before taking its fibrant replacement. Applying the forgetful functor $U : \mathsf{Alg}(T;\mathcal{M}) \longrightarrow \mathcal{M}$ entrywise to $f_\bullet$ yields a simplicial resolution of $Uf : UA \longrightarrow UB \in \mathcal{M}$ because weak equivalences and fibrations in $\mathsf{Alg}(T;\mathcal{M})$ are defined by the forgetful functor U.

Pick any object $X \in K$. Note that $T(X)$ is a cofibrant object in $\mathsf{Alg}(T;\mathcal{M})$ because $X \in \mathcal{M}$ is cofibrant and $T : \mathcal{M} \longrightarrow \mathsf{Alg}(T;\mathcal{M})$ is a left Quillen functor of semi-model categories [16] (12.1.8). Such functors preserve cofibrations between cofibrant objects, and hence preserve cofibrant objects. There is a commutative diagram of simplicial sets using [2] (Scholium 3.64):

$$
\begin{array}{ccc}
\mathrm{map}_{\mathsf{Alg}(T;\mathcal{M})}(T(X), A) & \xrightarrow{\mathrm{map}_{\mathsf{Alg}(T;\mathcal{M})}(T(X),f)} & \mathrm{map}_{\mathsf{Alg}(T;\mathcal{M})}(T(X), B) \\
\| & & \| \\
\mathsf{Alg}(T;\mathcal{M})\big(T(X), A_\bullet\big) & \xrightarrow{\mathsf{Alg}(T;\mathcal{M})(T(X),f_\bullet)} & \mathsf{Alg}(T;\mathcal{M})\big(T(X), B_\bullet\big) \\
\cong \downarrow & & \cong \downarrow \\
\mathcal{M}\big(X, (UA)_\bullet\big) & \xrightarrow{\mathcal{M}(X,Uf_\bullet)} & \mathcal{M}\big(X, (UB)_\bullet\big) \\
\| & & \| \\
\mathrm{map}_{\mathcal{M}}(X, UA) & \xrightarrow{\mathrm{map}_{\mathcal{M}}(X,Uf)} & \mathrm{map}_{\mathcal{M}}(X, UB)
\end{array}
$$

The middle isomorphisms are given by the free-forgetful adjunction between $\mathcal{M}$ and $\mathsf{Alg}(T;\mathcal{M})$ and the remark above about $U(f_\bullet) = (Uf)_\bullet$ being a simplicial resolution of Uf. Now the map $f \in \mathsf{Alg}(T;\mathcal{M})$ is a $T(K)$-colocal equivalence if and only if the top horizontal map in the above diagram is a weak equivalence for all $X \in K$. By commutativity this is equivalent to the bottom horizontal map being a weak equivalence for all $X \in K$. This in turn is equivalent to Uf being a K-colocal equivalence in $\mathcal{M}$. $\square$

The following result has versions for both semi-model structures and full model structures. We prove the latter first, as the statement is valuable in its own right, and the proof will be easier for the reader to follow.

Theorem 2.1. *Under Assumption 2.4 and the additional assumption that $\mathsf{Alg}(T;\mathcal{M})$ is admissible over $\mathcal{M}$, the following two statements are equivalent.*

1. *$\mathsf{Alg}(T;R_K\mathcal{M})$ is admissible over $R_K\mathcal{M}$.*
2. *The right Bousfield localization $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ exists.*

Furthermore, if either statement is true, then the model categories $\mathsf{Alg}(T;R_K\mathcal{M})$ and $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ are equal.

Theorem 2.1 says that, in the diagram

$$
\begin{array}{ccccc}
\mathsf{Alg}(T;\mathcal{M}) & \xrightarrow[\text{exists ?}]{R_{T(K)}} & R_{T(K)}\mathsf{Alg}(T;\mathcal{M}) & \xlongequal{\;?\;} & \mathsf{Alg}(T;R_K\mathcal{M}) \\
{\scriptstyle\text{transfer}}\uparrow & & & & \uparrow{\scriptstyle\text{transfer}}\;\vdots\;{\scriptstyle\text{exists ?}} \\
\mathcal{M} & \xrightarrow{\hspace{3cm}R_K\hspace{3cm}} & & & R_K\mathcal{M}
\end{array}
\tag{2.5.1}
$$

the ability to go counter-clockwise is equivalent to the ability to go clockwise. Furthermore, when either one is possible, the results are equal.

Proof. The categories $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ and $\mathsf{Alg}(T;R_K\mathcal{M})$ are both equal to $\mathsf{Alg}(T;\mathcal{M})$. Observe that $\mathsf{Alg}(T;R_K\mathcal{M})$ and $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ have the same fibrations, namely, fibrations in $\mathsf{Alg}(T;\mathcal{M})$. Moreover, Lemma 2.5 says that the classes of weak equivalences in $\mathsf{Alg}(T;R_K\mathcal{M})$ and $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ coincide. Hence, the classes of cofibrations coincide, and the axioms of a model category are satisfied in $\mathsf{Alg}(T;R_K\mathcal{M})$ if and only if they are satisfied in $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$. □

Remark 2.6. In [10] (7.7) the equality

$$
\mathsf{Alg}(T;R_K\mathcal{M}) = R_{T(K)}\mathsf{Alg}(T;\mathcal{M})
\tag{2.6.1}
$$

of model categories was observed under assumption (2)–i.e., $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ exists–*and* that R_K lifts to the homotopy category of T-algebras (see Def. 5.1). Moreover, the same equality was also observed in [21] when T is the monad of a colored operad.

