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β-EXPONENTIAL STABILITY OF TIME-DELAY SYSTEMS BASED ON SLIDING MODE CONTROL

Nassim Athmouni, Nejib Brahmia, Tarek Fajraoui, and Fehmi Mabrouk

This paper investigates sliding mode control for one-sided Lipschitz non-linear systems with time-delays and uncertainties. A suitable integral sliding surface is introduced, explicitly accounting for delay terms and distinguishing itself from existing approaches. To guarantee the β-exponential stability, a new sufficient condition is derived in the form of a linear matrix inequality. Furthermore, an appropriate sliding mode control law is developed to enforce finite-time convergence of the system states to the sliding surface and guarantee their persistence on it. Finally, a comparative numerical example is conducted to evaluate the performance and practicality of the proposed control strategy.

Keywords: time-delay systems, β-exponentially stable, one-sided Lipschitz, sliding mode control, linear matrix inequality

Classification: 93D05, 93D15, 93D23

1. INTRODUCTION

Over the past several decades, time-delay systems have been frequently encountered in various scientific and technical fields. This is due to the fact that delays frequently occur in many real systems, including engineering [7], biology [13], chemical reactors [9], population dynamics models [19] and an economic model [5]. The analysis of systems without delay is relatively straightforward compared to systems under time-delay. Although delay-dependent terms can lead to system stability in some cases, they are more often a source of instability and performance degradation in control problems, especially when the delays are time-varying, unknown, or unmodeled.

A time-varying delay can be considered as a source of external uncertainty in a dynamical system. It may not only introduce external uncertainties but also amplify internal ones, making the prediction of the system's behavior more challenging. This relationship between uncertainty and time delay reflects the need for the use of robust control strategies, which must deal simultaneously with modeling uncertainties and the effect of delay to ensure system stability and desired performance. This intrinsic connection between delay and uncertainty has been widely recognized in the literature [6, 24].

In order to overcome the challenges of stability, different effective control design strategies have been introduced in the literature, e. g., adaptive and robust control tech-

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nique [26, 27], the classical proportional-integral, and control of nonlinear systems via linear approximations [10], but these strategies are vulnerable to the uncertainty, which immediately impacts validity of the control law.

For the purpose of enhancing the efficiency of the system, different advanced nonlinear control methodologies have emerged in order to boost the robustness as well as the control strategies of uncertain nonlinear systems. These encompass intelligent control, model predictive control, observer control method [1, 2, 3, 18] and sliding mode control (SMC) [20, 23, 25]. Intelligent control has the benefit of not relying on an accurate mathematical model of the system. Nevertheless, its design and implementation processes tend to be complicated and computationally demanding. Model predictive control provides an intuitive framework for handling multi-variable and nonlinear systems and can explicitly incorporate constraints.

Observer-based methods improve system robustness by estimating and compensating for disturbances and uncertainties, to attain better tracking performance and improved system stability. Observer-based methods improve system robustness by estimating and compensating for disturbances and uncertainties, to attain better tracking performance and improved system stability. To enhance the performance of this strategy, many advanced nonlinear methods have been applied, such as neural networks [16, 17]. On the other hand, sliding mode control can be easily implemented and provides satisfying robustness for uncertainties of the model, so it can be used as a possible method of handling uncertainties and disturbance of nonlinear systems. Throughout the last few decades, SMC has gained significant attention due to its high robustness and accuracy. Noted in [23, 25], SMC comes across as an attractive strategy for managing nonlinear dynamic systems, particularly for systems that are subjected to some level of uncertainty or change. SMC have been extensively applied in the design of control laws for uncertain systems due to their numerous advantageous properties, including fast response, enhanced transient performance, and order reduction. Nonetheless, conventional SMC schemes employ a switching function to produce the control signals, which often leads to chattering. To address this problem, a vast number of seminal solutions have been suggested for example, [14] introduced a fixed-time sliding mode control system that ensures convergence within a specified time frame, independent of initial conditions, for DC/DC buck converters dealing with mismatched uncertainty. In addition, a fixed-time convergent sliding mode controller based on an observer has been adopted to permanent-magnet synchronous motors, targeting both the inner current loop and the outer speed loop [15].

It is well known that most nonlinear systems satisfy the Lipschitz condition [11, 12]. However, the Lipschitz condition is rarely satisfied globally, which limits the application of results based on a Lipschitz continuity condition. Several important results on Lipschitz systems have been reported, such as the output feedback stabilization [4], observer design, and practical stability of Lipschitz nonlinear systems [2]. It is worth mentioning that studies have indicated that design methods based on Lipschitz conditions are often effective only when Lipschitz constant is small. When the Lipschitz constant is large, or must be chosen to be large, the well-known Lipschitz conditions will fail to be effective. To overcome this limitation, the one-side Lipschitz condition has been suggested in various articles for control design . In [1], this concept was further expanded by incorporating the quadratically inner-bounded condition. Significant research efforts have been devoted to the study of one-sided Lipschitz non-linear systems [3]. Moreover, the issue of SMC for one-sided Lipschitz nonlinear systems without delay was investigated in [22].

At present, there are few studies on sliding mode control for time delay-varying nonlinear systems with uncertainties under one-sided Lipschitz and quadratic inner-bounded conditions. In [21], the authors studied a class of non-linear uncertain dynamical systems with time-varying delay using sliding mode control and the Lyapunov-Razumikhin approach, but their analysis is limited to asymptotic stability, and does not incorporate the one-sided Lipschitz condition. In [8], the authors establish the robust control problem for a class of one-sided Lipschitz nonlinear systems with time-varying delays and parameter uncertainties based on a state observer. They derived sufficient conditions to guarantee the asymptotic stability of the resulting closed-loop system. Motivated by SMC techniques, this work investigates the β-exponential stability for time-delay systems incorporating a nonlinear component subject to the one-sided Lipschitz condition. The main idea of this paper is to use a discontinuous control law to ensure that the system state trajectories reach and remain on the sliding surface. Specifically, the integral sliding mode control law is formulated based on the constructed integral sliding surface and the reaching condition. The outline of this paper is as follows: Section 2 presents the problem statement and preliminaries. Section 3 discusses the sliding surface design for time-varying delay systems. Section 4 provides the stability analysis. Section 5 details the construction of the control law. Finally, Section 6 provides a comparative numerical example to illustrate the results.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider the nonlinear time-varying system with time delays:

