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A STATELESS ORTHOMODULAR POSET WITH THE AUGMENTATION PROPERTY

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In the matrix representation of an effect algebra L (in particular, an orthomodular poset), if L admits a state, then its matrix $M(L)$ has the augmentation property $(\mathrm{rank}(M(L)) = \mathrm{rank}(M(L)|1))$. We show that the reverse implication is not true, thus answering an open question published on effect algebras.

Keywords: effect algebra, orthomodular poset, state, rank of a matrix

Classification: 06C15, 81P15, 81P10

1. INTRODUCTION

In the paper [1], the authors assign a matrix to an effect algebra and give necessary and sufficient conditions for a matrix to represent an effect algebra. They show that if it has a state then the assigned matrix and its augmented matrix have the same ranks. They further ask whether the reverse implication is also true (see [1, p. 749, Problem]). In this note we disprove it even in a special class of orthomodular posets and lattices.

2. BASIC NOTIONS

Effect algebras were introduced as a rather general mathematical structure that describes the events (effects) in quantum systems [4]. In this paper we take up a problem formulated in [1]. We show that to present a counterexample to the conjecture formulated therein, it suffices to consider special cases of effect algebras: orthomodular posets or lattices [4, 7, 11, 12]. The reader acquainted with the Greechie paste job [5, 12] may have an advantage in understanding the technical part of the paper, though it is not absolutely necessary since we stuck to the presentation of [1].

Definition 2.1. Let $(L, \leq, 0, 1, ')$ be an orthocomplemented poset that fulfills the orthomodular law $(a \leq b \Rightarrow b = a \vee (b \wedge a'))$. Then the pentuple $(L, \leq, 0, 1, ')$ is said to be an *orthomodular poset* (OMP) [12]. Its elements a, b are called *orthogonal* if $a \leq b'$. A mapping $s\colon L \to [0, 1]$ is called a *state* if

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1. $s(1) = 1$,
2. if $a \le b'$, then $s(a \vee b) = s(a) + s(b)$.

Throughout this note, we shall deal with a finite orthomodular poset, $(L, \le, 0, 1, ')$, denoted briefly by L. An *atom* is a minimal non-zero element. Every element can be expressed (not necessarily uniquely) as a supremum of mutually orthogonal atoms. It is known that a finite OMP is a union of its maximal Boolean subalgebras (called *blocks*), which correspond to maximal sets of mutually orthogonal atoms.

For OMPs, the matrix representation of effect algebras, introduced in [1], can be simplified to the following form:

Definition 2.2. (Bińczak et al. [1]) Let L be a finite OMP with n atoms and m blocks. The matrix $M(L) \in \mathbb{R}^{m \times n}$ associated with L has entries

$$(M(L))_{i,j} = \begin{cases} 1 & \text{if the } i\text{th block contains the } j\text{th atom,} \\ 0 & \text{otherwise.} \end{cases}$$

By $(M(L)|1)$ we denote the augmented matrix with the additional right-hand column $\underbrace{(1, 1, 1, \dots, 1)}_{m\times}{}^T$.

Let us note that in [1], a more general matrix representation of effect algebras is defined. It admits higher integers as entries. As one of main contributions, they give necessary and sufficient conditions for a matrix to represent an effect algebra. This result is based on [6]. To simplify the representation, let us formulate the rest of this note in terms of OMPs only.

Remark 2.3. A finite OMP L can be represented by its *Greechie diagram* [5]. It is a hypergraph whose vertices are atoms and whose edges are maximal orthogonal sets of atoms (blocks). In expressing this by figures, the edges are drawn as smooth lines (see figures below).

The states then are in a one-to-one correspondence with those non-negative labellings of vertices which sum up to 1 over all atoms of each block.

According to [5], the conjunction of the following conditions is sufficient for a hypergraph to be a Greechie diagram of an OMP:

- Each edge has at least three vertices.
- Every two edges have at most one common vertex.
- There is no cycle of length three.

If, moreover, there is no cycle of length four, the corresponding OMP is an OML.

3. RESULTS

Not all OMPs or OMLs admit a state [5]. In [1], the following observation was made:

Proposition 3.1. If an OMP L has a state, then $\mathrm{rank}(M(L)) = \mathrm{rank}(M(L)|1)$.

Proof. If s is a state, the state values at all atoms could be viewed as a vector x solving the system of linear equations $M(L) \cdot x = (1, 1, 1, \ldots, 1)^T$. So the result follows from the Rouché–Capelli theorem (also called Frobenius theorem, etc.), see also [1]. □

The authors of [1] ask if the necessary condition from Prop. 3.1 happens to be sufficient. We disprove this conjecture by the following counterexample.

Example 3.2. Let L be the orthomodular poset represented by the matrix

$$M(L) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The augmented matrix is

$$(M(L)|1) = \left(\begin{array}{ccccccccccccc|c} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right).$$

They both are of full rank, $\mathrm{rank}(M(L)) = \mathrm{rank}(M(L \mid 1)) = 7$.