The proof of Theorem 2.1 also proves the following corollary, since Lemma 2.5 only requires $\mathsf{Alg}(T;\mathcal{M})$ to be a semi-model category. By (2) below, we mean that the three classes of maps defining $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ [22] (Thm. 5.1.1) satisfy the axioms of a semi-model category.

Corollary 2.7. *Under Assumption 2.4, the following two statements are equivalent.*
  1. $\mathsf{Alg}(T;R_K\mathcal{M})$ *is semi-admissible over* $R_K\mathcal{M}$.
  2. The right Bousfield localization $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ *exists as a semi-model category.*
Furthermore, if either statement is true, then $\mathsf{Alg}(T;R_K\mathcal{M})$ *and* $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ *are equal as semi-model categories.*

Proof. When $\mathsf{Alg}(T;\mathcal{M})$ is only a semi-model category, we do not have a general theorem telling us that $\mathsf{Alg}(T;R_K\mathcal{M})$ or $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ are semi-model categories. However, both still have the same underlying category, we can still define classes of fibrations and weak equivalences in both settings, and we can prove that they coincide, exactly as in the proof of Theorem 2.1. As mentioned after Definition 2.1, the weak equivalences and fibrations determine the semi-model structure, so the axioms are satisfied in $\mathsf{Alg}(T;R_K\mathcal{M})$ if and only if they are satisfied in $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$.

□

Conditions under which (1) holds (hence (2) as well) are provided in Theorem 3.1 below. Conditions under which (2) has the structure of a *right* semi-model category (dual to Definition 2.1) are provided in [2], but we do not make use of this structure. As conditions regarding semi-model category existence are more easily verified than full model category existence, we prefer to work in the setting of semi-model categories, and the conclusions of Corollary 2.7 are sufficient for our needs.

Corollary 2.8. *Under Assumption 2.4 suppose either one of the two equivalent conditions in Corollary 2.7 is true. Then the functor $T : \mathcal{M} \longrightarrow \mathsf{Alg}(T; \mathcal{M})$ sends K-colocal equivalences between K-colocal objects in $\mathcal{M}$ to $T(K)$-colocal equivalences between $T(K)$-colocal objects in $\mathsf{Alg}(T; \mathcal{M})$.*

Proof. By Corollary 2.7 the free-forgetful adjunction

$$R_K \mathcal{M} \underset{U}{\overset{T}{\rightleftarrows}} \mathsf{Alg}(T; R_K \mathcal{M}) = R_{T(K)} \mathsf{Alg}(T; \mathcal{M})$$

is a Quillen adjunction. By Ken Brown's Lemma ([23] 1.1.12 and [16] 12.1.6) the left Quillen functor T sends weak equivalences between cofibrant objects to weak equivalences between cofibrant objects.   □

3. Admissibility over Right Bousfield Localization

Definition 3.1. Suppose $\mathcal{M}$ is a category, C is a class of maps in $\mathcal{M}$, and T is a monad on $\mathcal{M}$.
1. We call the class C *saturated* if it is closed under retracts, pushouts, and transfinite compositions.
2. We call T *finitary* if it preserves filtered colimits.

Definition 3.2. Suppose $\mathcal{M}$ is a model category, C is a saturated class of maps in $\mathcal{M}$, and T is a monad on $\mathcal{M}$.
1. We say that $\mathcal{M}$ is *C-perfect* [3] (Def. 2.1) if weak equivalences in $\mathcal{M}$ are closed under filtered colimits along maps in C. This means that, given a family of weak equivalences $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ indexed by a filtered partially ordered set, where all maps $X_\alpha \longrightarrow X_\beta$ and $Y_\alpha \longrightarrow Y_\beta$ are in C, then the map $f : \operatorname{colim} X_\alpha \longrightarrow \operatorname{colim} Y_\alpha$ is a weak equivalence.
2. We say that $\mathcal{M}$ is *C-compactly generated* [3] (Def. 2.4) if it is cofibrantly generated and C-perfect and if every object is small with respect to C [23] (Def. 2.1.3).
3. We say that T is *C-admissible on $\mathcal{M}$* [3] (Def. 2.9) if for each cofibration (resp., trivial cofibration) $f : A \longrightarrow B \in \mathcal{M}$ and each pushout of the form

$$
\begin{array}{ccc}
T(A) & \longrightarrow & X \\
{\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle g} \\
T(B) & \longrightarrow & Y
\end{array}
\tag{3.2.1}
$$

   in $\mathsf{Alg}(T; \mathcal{M})$, the underlying map $Ug \in \mathcal{M}$ is in C (resp., C and the class of weak equivalences), where $U : \mathsf{Alg}(T; \mathcal{M}) \longrightarrow \mathcal{M}$ is the forgetful functor.
4. We say that T is *C-semi-admissible on $\mathcal{M}$* [7] (Def. 2.4) if the previous statement holds whenever UX is cofibrant in $\mathcal{M}$.

The following observation provides conditions under which the equivalent statements in Corollary 2.7 are true.

Theorem 3.1. *Suppose:*
- *$\mathcal{M}$ is a cofibrantly generated model category.*
- *$C \subseteq \mathcal{M}$ is a saturated class of maps such that $\mathcal{M}$ is C-compactly generated.*
- *$K \subseteq \mathcal{M}$ is a set of cofibrant objects such that $R_K \mathcal{M}$ exists and is C-compactly generated.*

- *T is a finitary C-(semi-)admissible monad on $\mathcal{M}$.*

Then $\mathsf{Alg}(T; R_K\mathcal{M})$ *is (semi-)admissible over* $R_K\mathcal{M}$.

