ON PSEUDOPARALLEL SUBMANIFOLD OF TRANS-SASAKIAN MANIFOLDS ADMITTING SEMISYMMETRIC METRIC CONNECTION

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Abstract. The aim of the present paper is to study invariant pseudoparallel submanifolds of a trans-Sasakian manifold equipped with semisymmetric metric connection. We have searched the necessary and sufficient conditions for an invariant pseudoparallel submanifold to be totally geodesic with respect to semisymmetric metric connection and structures reduced by the structures on the ambient manifold under the same conditions.

Keywords: trans-Sasakian manifold; pseudoparallel and Ricci pseudoparallel; Ricci-generalized pseudoparallel and 2-pseudoparallel submanifold, semisymmetric metric connection

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1. Introduction

An almost contact manifold is odd-dimensional manifold $\widetilde{M}^{2n+1}$ which carries a field ϕ of endomorphism of the tangent space, called the structure vector field ξ, and a 1-form η-satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

where I denotes the identity mapping of tangent space of each point at M. From (1) it follows

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \ \operatorname{rank}(\phi) = 2n. \tag{2}$$

In this case, $\widetilde{M}^{2n+1}(\phi, \xi, \eta)$ is said to be almost contact manifold [3]. An almost contact manifold $\widetilde{M}^{2n+1}$ is called an almost contact metric manifold if a Riemannian

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metric tensor g satisfies

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\qquad \eta(X)=g(X,\xi),\quad g(\phi X,Y)+g(X,\phi Y)=0 \tag{3}$$

for any vector fields X, Y on $\widetilde{M}$. The structure (φ,ξ,η,g) on $\widetilde{M}$ is said to be almost contact metric structure. In an almost contact metric structure (φ,ξ,η,g), the Nijenhuis tensor and the fundamental form are, respectively, defined by

$$N_\varphi(X,Y)=\varphi^2[X,Y]+[\varphi X,\varphi Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y],$$

and

$$\Phi(X,Y)=g(X,\phi Y)$$

for all $X,Y\in\Gamma(T\widetilde{M})$. An almost contact metric structure (φ,ξ,η,g) is said to be normal if

$$N_\varphi(X,Y)+2\,\mathrm{d}\eta(X,Y)\xi=0.$$

If an almost contact metric structure (ϕ,ξ,η,g) is normal and satisfies

$$\mathrm{d}\eta=\alpha\Phi,\ \mathrm{d}\Phi=2\beta\eta\wedge\Phi,$$

then it is called trans-Sasakian structure, where $\alpha=\delta\Phi(\xi)/(2n)$, $\beta=\operatorname{div}(\xi)/(2n)$ and δ is the codifferential of g, see [11].

It is well known that trans-Sasakian condition may be characterized as an almost contact metric structure satisfying

$$(\widetilde{\nabla}_X\phi)Y=\alpha\{g(X,Y)\xi-\eta(Y)X\}+\beta\{g(\phi X,Y)\xi-\eta(Y)\phi X\} \tag{4}$$

for all $X,Y\in\Gamma(T\widetilde{M})$, see [4]. It follows that

$$\widetilde{\nabla}_X\xi=-\alpha\phi X-\beta\phi^2X. \tag{5}$$

A trans-Sasakian manifold of type $(0,0)$, $(0,\beta)$ and $(\alpha,0)$ are, respectively, known as Cosymplectic, β-Kenmotsu and α-Sasakian manifold. The different geometrical properties of trans-Sasakian manifolds have been studied by De and Tripathi [6], [7], [8], see also [2], [3], [11].

In a trans-Sasakian manifold the following holds for all $X,Y\in\Gamma(T\widetilde{M})$:

$$\widetilde{R}(X,Y)\xi=(\alpha^2-\beta^2)\{\eta(Y)X-\eta(X)Y\}+2\alpha\beta\{\eta(Y)\phi X-\eta(X)\phi Y\}+Y(\alpha)\phi X-X(\alpha)\phi Y+Y(\beta)\phi^2X-X(\beta)\phi^2Y \tag{6}$$

$$\eta(\widetilde{R}(X,Y)Z)=(\alpha^2-\beta^2)\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\} \tag{7}$$

$$\widetilde{S}(X,\xi)=\{2n(\alpha^2-\beta^2)-\xi(\beta)\}\eta(X)-(2n-1)X(\beta)-(\phi X)(\alpha) \tag{8}$$

$$\xi(\alpha)+2\alpha\beta=0, \tag{9}$$

where $\widetilde{R}$ and $\widetilde{S}$ denote the Riemannian curvature tensor and Ricci tensor of $\widetilde{M}$ with respect to the Levi-Civita connection $\widetilde{\nabla}$, respectively [5].