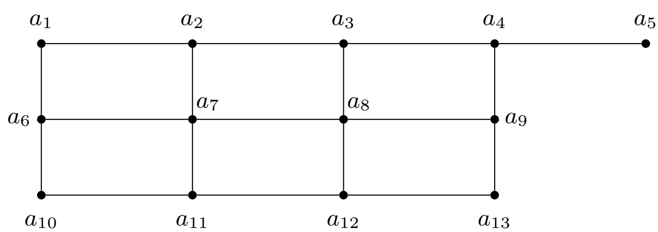

Fig. 1. The Greechie diagram of an orthomodular *poset* representing a counterexample.

There is no state on L because the system of linear equations represented by the augmented matrix does not have a non-negative solution. This can be easily seen from the Greechie diagram of L in Figure 1. (Observe that the cycles therein are of order 4 and therefore the diagram defines an OMP, see [5].) There are three horizontal edges and four vertical edges. If s is a state of L, we see that

$$\underbrace{\sum_{i=1}^{5} s(a_i)}_{1} + \underbrace{\sum_{i=6}^{9} s(a_i)}_{1} + \underbrace{\sum_{i=10}^{13} s(a_i)}_{1}$$
$$= \underbrace{s(a_1) + s(a_6) + s(a_{10})}_{1} + \underbrace{s(a_2) + s(a_7) + s(a_{11})}_{1}$$
$$+ \underbrace{s(a_3) + s(a_8) + s(a_{12})}_{1} + \underbrace{s(a_4) + s(a_9) + s(a_{13})}_{1} + s(a_5)$$

and therefore $3 = 4 + s(a_5)$. Thus $s(a_5) = -1$ and this is absurd. So L is stateless.

This manifests the conditions of the counterexample and we are done.

The atom a_5 is necessary for the augmentation property.

A similar counterexample can be constructed even for orthomodular lattices.

Example 3.3. The smallest OMLs without states are listed in [10], the first one comes from [8]. Again, we can add one vertex, a_5, that would play the same role as in Example 3.2. We obtain the Greechie diagram of an OML, drawn in Figure 2. Following Remark 2.3, it represents an OML, while the hypergraph of Figure 1 does not because it contains cycles of length four.

Each state attains the value of -1 at a_5. The proof follows verbatim the same reasoning. We can cover all vertices by 9 horizontal edges and all vertices different from a_5 by 10 vertical edges.

Let us shortly comment on the question we pursued here. Let us call a mapping $s\colon L \to \mathbb{R}$ a *signed state* if $s(1) = 1$, and $a \leq b' \implies s(a) + s(b) = s(a \vee b)$. The idea used above (the application of the Rouché–Capelli theorem) allows us to spell out the following result:

Proposition 3.4. A finite OMP (even any finite effect algebra) L admits a signed state if and only if $\operatorname{rank} M(L) = \operatorname{rank}(M(L)|1)$.

In Examples 3.2 and 3.3, the atom a_5 was needed for the existence of a signed state. E.g., in Example 3.2 we may take

$$s(a_1) = s(a_2) = s(a_8) = s(a_{13}) = -s(a_5) = 1$$

and $s(a_i) = 0$ for all other vertices.

In view of the results of [9, 13], if we needed to deal with the group-valued states (and there might be the need for that in quantum measurement [2, 3]) we are in the position to generalize the above-stated result to the group-valued states.

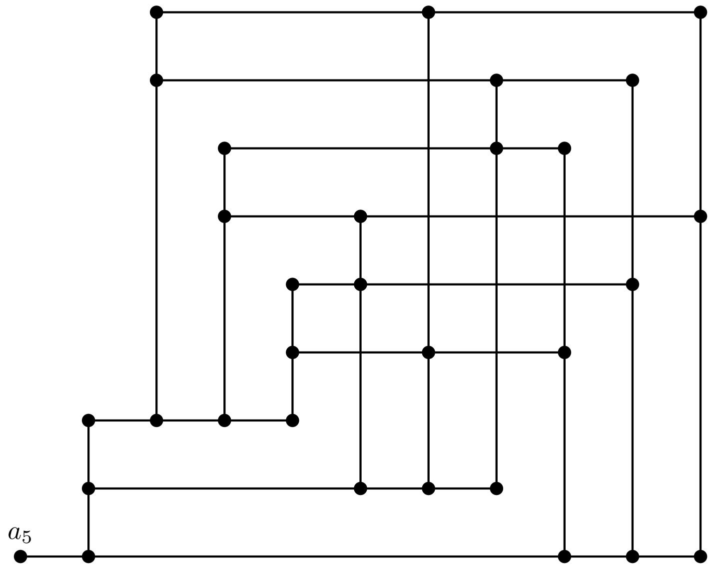

Fig. 2. The Greechie diagram of an orthomodular *lattice* representing a counterexample.

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