Proof. First consider the case of full model categories. To show that $\mathsf{Alg}(T; R_K\mathcal{M})$ is admissible, by [3] (Theorem 2.11) it is enough to show that T is C-admissible on $R_K\mathcal{M}$. Suppose $f : A \longrightarrow B \in R_K\mathcal{M}$, and consider the pushout (3.2.1).

1. If f is a trivial cofibration in $R_K\mathcal{M}$, then it is also a trivial cofibration in $\mathcal{M}$ because $\mathcal{M}$ and $R_K\mathcal{M}$ have the same fibrations and hence the same trivial cofibrations. Since T is C-admissible on $\mathcal{M}$, in the pushout (3.2.1) the map Ug is in C and is a weak equivalence in $\mathcal{M}$, hence also a weak equivalence in $R_K\mathcal{M}$.
2. If f is a cofibration in $R_K\mathcal{M}$, then it is also a cofibration in $\mathcal{M}$. So T being C-admissible on $\mathcal{M}$ implies that the map Ug is in C.

For the semi-model category case, we use the same argument with [17] (12.1.4 and 12.1.9) instead of [3] (2.11) and assume that UX is cofibrant in $R_K\mathcal{M}$, hence also cofibrant in $\mathcal{M}$. $\qquad\square$

Corollary 3.3. *Under the assumptions of Theorem 3.1 in which T is C-semi-admissible on $\mathcal{M}$, conditions (1) and (2) in Corollary 2.7 and the equality (2.6.1) are all true.*

Proof. $\mathsf{Alg}(T; \mathcal{M})$ is admissible over $\mathcal{M}$ by [3] (2.11), so Assumption 2.4 is satisfied. As $\mathsf{Alg}(T; R_K\mathcal{M})$ is semi-admissible over $R_K\mathcal{M}$ by Theorem 3.1, condition (1) in Corollary 2.7 is true, hence so are condition (2) and (2.6.1). $\qquad\square$

4. Compact Generation of Right Bousfield Localization

In Theorem 3.1, the assumption that $R_K\mathcal{M}$ is C-compactly generated means that:

1. *Every object is small with respect to C.* This is part of the assumptions of $\mathcal{M}$ being C-compactly generated.
2. *$R_K\mathcal{M}$ is a cofibrantly generated model category.* For example, if $\mathcal{M}$ is cellular in which every object is fibrant, then $R_K\mathcal{M}$ is also a cellular, hence cofibrantly generated, model category by [22] Theorem 5.1.1.
3. *The class of K-colocal equivalences is closed under filtered colimits along maps in C.* Below we will provide reasonable conditions under which this is true.

Definition 4.1. Suppose $\mathcal{M}$ is a model category with a distinguished set of maps J, and K is a set of objects in $\mathcal{M}$.

1. Define the set of maps

$$\Lambda(K) = \left\{ A^{\bullet} \otimes \partial\Delta[n] \longrightarrow A^{\bullet} \otimes \Delta[n] : A \in K, n \geq 0 \right\}$$

   in which $A^{\bullet}$ is a choice of a cosimplicial resolution of A [22] (Def. 16.1.2(1) and 16.3.1(1)).
2. Define the set $\overline{\Lambda(K)} = \Lambda(K) \cup J$ [22] (Def. 5.2.1).

$R_K\mathcal{M}$ can be made cofibrantly generated even if $\mathcal{M}$ is not cellular. One approach is given in [12] (Theorem 2.6). We provide here another approach, resulting in a combinatorial model structure on $R_K\mathcal{M}$.

Proposition 4.2. *Suppose $\mathcal{M}$ is a combinatorial model category, K is a set of objects, and every object of $\mathcal{M}$ is fibrant. Then $R_K\mathcal{M}$ is a combinatorial model category in which every object is fibrant.*

Proof. First, $\mathcal{M}$ is right proper because all objects are fibrant. The existence of the model structure on $R_K\mathcal{M}$ is Proposition 5.13 in [2], and all objects are fibrant because the fibrations in $R_K\mathcal{M}$ are the same as the fibrations in $\mathcal{M}$. A characterization of the cofibrations in $R_K\mathcal{M}$, that matches the characterization in Proposition 5.3.6 of [22], is given in [2] (Lemma 5.8). Lastly, [12] (Lemma 2.3) proves that $R_K\mathcal{M}$ is cofibrantly generated, with trivial cofibrations as in $\mathcal{M}$ and generating cofibrations $\overline{\Lambda(K)}$, if all objects are fibrant. The key point here is that cellularity is not required for the arguments regarding $\overline{\Lambda(K)}$ [22] (5.2.4-5.2.6) to work. $\square$

Proposition 4.3. *Suppose:*
- $\mathcal{M}$ *is a cofibrantly generated simplicial model category with the set* J *of generating trivial cofibrations, and* C *is a saturated class of maps in* $\mathcal{M}$.
- K *is a set of cofibrant objects in* $\mathcal{M}$ *such that* $R_K\mathcal{M}$ *is a cofibrantly generated model category with generating cofibrations* $\overline{\Lambda(K)}$.
- *All the objects in* $K \otimes \Delta[n]$ *for* $n \geq 0$ *and all the (co)domains of the maps in* J *are finite with respect to* C *[23] (7.4).*