$$\begin{cases} \dot{x}(t) = (A+\delta A)x(t) + (A_d+\delta A_d)x(t-h(t)) + f(t,x(t),x(t-h(t))) \\ \qquad\qquad + \delta f(t,x(t),x(t-h(t))) + Bu(t), \\ x(t) = \phi(t) \quad \text{for } t \in [-h,0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^s$ is the input. Moreover, $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times s}$ with $rank(B) = s$, are known constant matrices, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is nonlinear functions, $h(t)$ represents time-varying delay which satisfies:

$$0 \leq h(t) \leq h, \quad \dot{h}(t) \leq \dot{h} < 1,$$

where h, $\dot{h}$ are known scalars and $\phi \in \mathcal{C} = (C([-h,0],\mathbb{R}^n), \|.\|_\infty)$ is the initial condition with $\|.\|_\infty$ denoting the norm of uniform convergence. The nonlinear uncertainty function δf satisfies:

$$\delta f^T \delta f \leq \alpha x^T x, \text{ where } \alpha \geq 0. \tag{2}$$

The time-varying parameter uncertainties δA, δA_d , are structured as:

$$\delta A = M_0 H_0(t) N_0 \text{ and } \delta A_d = M_1 H_1(t) N_1,$$

where M_0, M_1, N_0 and N_1 are known real constant matrices of appropriate dimensions, while $H_0(t) \in \mathbb{R}^{k_0 \times l_0}$ and $H_1(t) \in \mathbb{R}^{k_1 \times l_1}$ are unknown matrices, satisfying:

$$H_i^T(t)H_i(t) \leq I, \quad \text{for all} \quad i = 0, 1.$$

It is worth mentioning that the following well-known facts and lemmas are essential for deriving the main results.

Remark 2.1. The condition assumed on the delay function is crucial to guarantee the causality of the system. Without this restriction, time-varying delay could lead to situations where the system depends on future states, which is physically unrealistic. Moreover, this assumption is commonly adopted to justify the applicability of Lyapunov–Krasovskii functional.

Remark 2.2. Under the assumption for the rank of the matrix B it is clear that the square matrix $B^T B$ is nonsingular.

Definition 2.3. Given $\beta > 0$. System (1) is β-exponentially stable if there exists a positive number $\eta > 0$ such that every solution $x(t, \phi)$ satisfies:

$$\|x(t, \phi)\| \leqslant \eta \|\phi\|_\infty e^{-\beta t}, \quad \forall t \in \mathbb{R}^+$$

and for all the uncertainties $\delta A(t)$, $\delta A_d(t)$.

Lemma 2.4. (Wang et al. [26]) For given matrices M, N, H with $H^T H \leq I$ and scalar $\varepsilon > 0$, the following inequality is satisfied:

$$MH(t)N + N^T H(t)^T M^T \leq \varepsilon M M^T + \varepsilon^{-1} N^T N.$$

Lemma 2.5. (Schur Complement)
Given constant symmetric matrices Σ_1, Σ_2, Σ_3 where $\Sigma_1 = \Sigma_1^T$ and $\Sigma_2 = \Sigma_2^T$, then the following equivalence holds:

$$\Sigma_2 < 0 \text{ and } \Sigma_1 - \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0 \text{ if and only if } \begin{pmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & \Sigma_2 \end{pmatrix} < 0.$$

Lemma 2.6. [18] For any constant matrix $M \in \mathbb{R}^{n \times n}$, $M = M^T > 0$, scalar $\gamma > 0$, and vector function $\omega : [0, \gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, it holds that:

$$\left(\int_0^\gamma \omega(s) \, \mathrm{d}s \right)^T M \left(\int_0^\gamma \omega(s) \, \mathrm{d}s \right) \leq \gamma \int_0^\gamma \omega^T(s) M \omega(s) \, \mathrm{d}s.$$

Definition 2.7. (Abbaszadeh and Marquez [1]) The nonlinear function f is said to be one-sided Lipschitz in the region $\mathcal{D} \subset \mathbb{R}^n$ if there exist $a_1, a_2 \in \mathbb{R}$ such that $\forall\, x,\ \bar{x},\ y,\ \bar{y} \in \mathcal{D}$, we have

$$(x - \bar{x})^T (f(t, x, y) - f(t, \bar{x}, \bar{y})) \quad \leq \quad a_1 (x - \bar{x})^T (x - \bar{x}) + a_2 (y - \bar{y})^T (y - \bar{y})$$

a_1 and a_2 are the one-sided Lipschitz constants which can be positive or negative.

Definition 2.8. (Abbaszadeh and Marquez [1]) The nonlinear function f is said to be quadratically inner-bounded in the region $\mathcal{D} \subset \mathbb{R}^n$, if there exist $b_1,\ b_2,\ c_1,\ c_2 \in \mathbb{R}$ such that $\forall\ x,\ \bar{x},\ y,\ \bar{y} \in \mathcal{D}$, the following inequality holds:

$$\begin{aligned}(f(t,x,y) - f(t,\bar{x},\bar{y}))^T(f(t,x,y) - f(t,\bar{x},\bar{y})) \leq b_1(x-\bar{x})^T(x-\bar{x}) + b_2(y-\bar{y})^T(\bar{x}-\bar{y})\\ +c_1(x-\bar{x})^T(f(t,x,y) - f(t,\bar{x},\bar{y}))\\ +c_2(y-\bar{y})^T(f(t,x,y) - f(t,\bar{x},\bar{y})).\end{aligned}$$

Assumption 2.1. The nonlinear function f satisfies the one-sided lipshitz condition.

Assumption 2.2. The nonlinear function f satisfies the quadratically inner-bounded condition.