By $\overline{\nabla}$, we denote semisymmetric metric connection on a contact metric manifold $(\widetilde{M}, \xi, \phi, \eta)$; then it is given by

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y - g(X,Y)\xi + \eta(Y)X \tag{10}$$

for all $X, Y \in \Gamma(T\overline{M})$, see [1]. In a trans-Sasakian manifold, we have

$$\begin{aligned} \overline{R}(X,Y)Z = \widetilde{R}(X,Y)Z &+ 3\{g(X,Z)Y - g(Y,Z)X\} \\ &+ 2\eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ &+ 2\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\xi, \end{aligned} \tag{11}$$

where $\overline{R}$ and $\overline{S}$ denote the Riemannian curvature tensor of $(\widetilde{M}, \xi, \phi, \eta)$ with respect to $\overline{\nabla}$. From (6) and (11), one can easily see

$$\begin{aligned} \overline{R}(X,Y)\xi = (\alpha^2 - \beta^2 - 1)\{\eta(Y)X - \eta(X)Y\} &+ 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\} \\ + Y(\alpha)\phi X - X(\alpha)\phi Y &+ Y(\beta)\phi^2 X - X(\beta)\phi^2 Y, \end{aligned} \tag{12}$$

$$\overline{S}(X,\xi) = \eta(X)[2n(\alpha^2 - \beta^2 - 1) - \xi(\beta)] - (2n-1)X(\beta) - (\phi X)(\alpha) \tag{13}$$

and

$$\overline{\nabla}_X \xi = -\alpha\phi X - (\beta + 1)\phi^2 X. \tag{14}$$

Now, we calculate the covariant derivative of ϕ with respect to semisymmetric metric connection. For this, by using (4) and (10), we have

$$\begin{aligned} (\overline{\nabla}_X \phi)Y &= \overline{\nabla}_X \phi Y - \phi \overline{\nabla}_X Y \\ &= \widetilde{\nabla}_X \phi Y - g(X, \phi Y)\xi + \eta(\phi Y)X - \phi\{\widetilde{\nabla}_X Y - g(X,Y)\xi + \eta(Y)X\} \\ &= (\widetilde{\nabla}_X \phi)Y - g(X, \phi Y)\xi - \eta(Y)X \\ &= \alpha\{g(X,Y)\xi - \eta(Y)X\} + (\beta+1)\{g(\phi X, Y)\xi - \eta(Y)\phi X\} \end{aligned} \tag{15}$$

for all $X, Y \in \Gamma(T\overline{M})$. This tells us $(\widetilde{M}, \xi, \phi, \eta)$ is a $(\alpha, \beta + 1)$-type trans-Sasakian manifold with respect to semisymmetric metric connection $\overline{\nabla}$.

Now, let M be an immersed submanifold of a semi-Riemannian manifold $(\widetilde{M}, g)$. Then the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{16}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{17}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are induced connections on M, $\Gamma(T^{\perp}M)$ and σ, A denote the second fundamental form, shape operatory of M, respectively. The covariant derivative of σ is defined by

$$(\widetilde{\nabla}_X \sigma)(Y, Z) = \nabla^{\perp}\sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{18}$$

for all $X, Y, Z \in \Gamma(TM)$.

A submanifold M of a semi-Riemannian manifold $(\widetilde{M}, g)$ is called Chaki-pseudo parallel if its second fundamental form σ satisfies

$$(\widetilde{\nabla}_X \sigma)(Y, Z) = 2\gamma(X)\sigma(Y, Z) + \gamma(Y)\sigma(X, Z) + \gamma(Z)\sigma(X, Y) \tag{19}$$

for all $X, Y, Z \in \Gamma(TM)$ and γ is a nowhere vanishing 1-form.

In particular, if $\gamma = 0$, then M is said to be a parallel submanifold of $\widetilde{M}$.

For a submanifold M of a semi-Riemannian manifold $(\widetilde{M}, g)$, the Gauss and Weingarten equations are given by

$$\begin{aligned} \widetilde{R}(X, Y)Z = R(X, Y)Z &+ A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X \\ &+ (\widetilde{\nabla}_X \sigma)(Y, Z) - (\widetilde{\nabla}_Y \sigma)(X, Z) \end{aligned} \tag{20}$$

and

$$g(\widetilde{R}(X, Y)U, V) = g(R^{\perp}(X, Y)U, V) + g([A_V, A_U]X, Y) \tag{21}$$

for all $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$, where R and $R^{\perp}$ are the Riemannian curvature tensors of M and $\Gamma(T^{\perp}M)$, respectively.