Then K-*colocal equivalences are closed under filtered colimits along* C. *It follows that* $R_K\mathcal{M}$ *is* C-*compactly generated if all objects are small with respect to* C *(e.g. if* $\mathcal{M}$ *is locally presentable).*

Proof. By Hovey's argument [23] (Cor. 7.4.2) (see also [3] Remark 2.2) it suffices to show that the domains and codomains of the maps in $\overline{\Lambda(K)} = \Lambda(K) \cup J$ are finite with respect to C. This is true for the maps in J by assumption. For $\Lambda(K)$ suppose $A \in K$. Using the simplicial model structure of $\mathcal{M}$, since A is cofibrant in $\mathcal{M}$, by [22] (Cor. 16.1.4(1)) a choice of a cosimplicial resolution of A is given by

$$A^\bullet = \big\{ A \otimes \Delta[n] : n \geq 0 \big\}.$$

In order for $A^\bullet \otimes \partial\Delta[n]$ and $A^\bullet \otimes \Delta[n]$–both of which are finite colimits of the various $A \otimes \Delta[n]$ [22] (Def. 16.3.1(1))–to be C-finite, it is enough for $A \otimes \Delta[n]$ to be C-finite, which is true by assumption. $\square$

5. Equivalent Approaches to Preservation of Algebras Under Right Bousfield Localization

The next definition is [10] (Def. 7.9) for right Bousfield localization. It provides one approach to preservation of monadic algebras under right Bousfield localization. Here we let C^T denote the comonad on $\mathsf{Ho}(\mathsf{Alg}(T;\mathcal{M}))$ determined by R_K.

Definition 5.1. Under Assumption 2.4 we say that R_K *lifts to the homotopy category of* T-*algebras* if:
1. There exists a coaugmented endofunctor $c^T : C^T \longrightarrow \mathrm{Id}$ on $\mathsf{Ho}(\mathsf{Alg}(T;\mathcal{M}))$.
2. There exists a natural isomorphism $h : R_K U \longrightarrow U C^T$ such that

$$U c^T \circ h = cU$$

   in $\mathsf{Ho}(\mathcal{M})$, where U is the forgetful functor and $c : R_K \longrightarrow \mathrm{Id}$ is the counit of the derived adjunction $\mathsf{Ho}(R_K\mathcal{M}) \rightleftarrows \mathsf{Ho}(\mathcal{M})$.

Another approach to preservation of monadic algebras under right Bousfield localization was proposed by the authors in [39], from which the following definition can be extracted.

Definition 5.2. Under Assumption 2.4 we say that R_K *preserves T-algebras* if:

1. When X is a T-algebra there is some T-algebra $\widetilde{X}$ that is weakly equivalent in $\mathcal{M}$ to $R_K U X$.

2. In addition, when X is a fibrant T-algebra, there is a choice of $\widetilde{X}$ in $\mathsf{Alg}(T; \mathcal{M})$, with $U(\widetilde{X})$ colocal in $\mathcal{M}$, there is a T-algebra homomorphism $c_X : \widetilde{X} \longrightarrow X$ lifting the colocalization map $q : R_K U X \longrightarrow U X$ up to homotopy, and there is a weak equivalence $\beta_X : U(\widetilde{X}) \longrightarrow R_K U X$ such that $q \circ \beta_X \cong U c_X$ in $\mathrm{Ho}(\mathcal{M})$.

A related concept is whether the forgetful functor preserves right Bousfield localization in the following sense.

Definition 5.3. Under Assumption 2.4 suppose $R_{T(K)}\mathsf{Alg}(T; \mathcal{M})$ exists as a semi-model category. Then we say that the forgetful functor

$$R_{T(K)}\mathsf{Alg}(T; \mathcal{M}) \xrightarrow{U} R_K \mathcal{M}$$

preserves right Bousfield localization if, given any map $c : R_{T(K)}X \longrightarrow X \in \mathsf{Alg}(T; \mathcal{M})$ that is a $T(K)$-colocal equivalence with $T(K)$-colocal domain, the map $Uc \in \mathcal{M}$ is a K-colocal equivalence with K-colocal domain.

We now observe that these three approaches to preservation of algebras under right Bousfield localization are equivalent. The following omnibus theorem should be compared to [7] (Theorem 5.6), which deals with different approaches to preservation of algebras under left Bousfield localization.

Theorem 5.1. *Under Assumption 2.4 suppose further that* $\mathsf{Alg}(T; R_K\mathcal{M})$ *is semi-admissible over* $R_K\mathcal{M}$. *Then the following statements are equivalent.*

1. *The forgetful functor*

$$\mathsf{Alg}(T; R_K\mathcal{M}) = R_{T(K)}\mathsf{Alg}(T; \mathcal{M}) \xrightarrow{U} R_K\mathcal{M}$$

preserves weak equivalences and cofibrant objects, in which the equality on the left is from Corollary 2.7.