Remark 2.9. Assumptions 2.1 and 2.2 are widely applicable to practical time-delay systems. In particular, the one-sided Lipschitz condition is less conservative than the classical global Lipschitz condition and captures the dissipative nature of physical systems exhibiting nonlinear characteristics such as saturation or dead zones. Similarly, the quadratic inner-boundedness offers a general and flexible representation for uncertain nonlinearities through tractable quadratic constraints, which is highly useful in the design of robust observers and controllers. It is worth noting that one-sided Lipschitz nonlinear systems include Lipschitz systems as a special case. Unlike the Lipschitz condition, the constants $a_1,\ a_2,\ b_1,\ b_2,\ c_1,$ and c_2 in Assumptions 2.1 and 2.2 can take positive, negative, or zero values. Moreover, if the function $f(x(t), x(t-h(t)))$ satisfies the global Lipschitz condition, then it also satisfies both the one-sided Lipschitz and the quadratic inner-boundedness conditions.

3. SLIDING SURFACE DESIGN

In this section, we shall design the switching manifold, so that the sliding motion on that manifold has the desired dynamics.

Remark 3.1. In SMC, there are two phases: the reaching phase, where states are driven to reach the sliding surface and the sliding phase where the system states must remain on the surface due to the control.

Define the following sliding mode switching function

$$s(t) = Gx(t) - \int_0^t G(A+BK)x(s)\,\mathrm{d}s - \int_0^t GA_d x(s-h(s))\,\mathrm{d}s, \tag{3}$$

where G $\in \mathbb{R}^{m\times n}$ and satisfy GB is nonsingular. Note that G is not unique but any choice is acceptable for given matrix B.

The time derivative of $s(t)$ along the trajectories of (1) is

$$\begin{aligned}\dot{s}(t) &= G\dot{x}(t) - G(A+BK)x(t) - GA_d x(t-h(t))\\ &= G[(A+\delta A)x(t) + (A_d+\delta A_d)x(t-h(t)) + f(t,x(t),x(t-h(t)))\\ &\quad + \delta f(t,x(t),x(t-h(t))) + Bu(t)] - G(A+BK)x(t) - GA_d x(t-h(t))\end{aligned}$$

$$
\begin{aligned}
= \quad & -G(BK)x(t) + G\delta Ax(t) + G\delta A_d x(t-h(t)) + Gf(t,x(t),x(t-h(t))) \\
& + G\delta f(t,x(t),x(t-h(t))) + GBu(t).
\end{aligned}
$$

Remark 3.2. The performance of the SMC system depends on the design of the sliding surfaces, which is constructed to facilitate sliding mode dynamics. It should be noted that the surface function differs from the previous works without delay like in [20, 22]. In this work, a third term was added due to the influence of the time-delay.

Based on the sliding mode approach, when the system reaches the sliding surface in $t = t_r$. It follows that $s(t) = 0$ and $\dot{s}(t) = 0$ for any $t \geq t_r$.
To ensure the trajectory remains on the sliding surface, we will choose a control called equivalent control described as:

$$
\begin{aligned}
u_{eq}(x(t)) :=& (GB)^{-1}G[BKx(t) - \delta Ax(t) - \delta A_d x(t-h(t)) \\
& - f(t,x(t),x(t-h(t))) - \delta f(t,x(t),x(t-h(t)))].
\end{aligned}
$$

Substituting (3) into (1), results in the sliding motion described below:

$$
\begin{aligned}
\dot{x}(t) = & (A+\delta A)x(t) + (A_d + \delta A_d)\, x(t-h(t)) \\
& + (f(t,x(t),x(t-h(t))) + \delta f(t,x(t),x(t-h(t)))) \\
& + B(GB)^{-1}G[BKx(t) - \delta Ax(t) - \delta A_d x(t-h(t)) - f(t,x(t),x(t-h(t))) \\
& - \delta f(t,x(t),x(t-h(t)))].
\end{aligned}
$$

Hence, it will lead to:

$$
\begin{aligned}
\dot{x}(t) = & (A + BK + (I - B(GB)^{-1}G)\delta A)x(t) + (A_d + (I - B(GB)^{-1}G)\delta A_d)xx(t-h(t)) \\
& + (I - B(GB)^{-1}G)(f(t,x(t),x(t-h(t))) + \delta f(t,x(t),x(t-h(t)))).
\end{aligned}
$$

To simplify the calculation, $x(t)$ will be replaced with x, $x(t-h(t))$ with x_h, and $f(t,x(t),x(t-h(t)))$ with f.
Consider that:

$$
\begin{aligned}
\tilde{G} &:= (I - B(GB)^{-1}G), \\
\tilde{A} &:= A + BK + \tilde{G}\delta A, \\
\tilde{A}_d &:= A_d + \tilde{G}\delta A_d, \\
\tilde{f} &:= \tilde{G}(f + \delta f).
\end{aligned}
$$

Thus, on the sliding surface, the system (1) holds:

$$
\dot{x} = \tilde{A}x + \tilde{A}_d x_h + \tilde{f}. \tag{4}
$$

4. STABILITY ANALYSIS

Theorem 4.1. Under assumption 2.1 and assumption 2.2, if there exist positive constants ε_1, ε_2, ρ_0, ρ_1, symmetric positive definite matrices P, Q, Q_1 and a square matrix

Y can be found to satisfy the following LMI:

$$\Omega = \begin{pmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & 0 & \Xi_{15} & \Xi_{16} & \Xi_{17} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & 0 & 0 \\ * & * & \Xi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -\rho_0 I & 0 \\ * & * & * & * & * & * & -\rho_1 I \end{pmatrix} < 0, \tag{5}$$

where
$\Xi_{11} = A^T P + Y^T + PA + Y + hQ + Q_1 + \alpha I + 2\beta P + \epsilon_2 b_1 I + \epsilon_1 a_1 I + \rho_0 N_0^T N_0$, $\Xi_{12} = PA_d$,
$\Xi_{13} = P\tilde{G} - \frac{1}{2}\varepsilon_1 I + \frac{\varepsilon_2}{2} c_1 I$, $\Xi_{15} = P\tilde{G}$, $\Xi_{16} = P\tilde{G}M_0$, $\Xi_{17} = P\tilde{G}M_1$,
$\Xi_{22} = \rho_1 N_1^T N_1 + \varepsilon_1 a_2 I - (1-\dot{h})e^{-2\beta h} Q_1 + \varepsilon_2 b_2 I$, $\Xi_{23} = \frac{1}{2}\varepsilon_2 c_2 I$,
$\Xi_{33} = -\varepsilon_2 I$, $\Xi_{44} = -\frac{e^{-2\beta h}}{h} Q$,
then, the sliding mode dynamic (4) will be β-exponentially stable under the gain matrix $K = (B^T B)^{-1} B^T P^{-1} Y$.