As a generalization of Ricci semisymmetric manifolds, the notion of Ricci pseudo symmetric manifolds was introduced [6]. We may call it as Ricci pseudoparallel. A Riemannian manifold (M, g) $(n > 2)$ is said to be Deszcz-pseudo Ricci-symmetric if

$$(\widetilde{R}(X, Y) \cdot \widetilde{S})(U, V) = L_s Q(g, \widetilde{S})(U, V; X, Y) \tag{22}$$

holds on $U_s = \{x \in M \colon (\widetilde{S} - g\tau/n) \neq 0\}$ for each point $x \in M$ and $X, Y, U, V \in \Gamma(TM)$, where Q denotes the Tachibana tensor, L_s is a function on U_s, $\widetilde{R}$ is the Riemannian curvature tensor, $\widetilde{S}$ is the Ricci tensor and τ is the scalar curvature of the ambient manifold.

Definition 1. A submanifold M of a semi-Riemannian manifold $(\widetilde{M}, g)$ is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel if

$$\widetilde{R} \cdot \sigma, \quad Q(g, \sigma), \tag{23}$$

$$\widetilde{R} \cdot \widetilde{\nabla}\sigma, \quad Q(g, \widetilde{\nabla}\sigma), \tag{24}$$

$$\widetilde{R} \cdot \sigma, \quad Q(\widetilde{S}, \sigma), \tag{25}$$

and

$$\widetilde{R} \cdot \widetilde{\nabla}\sigma, \quad Q(\widetilde{S}, \widetilde{\nabla}\sigma), \tag{26}$$

respectively, are linearly dependent, where Q denotes the Tachibana tensor [9].

In [4], almost contact metric structures on 3-dimensional Lie algebras and the class of left invariant almost contact metric structures on the corresponding Lie groups have been studied. In [8], the authors studied 3-dimensional trans-Sasakian manifolds which are locally ϕ-symmetric and have been a η-parallel Ricci tensor. Also [5], Chaubey searched the properties of special weakly Ricci symmetric and generalized Ricci recurrent trans-Sasakian manifolds. On the other hand, the geometry of pseudosubmanifolds continues to be worked on at different structures (see [9], [10]).

Motivated by the above studies, in this paper, the geometry of invariant pseudoparallel submanifolds with respect to semisymmetric metric connection of a trans-Sasakian manifold is studied. The necessary and sufficient conditions are investigated under some hypotheses. The obtained results have been compared.

From a submanifold point of view, parallel submanifolds are the simplest Riemannian submanifolds next to totally geodesic ones. Parallel submanifolds form an important class of Riemannian submanifolds since extrinsic invariants of a parallel submanifold do not vary from point to point. In this paper, we provide a comprehensive survey on this important class of submanifolds.

2. Invariant pseudoparallel submanifolds of a trans-Sasakian

The geometry of submanifolds of a contact metric manifold depends on the behaviour of contact metric structure ϕ. Namely, a submanifold M of a trans-Sasakian manifold $\widetilde{M}$ is said to be invariant if the structure vector field ξ is tangent to M at every point of M and ϕX is tangent to M for any vector feld X tangent to M at every point of M. In other words, $\phi(TM) \subset (TM)$ at each point of M.

We note that in theory of submanifolds of differential geometry, all of the properties an invariant submanifold inherits are almost all properties of the ambient manifold. Therefore, invariant submanifolds are an active and fruitful research field playing a significant role in the development of modern differential geometry. In this connection, the papers related to invariant submanifolds have been studied and continues to be studied at different manifold types.

In the rest of this paper, we will assume that M is an invariant submanifold of trans-Sasakian manifold $\widetilde{M}$ unless stated otherwise.

Let M be an immersed submanifold of a trans-Sasakian manifold $\widetilde{M}(\phi, \xi, \eta, g)$. By R, we denote the Riemannian curvature tensor of submanifold M. Then the following relations hold:

$$\widetilde{R}(X, Y)\xi = R(X, Y)\xi, \tag{27}$$

$$\sigma(X, \phi Y) = \sigma(\phi X, Y) = \phi\sigma(X, Y), \tag{28}$$

$$\sigma(X, \xi) = 0. \tag{29}$$

P r o o f. We will not give the proof since it is a result of direct calculations. □

Theorem 2. *Let M be an invariant submanifold of a trans-Sasakian manifold $\widetilde{M}(\phi, \xi, \eta, g)$. If M is Chaki pseudoparallel with respect to semisymmetric metric connection, then at least one of the following is true:*

(1) *$\widetilde{M}(\phi, \xi, \eta, g)$ is a β-Kenmotsu manifold with $\beta = -\gamma(\xi) - 1$,*
(2) *M is a totally geodesic submanifold.*

P r o o f. Let us suppose that M is Chaki pseudoparallel. From (18) and (19) we have

$$(\overline{\nabla}_X\sigma)(Y, Z) = 2\gamma(X)\sigma(Y, Z) + \gamma(Y)\sigma(X, Z) + \gamma(Z)\sigma(X, Y)$$

for all $X, Y \in (TM)$. In the last equality, taking $Y = \xi$ and by using (5) and (14), we observe

$$\begin{aligned}\gamma(\xi)\sigma(X, Z) = &-\sigma(\nabla'_X\xi, Z) = \sigma(\alpha\phi X + (\beta+1)\phi^2X, Z)\\ &= \alpha\phi\sigma(X, Z) - (\beta+1)\sigma(X, Z),\end{aligned}$$

where ∇' denotes the induced connection on M by $\overline{\nabla}$. Since vector fields σ and $\phi\sigma$ are linearly independent and the ambient space is a trans-Sasakian manifold, we conclude that $\sigma = 0$ or $\alpha = \gamma(\xi) + \beta + 1 = 0$. This completes the proof. □

We have the following corollary since every totally geodesic submanifold is Chaki pseudoparallel.