2. R_K *preserves T-algebras (Def. 5.2).*
3. R_K *lifts to the homotopy category of T-algebras (Def. 5.1).*
4. *The forgetful functor U preserves right Bousfield localization (Def. 5.3).*

Proof. $(1) \implies (2)$ is proven in Theorem 6.2 in [39]. To be self-contained, we recall the details in the case when X is a fibrant T-algebra. We define $\widetilde{X}$ to be $Q_{K,T}Q_T X$ where Q_T (resp. $Q_{K,T}$) denotes the cofibrant replacement functor in $\mathsf{Alg}(T; \mathcal{M})$ (resp. $\mathsf{Alg}(T; R_K\mathcal{M})$), and the map c_X is simply the composite of cofibrant replacement maps $Q_{K,T}Q_T X \longrightarrow Q_T X \longrightarrow X$. The map β_X is defined by the following lifting diagram in $R_K\mathcal{M}$, where the right vertical map is cofibrant replacement in $R_K\mathcal{M}$:

Using that U preserves cofibrant objects we conclude that $UQ_{K,T}Q_TX$ is cofibrant in $R_K\mathcal{M}$. It easily follows that β is a weak equivalence, since it is a K-colocal equivalence between K-colocal objects (by the 2-out-of-3 property).

For (2) $\Longrightarrow$ (3) we assume that R_K preserves T-algebras. We must produce a coaugmented endofunctor $c^T : C^T \longrightarrow Id$ on $\mathsf{Ho}(\mathsf{Alg}(T;\mathcal{M}))$ and a natural isomorphism $h : R_KU \longrightarrow UC^T$, satisfying the assumptions of Def. 5.1. We will use our assumption that the transferred semi-model structure on $\mathsf{Alg}(T;R_K\mathcal{M})$ exists, and we make use of the notation for c_X from above. We take C^T to be the image in $\mathsf{Ho}(\mathsf{Alg}(T;\mathcal{M}))$ of $\widetilde{(-)}$ (i.e. of $Q_{K,T}Q_T(-)$), so that c^T is the image in $\mathsf{Ho}(\mathsf{Alg}(T;\mathcal{M}))$ of c_X. To construct $h : R_KU \longrightarrow UC^T$, consider the following lifting diagram, where X is a fibrant T-algebra:

$$
\begin{array}{ccc}
\varnothing & \longrightarrow & UQ_{K,T}Q_TX \\
\downarrow & \overset{\beta}{\nearrow} & \simeq \downarrow Uc_X \\
R_KUX & \underset{\simeq}{\overset{q}{\longrightarrow}} & UX
\end{array}
$$

The left vertical map above is a cofibration in $R_K\mathcal{M}$, because q is a cofibrant replacement map in $R_K\mathcal{M}$. The right vertical map is a trivial fibration in $R_K\mathcal{M}$ because U preserves trivial fibrations and c_X is a trivial fibration in $\mathsf{Alg}(T;R_K\mathcal{M})$. It follows that β is a weak equivalence in $\mathcal{M}$. We take h to be the image of β in $\mathsf{Ho}(\mathcal{M})$, and immediately deduce that h is an isomorphism in $\mathsf{Ho}(\mathcal{M})$. Furthermore, $Uc^T \circ h = cU$ in $\mathsf{Ho}(\mathcal{M})$ (see Def. 5.1) by commutativity of the lower triangle above, since c is the image of q in $\mathsf{Ho}(\mathcal{M})$. Furthermore, the lift β is unique in $\mathsf{Ho}(\mathcal{M})$ by the universal property of right localization, since any other lift would necessarily be a weak equivalence between the same K-colocal objects. Finally, this lift is natural in $\mathsf{Ho}(\mathcal{M})$ because if we began with a map $X \longrightarrow Y$ and constructed lifts β_X and β_Y then we would in addition construct a homotopy unique lift from R_KUX to $UQ_{K,T}Q_TY$, so commutativity of the relevant naturality square in $\mathsf{Ho}(\mathcal{M})$ follows from uniqueness.

The implication (3) $\Longrightarrow$ (1) is [10] (Theorem 7.10(iii)) in the case of model categories. For semi-model categories, to see that the forgetful functor preserves cofibrant objects, we could just use the assumed natural isomorphism $h : R_KU \longrightarrow UC^T$ in Def. 5.1(2) and the equality $\mathsf{Alg}(T;R_K\mathcal{M}) = R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ in Corollary 2.7. The forgetful functor preserves weak equivalences because $\mathsf{Alg}(T;R_K\mathcal{M})$ is semi-admissible over $R_K\mathcal{M}$. We have shown that the first three statements are equivalent.

To see that (1) $\Longrightarrow$ (4), simply observe that the map c in Def. 5.3 is a weak equivalence with cofibrant domain in $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$. So (1) guarantees that the map $Uc \in R_K\mathcal{M}$ is also a weak equivalence with cofibrant domain, i.e., a K-colocal equivalence with K-colocal domain in $\mathcal{M}$.

Finally, we show that (4) $\Longrightarrow$ (2). Since $\mathsf{Alg}(T;R_K\mathcal{M}) = R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$ exists by Corollary 2.7, every T-algebra X has a functorial cofibrant replacement in $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$. In other words, we may take the given functorial factorization

$$
\varnothing \longrightarrow R_{T(K)}X \overset{c_X}{\underset{\simeq}{\twoheadrightarrow}} X \in R_{T(K)}\mathsf{Alg}(T;\mathcal{M})
$$

of the map $\varnothing \longrightarrow X$ into a cofibration followed by a trivial fibration in $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$. We take $R_{T(K)}X \in \mathsf{Alg}(T;\mathcal{M})$ as our choice of $\widetilde{X}$ in Def. 5.2. Since the above factorization is taken in $R_{T(K)}\mathsf{Alg}(T;\mathcal{M})$, the map c_X is a T-algebra map. Furthermore, (4) guarantees that the underlying map $Uc_X \in \mathcal{M}$ is a K-colocal equivalence with K-colocal domain.