P r o o f. Define the Lyapunov function as like $V_x = V_1 + V_2 + V_3$, where

$$V_1 = x^T P x, \quad V_2 = \int_{-h}^{0} \int_{t+\theta}^{t} x^T(s) e^{2\beta(s-t)} Q x(s) \, \mathrm{d}s \mathrm{d}\theta$$

and

$$V_3 = \int_{t-h(t)}^{t} x^T(s) e^{2\beta(s-t)} Q_1 x(s) \, \mathrm{d}s.$$

Calculating the time derivative of V_1, V_2, and V_3 along of (4)

$$\begin{aligned} \dot{V}_1 &= [\tilde{A}x + \tilde{A}_d x_h + \tilde{f}]^T P x + x^T P [\tilde{A}x + \tilde{A}_d x_h + \tilde{f}] \\ &= x^T \tilde{A}^T P x + x_h^T \tilde{A}_d^T P x + \tilde{f}^T P x + x^T P \tilde{A} x + x^T P \tilde{A}_d x_h + x^T P \tilde{f} \\ &= x^T [A + BK + \tilde{G} M_0 H_0 N_0]^T P x + x_h^T [A_d + \tilde{G} M_1 H_1 N_1]^T P x \\ &\quad + [\tilde{G}(f + \delta f)]^T P x + x^T P [A + BK + \tilde{G} M_0 H_0 N_0] x \\ &\quad + x^T P [A_d + \tilde{G} M_1 H_1 N_1] x_h + x^T P [\tilde{G}(f + \delta f)]. \end{aligned}$$

By using lemma 2.4 and for $\rho_0 > 0$, one gets:

$$\left(P\tilde{G}M_0\right) H_0 N_0 + \left(\left(P\tilde{G}M_0\right) H_0 N_0\right)^T \leqslant \rho_0^{-1} \left(P\tilde{G}M_0\right) \left(P\tilde{G}M_0\right)^T + \rho_0 N_0^T N_0.$$

Additionally, for $\rho_1 > 0$ the expression becomes:

$$\begin{aligned} 2x^T P\tilde{G} M_1 H_1 N_1 x_h &\leqslant \rho_1^{-1} x^T \left(P\tilde{G}M_1\right) \left(P\tilde{G}M_1\right)^T x + \rho_1 x_h^T N_1^T H_1^T H_1 N_1 x_h \\ &\leqslant \rho_1^{-1} x^T \left(P\tilde{G}M_1\right) \left(P\tilde{G}M_1\right)^T x + \rho_1 x_h^T N_1^T N_1 x_h, \end{aligned}$$

by the classical inequality one gets:

$$2x^T P\tilde{G}\delta f \quad \leq \quad x^T(P\tilde{G})(P\tilde{G})^T x + \delta f^T \delta f.$$

Under inequality (2) it follows that:

$$2x^T P\tilde{G}\delta f \quad \leq \quad x^T(P\tilde{G})(P\tilde{G})^T x + \alpha x^T x.$$

Then the following inequality results:

$$\begin{aligned}\dot{V}_1 \leqslant\ & x^T \left[(A+BK)^T P + P(A+BK)\right] x \\ & + 2\beta x^T P x - 2\beta V_1(t) + \rho_0^{-1} x^T \left(P\tilde{G}M_0\right)\left(P\tilde{G}M_0\right)^T x \\ & + \rho_0 x^T N_0^T N_0 x + 2x^T P A_d x_h + \rho_1^{-1} x^T \left(P\tilde{G}M_1\right)\left(P\tilde{G}M_1\right)^T x + \rho_1 x_h^T N_1^T N_1 x_h \\ & + 2x^T P\tilde{G} f + x^T(P\tilde{G})(P\tilde{G})^T x + \alpha x^T x.\end{aligned}$$

As for

$$\begin{aligned}\dot{V}_2 = -2\beta e^{-2\beta t}\int_{-h}^{0}\int_{t+\theta}^{t} x^T(s)e^{2\beta s}Qx(s)\,\mathrm{d}s\mathrm{d}\theta + e^{-2\beta t}[hx^T e^{2\beta t}Qx \\ - \int_{-h}^{0} x^T(t+\theta)e^{2\beta(t+\theta)}Qx(t+\theta)\,\mathrm{d}\theta]\end{aligned}$$

by using Lemma 2.6 and a variable change for the integral it is deduced that:

$$\dot{V}_2 \leqslant x^T hQx - 2\beta V_2(t) - \frac{1}{h}\left(\int_{t-h}^{t} x(s)e^{\beta(s-t)}\,\mathrm{d}s\right)^T Q\left(\int_{t-h}^{t} x(s)e^{\beta(s-t)}\,\mathrm{d}s\right).$$

Now as for

$$\dot{V}_3 = -2\beta V_3 + x^T Q_1 x - (1-\dot{h}(t))x_h^T e^{-2\beta h(t)} Q_1 x_h,$$

it can be written for $\varepsilon_2 > 0$:

$$\varepsilon_2 b_1 x^T x + \varepsilon_2 b_2 x_h^T x_h + \varepsilon_2 c_1 x^T f + \varepsilon_2 c_2 x_h^T f - \varepsilon_2 f^T f \geq 0$$

and by using the fact that f is one-sided Lipschitz, inner-bounded and $f(0,0) = 0$ we have that:

$$f^T x \leqslant a_1 x^T x + a_2 x_h^T x_h,$$

consequently one gets:

$$\forall \varepsilon_1 > 0 \text{ we have } \varepsilon_1 a_1 x^T x + \varepsilon_1 a_2 x_h^T x_h - \varepsilon_1 x^T f \geq 0$$

and

$$f^T f \leqslant b_1 x^T x + b_2 x_h^T x_h + c_1 x^T f + c_2 x_h^T f.$$

So the inequality becomes:

$$\begin{aligned}\dot{V}_3 \leqslant -2\beta V_3 + x^T Q_1 x - (1-\dot{h}(t))x_h^T e^{-2\beta h(t)} Q_1 x_h + \varepsilon_1 a_1 x^T x + \varepsilon_1 a_2 x_h^T x_h - \varepsilon_1 x^T f \\ + \varepsilon_2 b_1 x^T x + \varepsilon_2 b_2 x_h^T x_h + \varepsilon_2 c_1 x^T f + \varepsilon_2 c_2 x_h^T f - \varepsilon_2 f^T f.\end{aligned}$$

We infer:

$$\begin{aligned}
\dot{V}_x &= \dot{V}_1 + \dot{V}_2 + \dot{V}_3 \\
&\leqslant x^T \left[(A+BK)^T P + P(A+BK) + hQ\right] x + \rho_0^{-1} x^T \left(P\tilde{G}M_0\right)\left(P\tilde{G}M_0\right)^T x \\
&+ \rho_0 x^T N_0^T N_0 x + 2x^T P A_d x_h + \rho_1^{-1} x^T \left(P\tilde{G}M_1\right)\left(P\tilde{G}M_1\right)^T x + \rho_1 x_h^T N_1^T N_1 x_h \\
&+ 2x^T P\tilde{G} f + x^T (P\tilde{G})(P\tilde{G})^T x + \alpha x^T x + 2\beta x^T P x - 2\beta(V_1 + V_2) \\
&- \frac{e^{-2\beta h}}{h}\left(\int_{t-h}^{t} x(s) e^{\beta(s-t)}\,\mathrm{d}s\right)^T Q \left(\int_{t-h}^{t} x(s) e^{\beta(s-t)}\,\mathrm{d}s\right) \\
&- 2\beta V_3 + x^T Q_1 x - (1-\dot{h}(t)) x_h^T e^{-2\beta h(t)} Q_1 x_h + \varepsilon_1 a_1 x^T x + \varepsilon_1 a_2 x_h^T x_h - \varepsilon_1 x^T f \\
&+ \varepsilon_2 b_1 x^T x + \varepsilon_2 b_2 x_h^T x_h + \varepsilon_2 c_1 x^T f + \varepsilon_2 c_2 x_h^T f - \varepsilon_2 f^T f.
\end{aligned}$$

It yields that:

$$\dot{V}_x \leq \begin{pmatrix} x \\ x_h \\ f \\ \int_{t-h}^{t} x(s)\,\mathrm{d}s \end{pmatrix}^T \begin{pmatrix} \Xi'_{11} & \Xi_{12} & \Xi_{13} & 0 \\ * & \Xi_{22} & \Xi_{23} & 0 \\ * & * & \Xi_{33} & 0 \\ * & * & * & \Xi_{44} \end{pmatrix} \begin{pmatrix} x \\ x_h \\ f \\ \int_{t-h}^{t} x(s) e^{\beta(s-t)}\,\mathrm{d}s \end{pmatrix} - 2\beta V,$$

where

$$\begin{aligned}
\Xi'_{11} &= (A+BK)^T P + P(A+BK) + hQ + Q_1 + \rho_1^{-1}\left(P\tilde{G}M_1\right)\left(P\tilde{G}M_1\right)^T \\
&\quad + \rho_0^{-1}(P\tilde{G}M_0)(P\tilde{G}M_0)^T + (P\tilde{G})(P\tilde{G})^T + \alpha I + 2\beta P + \varepsilon_2 b_1 I + \varepsilon_1 a_1 I + \rho_0 N_0^T N_0.
\end{aligned}$$

Since $\Omega' < 0$ and by using Schur complement Lemma 2.5, it follows that:

$$\dot{V}_x \leqslant -2\beta V_x \tag{6}$$

by integrating (6) on $[0, t]$ results in

$$\int_0^t \frac{\dot{V}(x_s)}{V(x_s)} \leq -2\beta t.$$

So,

$$V(x_t) \leq V(x_0) e^{-2\beta t}.$$

Therefore the following holds:

$$\begin{aligned}
x^T P x &= x_t^T(0) P x_t(0) \\
&\leq \lambda_{\max}(P)\,\|x_t(0)\|^2 \\
&\leq \lambda_{\max}(P)\,\|x_t\|_\infty^2
\end{aligned}$$

and

$$\int_{t-h(t)}^{t} x^T(s) e^{2\beta(s-t)} Q_1 x(s)\,\mathrm{d}s \leq \lambda_{\max}(Q_1) \int_{t-h}^{t} x^T(s) e^{2\beta(s-t)} x(s)\,\mathrm{d}s$$

$$\begin{aligned}
&\leq \lambda_{\max}(Q_1)\int_{-h}^{0} x^T(\theta+t)e^{2\beta\theta}x(\theta+t)\,\mathrm{d}\theta \\
&\leqslant \lambda_{\max}(Q_1)\int_{-h}^{0} x_t^T(\theta)x_t(\theta)\,\mathrm{d}\theta \\
&\leqslant h\lambda_{\max}(Q_1)\,\|x_t\|_\infty^2
\end{aligned}$$

and

$$\begin{aligned}
\int_{-h}^{0}\int_{t+\theta}^{t} x^T(s)e^{2\beta(s-t)}Qx(s)\,\mathrm{d}s\mathrm{d}\theta &= \int_{-h}^{0}\int_{\theta}^{0} x^T(\tau+t)e^{2\beta\tau}Qx(\tau+t)\,\mathrm{d}\tau\mathrm{d}\theta \\
&\leq \lambda_{\max}(Q)\int_{-h}^{0}\int_{\theta}^{0} x_t^T(\tau)e^{2\beta\tau}Qx_t(\tau)\,\mathrm{d}\tau\mathrm{d}\theta \\
&\leq \frac{h^2}{2}\lambda_{\max}(Q)\,\|x_t\|_\infty^2 .
\end{aligned}$$

It can be established that:

$$V_x \leq \left[\lambda_{\max}(P) + h\lambda_{\max}(Q_1) + \frac{h^2}{2}\lambda_{\max}(Q)\right]\|x_t\|_\infty^2 .$$