It remains to show that $\widetilde{X} = R_{T(K)}X$ is weakly equivalent to $R_K UX$ in $\mathcal{M}$. Consider the diagram

$$
\begin{array}{ccc}
\varnothing & \longrightarrow & R_K UX \\
\downarrow & \overset{\beta}{\nearrow} & \ \downarrow{\simeq}\ q \\
UR_{T(K)}X & \underset{\simeq}{\overset{Uc_X}{\longrightarrow}} & UX
\end{array}
\tag{5.3.1}
$$

in $R_K\mathcal{M}$, in which $q : R_K UX \longrightarrow UX$ is the given functorial cofibrant replacement of UX in $R_K\mathcal{M}$. Since the map q is a trivial fibration and $UR_{T(K)}X$ is cofibrant in $R_K\mathcal{M}$, a dotted lift β exists, making the lower triangle commute as required. Since both Uc_X and q are weak equivalences in $R_K\mathcal{M}$, so is β by the 2-out-of-3 property. Therefore, β is a K-colocal equivalence between K-colocal objects in $\mathcal{M}$. So it is actually a weak equivalence in $\mathcal{M}$ by [22] (3.2.13(2)), as desired. $\square$

Remark 5.4. In the context of a colored operad in a cofibrantly generated simplicial monoidal model category, the implication (1) $\Longrightarrow$ (4) in Theorem 5.1 is a result in [21].

6. Applications

We conclude this paper with numerous applications of Theorem 5.1. First, in [10] it was proven that certain colocalizations lift to the homotopy category of modules over a ring spectrum. We can now use Theorem 5.1 to deduce that these colocalizations preserve ring and module structure. As the setting of spectra eluded us in [39], these are the first general results we are aware of regarding preservation of algebraic structure by right Bousfield localizations for spectra.

Next, several results regarding preservation of algebraic structure by various right Bousfield localizations were proven in [39] for chain complexes, spaces, equivariant spaces, and the stable module category. We can now use Theorem 5.1 to deduce that these right Bousfield localizations lift to homotopy categories of T-algebras and that the forgetful functor preserves these right Bousfield localizations.

Both sets of applications are in the context of algebras over colored operads, since this is the setting most amenable to proving (semi-)admissibility results. Given a colored operad P, P-algebras are the same as algebras over the associated monad $T = U \circ P$ on $\mathcal{M}$, where P is the free P-algebra functor and U is the forgetful functor. We mention that the generality of our approach allows for applications beyond the setting of algebras over operads. For example, [6] recently showed the power of the homotopy theory of substitudes, which encode more structure than colored operads do. Similarly, [24] (resp. [30]) proved admissibility results in the category of small categories (resp. simplicial sets) with respect to a broad class of monads that do not arise from operads. Our results hold for such settings.

6.1 Spectra Let $\mathcal{M}$ be the positive stable model structure on symmetric spectra. Let K be a cofibrant spectrum. We proved in [37] (Theorem 8.3.1) that all colored operads are admissible in this setting.

Example 6.2. If E be a connective ring spectrum (that is, (-1)-connected, so $\pi_k(E) = 0$ for all $k < 0$) that is cofibrant in $\mathcal{M}$. Then R_K preserves E-modules. This follows from the proof of Theorem 7.11 in [10] and Theorem 5.1, since R_K lifts to the homotopy category of E-modules.

Example 6.3. If K is connective, and E is any ring spectrum that is cofibrant in $\mathcal{M}$, then R_K preserves E-modules. This follows from the proof of Theorem 7.11 in [10] and Theorem 5.1, since R_K lifts to the homotopy category of E-modules.

Theorem 6.1. *Let P be a colored operad valued in simplicial sets or symmetric spectra. Let K be a set of cofibrant objects such that $R_K\mathcal{M}$ is a monoidal model category and the K-colocal objects are closed under smash product (e.g., if R_K is an $\mathcal{M}$-enriched colocalization in the sense of Definition 5.2 of [19]). Then R_K preserves P-algebras, R_K lifts to the homotopy category of P-algebras, and U preserves R_K.*

Proof. The key ingredient will be Theorem 5.8 in [19], together with the dual of Theorem 7.6 in [10]. We have slightly changed the notation from [19] to match our notation. We focus first on the case of simplicial colored operads. Let $\phi : P_\infty \longrightarrow P$ be a cofibrant replacement in the model category of simplicial operads. By Theorems 1.3 and 1.4 in [14], the categories of P_∞-algebras and P-algebras both inherit transferred model structures, and they are Quillen equivalent via ϕ^*. Thus, preservation for P-algebras is equivalent to preservation for P_∞-algebras.

Let X be a P_∞-algebra and let $r : X \longrightarrow X'$ be fibrant replacement in the model category of P_∞-algebras (so X' is colorwise fibrant as a spectrum). For every set of colors $c_1, \ldots, c_n$, the object $R_K(U(X'(c_1))) \otimes \cdots \otimes R_K(U(X'(c_n)))$ is K-colocal by our hypothesis on K-colocal objects. It follows from Theorem 5.8 in [19] that $R_K(U(X'))$ admits a P_∞-algebra structure. Furthermore, $R_K(U(X'))$ is K-colocally weakly equivalent to $R_K(U(X))$, by the 2-out-of-3 property, hence weakly equivalent by Theorem 3.2.13 of [22]. It follows that R_K preserves P-algebras, hence (by Theorem 5.1) that R_K lifts to the homotopy category of P-algebras and that U preserves R_K.