Since

$$V(x_0) \leq \left[\lambda_{\max}(P) + h\lambda_{\max}(Q_1) + \frac{h^2}{2}\lambda_{\max}(Q)\right]\|x_0\|_\infty^2$$

this yields the following inequality:

$$V(x_t) \leq \left[\lambda_{\max}(P) + h\lambda_{\max}(Q_1) + \frac{h^2}{2}\lambda_{\max}(Q)\right]\|\phi\|_\infty^2 e^{-2\beta t}.$$

Thus

$$\|x(t)\|^2 \leq \frac{\left[\lambda_{\max}(P) + h\lambda_{\max}(Q_1) + \frac{h^2}{2}\lambda_{\max}(Q)\right]}{\lambda_{\min}(P)}\|\phi\|_\infty^2 e^{-2\beta t}.$$

□

5. CONTROL LAW DESIGN

After constructing the integral sliding surface, we focus, in this section, on integrating the SMC law, through which the sliding motion is triggered within a finite time t_r by satisfying the reachability condition. Consequently the following theorem is declared.

Theorem 5.1. Consider the sliding surface function (3) and the gain matrix K acquired in Theorem 4.1. The following control law:

$$u(t) = Kx(t) - \nabla(t)sign(s(t)), \tag{7}$$

causes the state of the system (1) to reach the sliding surface from a random initial point within a finite time t_r, and subsequently remains on it, where

$$\nabla(t) = (GB)^{-1}(\beta + \|GM_1\|\|N_1\|\|x(t)\| + \|GM_2\|\|N_2\|\|x(t-h(t))\| + \|G\|\|f\| + G\|x\|)$$

and β is a positive scalar.

P r o o f. Let

$$s(t) = Gx(t) - \int_0^t G(A+BK)x(s)\,\mathrm{d}s - \int_0^t GA_d x(s-h(s))\,\mathrm{d}s.$$

So

$$\begin{aligned}\dot{s}(t) &= G\dot{x}(t) - G(A+BK)x(t) - GA_d x(t-h(t)) \\ &= G[(A+\delta A)x(t) + (A_d+\delta A_d)x(t-h(t)) + (f+\delta f) + Bu] \\ &\quad -G(A+BK)x(t) - GA_d x(t-h(t)) \\ &= -G(BK)x(t) + G\delta A x(t) + G\delta A_d x(t-h(t)) + Gf + G\delta f + GBu.\end{aligned}$$

Consider now the Lyapunov function candidate:

$$V_s(t) = \frac{1}{2}s^T(t)s(t).$$

Using the control law (7), $\dot{V}_s(t)$ is given by

$$\begin{aligned}\dot{V}_s(t) = s^T(t)[&G\delta A x(t) + G\delta A_d x(t-h(t)) + Gf + G\delta f \\ &+ (-\beta - \|GM_0\|\|N_0\|\|x(t)\| - \|GM_1\|\|N_1\|\|x(t-h(t))\| \\ &- \|G\|\|f\| - G\|x\|)sign(s(t))].\end{aligned}$$

Since for all $x, y \in \mathbb{R}^n$, we have $x^T y \leq \|x\| y.sign(y)$ where $sign(y) = \begin{pmatrix} sign(y_1) \\ \vdots \\ sign(y_n)\end{pmatrix}$.

It can be deduced that:

$$\dot{V}_s(t) \leq -\beta s^T(t) sign(s(t)) = -\beta \sum_{j=1}^{n} |s_j(t)|.$$

Furthermore we have $s^T(t)sign(s(t)) \geq \|s(t)\|$ and $V_s(t) = \frac{1}{2}s^T(t)s(t) = \frac{1}{2}\|s(t)\|^2$.
So the following holds:

$$\dot{V}_s(t) \leq -\sqrt{2}\beta\sqrt{V_s(t)}. \tag{8}$$

Integrating both sides of (8) from 0 to $t > 0$, this yields the following results:

$$\sqrt{V_s(t)} \leq \sqrt{V_s(0)} - \frac{\sqrt{2}}{2}\beta t.$$

Under the condition $V_s(t) \geq 0$ and when it takes the value 0 we reach the sliding surface $s(t) = 0$ and it is deduced that:

$$t_r \leq \frac{2}{\sqrt{2}}\sqrt{V_s(0)}.$$

□

6. ILLUSTRATIVE EXAMPLE

In this section, an example is given to show the effectiveness of our results.

Consider the uncertain nonlinear systems (1) with the following parameters:

$$A = \begin{bmatrix} 0.1 & -1 \\ 0.1 & -0.1 \end{bmatrix}, \quad \delta A = \begin{bmatrix} 0.02 & -0.04 \\ 0.01 & 0.03 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.2 & 0.1 \\ 0.6 & 0 \end{bmatrix}, \quad \delta A_d = \begin{bmatrix} -0.02 & 0.01 \\ 0.02 & -0.03 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad M_0 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}, \quad N_0 = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0.1 & 1 \end{bmatrix}, \quad f(t, x(t), x(t-h(t))) = \begin{bmatrix} x_1(t)\big(x_1^2(t) + x_2^2(t)\big) \\ x_2(t)\big(x_1^2(t) + x_2^2(t)\big) - 1.33\sin(x_2(t-h(t))) \end{bmatrix}.$$

Moreover, set

$$a_1 = 0, \quad a_2 = 1.33, \quad b_1 = -48, \quad b_2 = 1.7689, \quad c_1 = -16, \quad c_2 = 0, \quad \alpha = 0.001,$$

$$h(t) = 0.1\sin(t), \quad h = 0.2, \quad \dot{h} = 0.1.$$

Using MATLAB LMI Toolbox to solve the inequality (5), one obtains

$$P = \begin{bmatrix} 0.2609 & 0.0737 \\ 0.0737 & 0.1034 \end{bmatrix}, \quad Y = \begin{bmatrix} -8.1362 & -12.5033 \\ 11.1496 & -7.6350 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.8462 & -0.0000 \\ -0.0000 & 0.8462 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 4.5450 & 0.2500 \\ 0.2500 & 7.0861 \end{bmatrix}$$

and

$$\varepsilon_1 = 0.2777, \quad \varepsilon_2 = 0.0717, \quad \rho_0 = 3.9859, \quad \rho_1 = 2.8844, \quad K = \begin{bmatrix} -77.1987 & -33.8802 \end{bmatrix}.$$

Hence, the sliding variable (3) and the SMC law (7) are designed as

$$s(t) = [-0.2 \ \ 1]x(t) - \int_0^t [15.5197 \ \ 6.8760]x(s)\,\mathrm{d}s - \int_0^t [0.5600 \ \ -0.0200]x(s-h(s))\,\mathrm{d}s,$$
$$u(t) = [-77.1987 \ \ -33.8802]x(t) - \nabla(t) sign(s(t)).$$

Take $\beta = 0.15$, and $x_0 = \begin{bmatrix} -0.2 & 0.1 \end{bmatrix}^T$, the above curves are obtained: Figure 1, 2 and 3 depict the response of the state $x(t)$ for the open-loop and the closed-loop systems, respectively. It is clear that the open-loop system is unstable, while the closed-loop system stabilizes rapidly. Figure 4 shows the evolution responses of sliding variable $s(t)$ and Figure 5 displays the control signal $u(t)$. It is clear that the system is β-exponentially stable.

The proposed method is compared with that of L. Huang et al. [8], in which the authors addressed the robust control problem for a class of one-sided Lipschitz systems with time-varying delay and parameter uncertainties using a state observer. For comparison purposes, since the nonlinear uncertainty function δf was not considered in [8], the simulations in the previous section are carried out with

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \end{bmatrix}.$$

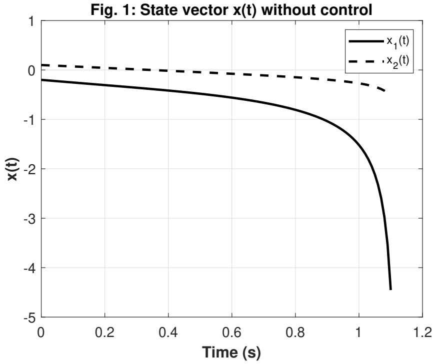

Fig. 1: State vector x(t) without control

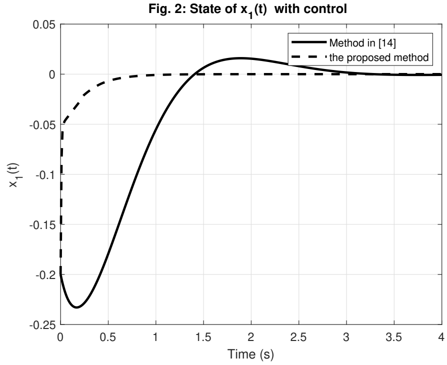

Fig. 2: State of x_1(t) with control

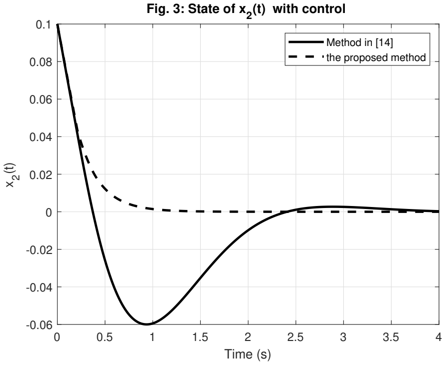

Fig. 3: State of x_2(t) with control

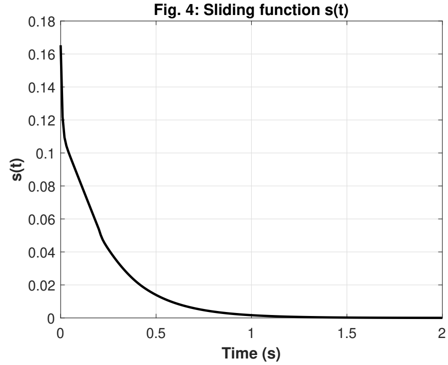

Fig. 4: Sliding function s(t)

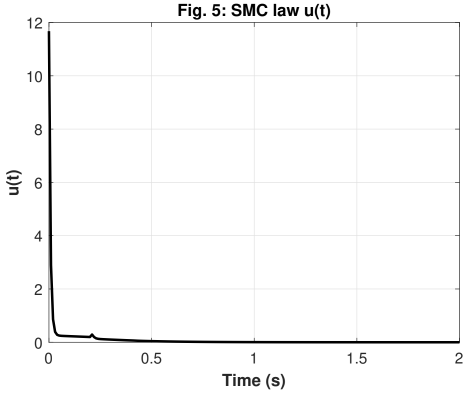

Fig. 5: SMC law u(t)

For $h = 0.2$ and $x(0) = [-0.2 \quad 0.1]^\top$, by applying both the proposed method and the method in [8], the simulation results are presented in Figs. 2–3. In these figures, the behaviors of $x_1(t)$ and $x_2(t)$ for the closed-loop systems (under both control methods) are compared. It can be observed that $x_1(t)$ and $x_2(t)$ under the proposed method exhibit better convergence characteristics than those obtained with the method in [8].

7. CONCLUSION

A new sliding mode control strategy is proposed to establish β-exponential stability for time-delay systems satisfying one-sided Lipschitz condition on the nonlinear part. The proposed control framework is based on an integral sliding surface specifically designed to incorporate time-delay effects, and it integrates a gain matrix derived using linear matrix inequality techniques, which are formulated through Lyapunov–Krasovskii functionals. The effectiveness and advantages of the proposed control method are demonstrated via a comparative numerical example. Finally, extending the approach to quasi-one-sided Lipschitz nonlinear systems could represent a promising direction for future research.