For the case of spectral colored operads, one cannot use the results in [14]. However, admissibility of P and P_∞ follow from Theorem 8.3.1 in [37], and rectification between P-algebras and P_∞-algebras can be verified as in Corollary 5.1.1 of [38]. The rest of the proof above works without changes, as [19] works for any enriching category, even $\mathcal{M}$ itself.

Lastly, note that if R_K is an $\mathcal{M}$-enriched colocalization (i.e., defined via internal hom objects rather than simplicial mapping spaces) then $R_K\mathcal{M}$ is a monoidal model category by Remark 5.3 in [19], and K-colocal objects are closed under the smash product by Lemma 5.6 of [19]. □

Example 6.4. Let $K = \{\Sigma^{n+1}S^0\}$ where S^0 is the sphere spectrum. Then K-colocal spectra are precisely the n-connective spectra (that is, $\pi_{\leq n} = 0$), and K-colocalization amounts to taking n-connective covers. This colocalization may be defined as an sSet-enriched colocalization, but not as an $\mathcal{M}$-enriched colocalization since K-colocal objects are not closed under smash product (see Section 7 of [19]). However, for $n = 0$, K-colocal objects *are* closed under smash product. Thus, Theorem 6.1 demonstrates that taking connective covers preserves P-algebra structure for every colored operad P, that the connective cover colocalization lifts to homotopy categories of P-algebras, and that the forgetful functor preserves connective cover colocalizations.

6.5   Spaces   Consider the Quillen model structure on Top, the category of compactly generated topological spaces. All colored operads are admissible in this setting, and in the setting of G-spaces for a compact Lie group G [39] (Theorem 5.11).

Example 6.6 (*n*-connected covers)**.** Let $K = \{S^m \mid m > n\}$, so that $R_K(X) = CW_A(X)$ where $A = S^n$ [11, 15]. The K-colocal objects are X with $\pi_{\leq n}(X) = 0$, and the K-colocal equivalences are maps f with $\pi_{>n}(f)$ an isomorphism. As this set K is closed under smashing with spheres,

Theorems 4.3, 4.5, and 9.3 in [39] demonstrate that the pushout product axiom is satisfied in $R_K(\mathsf{Top})$. The same is true for any CW complex A. For the case $K = \{S^1\}$, the K-colocal spaces are precisely those X with $\pi_0(X) = 0$; i.e., X is path connected. Any E_∞-operad O has path connected spaces (contractible even), so is entrywise K-colocal. It is easy to check that such an operad is in fact Σ-cofibrant in the K-colocal model structure on symmetric sequences, since the fixed-point property of $E\Sigma_n$ guarantees the existence of an equivariant lift in a lifting problem against a K-colocally trivial fibration. Theorem 6.2 in [39] proves that R_K preserves E_∞-algebras (the requisite smallness assumptions are verified in [33]). Theorem 5.1 now demonstrates that R_K lifts to the homotopy category of E_∞-algebras in such a way that the forgetful functor U preserves right Bousfield localization.

6.7 Equivariant spaces Let G be a compact Lie group, and let $\mathcal{M} = \mathsf{Top}^G$ denote the category of G-equivariant compactly generated spaces, with the fixed-point model structure where a map f is a weak equivalence (resp. fibration) if and only if f^H is a weak equivalence (resp. fibration) for every closed $H < G$. Consider the model structure on operads valued in $\mathcal{M}$ transferred along the free-operad functor from the product model structure on $\mathcal{M}^\Sigma = \prod_{n \geq 0} \mathcal{M}^{\Sigma_n}$, where each $\mathcal{M}^{\Sigma_n}$ has the projective model structure. Note that this is usually the wrong model structure for the study of G-equivariant operads (because one wants to study fixed spaces of subgroups of $G \times \Sigma_n$), but this model structure is the natural home for the G-equivariant E_∞-operad of [25]. In this model structure, the cofibrant replacement of the commutative operad Com is an equivariant E_∞-operad, which plays an important role in the search for algebraic models for equivariant spectra.

We will show a preservation result for this operad. Note, however, that this is the wrong operad to encode complete equivariant commutativity (including norms) in G-spectra, because it does not allow for mixing of the G-action with the Σ_n-action. The correct operads to study for norms are the N_∞-operads of [9].

Example 6.8. Suppose F is a nonempty set of subgroups of G and let $K(F) = \{(G/H)_+ \mid H \in F\}$. Colocalization with respect to $K(F)$ preserves algebras over any G-E_∞-operad $\mathcal{E}$, because $\mathcal{E}$ is Σ-cofibrant in $\prod_n (\mathsf{Top}^G)^{\Sigma_n}$, so $\mathcal{E}$-algebras inherit a semi-model structure in Top^G and in $R_{K(F)}(\mathsf{Top}^G)$. Theorem 5.1 now demonstrates that the colocalization $R_{K(F)}$ (that focuses on cells G/H for $H \in K(F)$) lifts to the homotopy category of E_∞-algebras and that the forgetful functor preserves $R_{K(F)}$.

Example 6.9. Let $K = \{S^{n+1}\}$, so that K-colocal objects are n-connected covers. Then both the operad $\mathcal{E}$ and the N_∞ operads of [9] are objectwise K-colocal and Σ_n-free. They, therefore, satisfy condition $\star^\mathsf{O}$ from [39], and hence their algebras are preserved by taking n-connected covers. Theorem 5.1 demonstrates that the n-connected cover colocalization lifts to the homotopy categories of E_∞ and N_∞-algebras.

6.10 Chain complexes For a ring R, the category of chain complexes of R-modules admits a projective model structure [23] (Thm. 2.3.11) which is also a monoidal model category [23] (Prop. 4.2.13) when R is commutative. When R is a field of characteristic zero, every operad is admissible [37] (Thm. 8.1.1). Otherwise, Σ-cofibrant colored operads are semi-admissible [37] (Thm. 6.3.1) but can fail to be admissible [7] (Ex. 2.8). For simplicity, we stick to the characteristic zero case below, but our techniques allow for applications (using semi-admissibility) for a much broader class of rings.

Theorem 6.2. *Let k be a field of characteristic zero. Then, for any set of cofibrant objects K, and any colored operad O that is objectwise K-colocal, R_K lifts to the homotopy category of O-algebras and the forgetful functor $U : \mathsf{Alg}_{\mathsf{O}}(\mathsf{Ch}(k)) \longrightarrow \mathsf{Ch}(k)$ preserves R_K.*

Proof. Theorem 11.7 in [39] demonstrates that every right Bousfield localization $R_K(\mathsf{Ch}(k))$ is monoidal. Theorem 11.12 in [39] demonstrates that $\mathsf{Ch}(k)$ satisfies condition ♦, which says all symmetric sequences are projectively cofibrant, since k has characteristic zero. Thus, any O that is objectwise K-colocal is Σ-cofibrant with respect to $R_K(\mathsf{Ch}(k))$, so there is a transferred semi-model structure on O-algebras. The preservation theorem (Theorem 6.2 in [39]) demonstrates that R_K preserves O-algebras, and Theorem 5.1 provides the required lift of R_K to O-algebras. □

Let $S(n)$ denote the chain complex that is k in degree n and 0 elsewhere.

Example 6.11. Let $K = \{S(n)\}$ for some n. Then the K-colocal objects are the X such that $H_{<n}(X) = 0$, and the K-colocal equivalences are maps f such that $H_{\geq n}(f)$ is an isomorphism. The functor R_K can be viewed as an n-connected cover. Suppose k has characteristic zero. For any n, R_K lifts to the homotopy category of O-algebras, for any E_∞-operad O, by Theorem 6.2, because all spaces $\mathsf{O}(n)$ have $H_i(\mathsf{O}(n)) = 0$ for all i, hence are K-colocal. Similarly, R_K lifts to the homotopy category of commutative differential graded algebras when k has characteristic zero.

Example 6.12. Suppose $R_K(\mathsf{Ch}(k))$ is a monoidal model category and that the K-colocalization functor can be chosen to be lax monoidal (e.g. see [19] (5.6)). Then for any colored operad O, the sequence $R_K\mathsf{O}$ defined by

$$(R_K\mathsf{O})\left(\begin{smallmatrix} d \\ [\underline{c}] \end{smallmatrix}\right) = R_K\left(\mathsf{O}\left(\begin{smallmatrix} d \\ [\underline{c}] \end{smallmatrix}\right)\right)$$

is a colored operad over $R_K\mathcal{M}$. By construction, this colored operad is objectwise K-colocal, so the right Bousfield localization R_K lifts to the homotopy category of $R_K\mathsf{O}$-algebras.

Remark 6.13. The category of simplicial abelian groups has a cofibrantly generated model structure [29] in which all objects are fibrant. This category is equivalent to the category of bounded below chain complexes, by the Dold-Kan Theorem. The normalized chains functor N is a natural isomorphism, compatible with the model structures, and is monoidal by [31] (4.1). Thus, all results about right Bousfield localizations lifting to categories of algebras in $\mathsf{Ch}(k)$ yield analogous results in the category of simplicial k-modules.

6.14   Stable module category   Let k be a field, let G be a finite group, and let $R = k[G]$. The category of R-modules has a monoidal model structure whose homotopy category is known as the stable module category [23] (Thm. 2.2.2). We proved that all operads are admissible in this context [39] (Thm. 12.2). It is worth remarking that this model structure can be defined for general rings R, as discussed in [39] (Sec. 12), and admissibility is not known outside the case where k is a field. For example, if k is a PID that is not a field, then we only know that Σ-cofibrant colored operads are semi-admissible. We provide the example below in the context where k is a field, and we encourage the reader to work out further applications when R is a more general ring.

Example 6.15. Let $\mathcal{M} = k[G]$-mod and let $K = \{k\}$ where k has the trivial G action. The colocalization of $\mathcal{M}$ with respect to this K is studied in [8] (IV.2.7). Since k is the monoidal unit,

the commutative operad is objectwise K-colocal. Let K' denote the stable, monoidal closure of K, following [1], and let K'' be the closure of K' and $\{k[\Sigma_n] \mid n = 1, 2, ...\}$. Corollary 12.3 of [39] demonstrates model structures on commutative and associative algebras in this setting, so Theorem 6.2 in [39] provides preservation results for such algebraic structures under right Bousfield localization. Theorem 5.1 then proves that $R_{K'}$ lifts to the homotopy category of commutative monoids and that $R_{K''}$ lifts to the homotopy categories of commutative monoids and associative monoids.

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