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REFERENCES

[1] M. Abbaszadeh and H. J. Marquez: Nonlinear observer design for one-sided Lipschitz systems. In: Proc. 2010 American Control Conference, IEEE 2010, pp. 5284–5289. DOI:10.1109/acc.2010.5530715

[2] N. Echi: Observer design and practical stability of nonlinear systems under unknown time-delay. Asian J. Control *23* (2021), 685–696. DOI:10.1002/asjc.2271

[3] N. Echi and F. Mabrouk: Observer based control for practical stabilization of one-sided Lipschitz nonlinear systems. Rocky Mt. J. Math. *54* (2024), 109–120. DOI:10.1216/rmj.2024.54.109

[4] M. Ekramian: Static output feedback problem for Lipschitz nonlinear systems. J. Franklin Inst. *357* (2020), 1457–1472. DOI:10.1016/j.jfranklin.2019.10.031

[5] M. Ghanes, J. De Leon, and J. Barbot: Observer design for nonlinear systems under unknown time-varying delays. IEEE Trans. Automat. Control *58* (2013), 1529–1534. DOI:10.1109/TAC.2012.2225554

[6] K. Gu, V. L. Kharitonov, and J. Chen: Stability of Time-Delay Systems. Birkhäuser, 2003.

[7] J. K. Hale and S. M. V. Lunel: Introduction to Functional Differential Equations. Springer, New York 1993.

[8] L. Huang, X. Lin, B. Zhong, and D. Xu: Robust control for one-sided Lipschitz nonlinear systems with time-varying delays and uncertainties. IET Control Theory Appl. *14* (2020), 2116–2126. DOI:10.1049/iet-cta.2019.1461

[9] X. Jia, X. Chen, S. Xu, B. Zhang, and Z. Zhang: Adaptive output feedback control of nonlinear time-delay systems with application to chemical reactor systems. IEEE Trans. Ind. Electron. *64* (2017), 4792–4799. DOI:10.1109/TIE.2017.2668996

[10] W. Kang: Approximate linearization of nonlinear control systems. Systems Control Lett. *23* (1994), 43–52. DOI:10.1016/0167-6911(94)90080-9

[11] H. K. Khalil: Nonlinear Syst. 3rd ed. Prentice-Hall, Upper Saddle River, NJ, USA 2001.

[12] Z. Kang, X. Lin, X. Shen, Z. Liu, Y. Gao, and J. Liu: Event-Triggered Generalized Super-Twisting Sliding Mode Control for Position Tracking of PMSMs. IEEE Trans. Ind. Informat. (2025). DOI:10.1109/tii.2025.3556086

[13] C. Lili, Z. Ying, and Z. Xian: Guaranteed cost control for uncertain genetic regulatory networks with interval time-varying delays. Neurocomputing *131* (2014), 105–112. DOI:10.1016/j.neucom.2013.10.035

[14] Z. Liu et al.: Fixed-time sliding mode control for DC/DC buck converters with mismatched uncertainties. IEEE Trans. Circuits Syst. I *70* (2023), 472–480. DOI:10.1109/TCSI.2022.3217612

[15] X. Lin, C. Wu, W. Yao, Z. Liu, X. Shen, R. Xu, G. Sun, and J. Liu: Observer-based fixed-time control for permanent-magnet synchronous motors with parameter uncertainties. IEEE Trans. Power Electron. *38* (2023), 4335–4344. DOI:10.1109/TPEL.2022.3226033

[16] X. Lin, R. Xu, W. Yao, Y. Gao, G. Sun, J. Liu, L. Peretti, and L. Wu: Observer-based prescribed performance speed control for PMSMs: A data-driven RBF neural network approach. IEEE Trans. Ind. Informat. *20* (2024), 7502–7512. DOI:10.1109/TII.2024.3357194

[17] X. Lin, J. Liu, Z. Liu, Y. Gao, L. Peretti, and L. Wu: Model-Free Current Predictive Control for PMSMs With Ultra-Local Model Employing Fixed-Time Observer and Extremum-Seeking Method. IEEE Trans. Power Electron. *40* (2025), 10682–10693. DOI:10.1109/TPEL.2025.3553685

[18] K. Mathiyalagan and R. Ragul: Observer-based finite-time dissipativity for parabolic systems with time-varying delays. Appl. Math. Comput. *413* (2022), 126605. DOI:10.1016/j.amc.2021.126605

[19] Y. Muroya, T. Kuniya, and J. L. Wang: Stability analysis of a delayed multi-group SIS epidemic model with nonlinear incidence rates and patch structure. J. Math. Anal. Appl. *425* (2015), 415–439. DOI:10.1016/j.jmaa.2014.12.019

[20] Y. Niu, W. C. Ho, and J. Lam: Robust integral sliding mode control for uncertain stochastic systems with time-varying delay. Automatica *41* (2005), 873–880. DOI:10.1016/j.automatica.2004.11.035

[21] A. E. Onyeka, X.-G. Yan, and J. Mu: Sliding mode control of time-delay systems with delayed nonlinear uncertainties. IFAC-PapersOnLine *50* (2017), 2696–2701. DOI:10.1016/j.ifacol.2017.08.555

[22] W. Saad, A. Sellami, and G. Garcia: Robust integral sliding mode-H_∞ control of one-sided Lipschitz non-linear systems. IET Control Theory Appl. *2* (2018), 2357–2367. DOI:10.1049/iet-cta.2018.5510

[23] X. Shen, J. Liu, Z. Liu, Y. Gao, J. I. L. S. Vazquez, L. Wu, and L. G. Franquelo: Sliding mode control of neutral-point-clamped power converters with gain adaptation. IEEE Trans. Power Electron. *1* (2024), 1–23. DOI:10.1109/tpel.2024.3386800

[24] R. Sipahi, S.-I. Niculescu, C. T. Abdallah, W. Michiels, and K. Gu: Stability and stabilization of systems with time delay. IEEE Control Syst. *31* (2011), 38–65. DOI:10.1109/MCS.2010.939135

[25] V. Utkin: Sliding Modes in Control and Optimization. Springer–Verlag, Berlin 1992.

[26] Y. Wang, L. Xie, and C. E. D. Souza: Robust control of a class of uncertain nonlinear system. Syst. Control Lett. *19* (1992), 139–149. DOI:10.1016/0167-6911(92)90097-C

[27] Z. Zhang, Y. Guo, S. Zhu, J. Liu, and D. Gong: Adaptive integral sliding-mode finite-time control with integrated extended state observer for uncertain nonlinear systems. Information Sciences *667* (2024), 120456. DOI:10.1016/j.ins.2024.120456

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