Corollary 3. *Let M be an invariant submanifold of a trans-Sasakian manifold $\widetilde{M}$ admitting semisymmetric metric connection. Then M is a parallel submanifold if and only if M is totally geodesic.*

Theorem 4. *Let M be an invariant submanifold of a trans-Sasakian manifold $\widetilde{M}$ admitting semisymmetric metric connection. M is generalized 2-recurrent with respect to semisymmetric metric connection if and only if M is either a totally geodesic submanifold or $\widetilde{M}$ is a type -1-Kenmotsu manifold.*

P r o o f. Since M is a 2-recurrent submanifold with respect to semisymmetric metric connection $\overline{\nabla}$, there are forms ψ and θ on $\widetilde{M}$ such that

$$(\overline{\nabla}_X\overline{\nabla}_Y\sigma)(Z,W)=\psi(X,Y)\sigma(Z,W)+\theta(X)(\overline{\nabla}_Y\sigma)(Z,W) \tag{30}$$

for all $X,Y,Z,W\in\Gamma(TM)$. For $Z=W=\xi$, this implies

$$\begin{aligned}\nabla^{\perp}_X(\overline{\nabla}_Y\sigma)(\xi,\xi)-(\overline{\nabla}_Y\sigma)(\nabla_X\xi,\xi)-(\overline{\nabla}_X\sigma)(\xi,\nabla_Y\xi)-(\overline{\nabla}_{\nabla_XY}\sigma)(\xi,\xi)\\ =\psi(X,Y)\sigma(\xi,\xi)+\theta(X)(\overline{\nabla}_Y\sigma)(\xi,\xi),\end{aligned}$$

that is,

$$\begin{aligned}\nabla^{\perp}_X\{\nabla^{\perp}_Y\sigma(\xi,\xi)-2\sigma(\nabla_Y\xi,\xi)\}-(\overline{\nabla}_Y\sigma)(-\alpha\phi X-(\beta+1)\phi^2X,\xi)\\ -(\overline{\nabla}_X\sigma)(\xi,-\alpha\phi Y-(\beta+1)\phi^2Y)-\{\nabla^{\perp}_{\nabla_XY}\sigma(\xi,\xi)-2\sigma(\nabla_{\nabla_XY}\xi,\xi)\}\\ =\theta(X)\{\nabla^{\perp}_Y\sigma(\xi,\xi)-2\sigma(\nabla_Y\xi,\xi)\}.\end{aligned}$$

The nonzero components of these expressions give us

$$\begin{aligned}(\overline{\nabla}_X\sigma)(\nabla_Y\xi,\xi)+(\overline{\nabla}_Y\sigma)(\xi,\nabla_Y\xi)=&-(\overline{\nabla}_Y\sigma)(-\alpha\phi X-(\beta+1)\phi^2X,\xi)\\ &-(\overline{\nabla}_X\sigma)(\xi,-\alpha\phi Y-(\beta+1)\phi^2Y)=0.\end{aligned}$$

By using (14), (18) and (29), we obtain

$$\sigma(\nabla_Y\xi,\alpha\phi X)+\sigma(\nabla_Y\xi,(\beta+1)\phi^2X)+\sigma(\nabla_X\xi,\alpha\phi Y)+\sigma(\nabla_X\xi,(\beta+1)\phi^2Y)=0,$$

that is,

$$[(\beta+1)^2-\alpha^2]\sigma(X,Y)-2\alpha(\beta+1)\phi\sigma(X,Y)=0. \tag{31}$$

Since σ and $\phi\sigma$ are linearly independent vectors, from (31), we can easily see that $\sigma=0$ or $(\alpha,\beta)=(0,-1)$. This completes the proof. $\square$

Theorem 5. *Let M be an invariant pseudoparallel submanifold of trans-Sasakian manifold $\widetilde{M}(\phi,\xi,\eta,g)$ admitting semisymmetric metric connection $\overline{\nabla}$. Then M is either totally geodesic submanifold or $\lambda=\alpha^2-\beta^2-\xi(\beta)-1$.*

P r o o f. Since M is pseudoparallel, there is a function λ on $\widetilde{M}$ such as

$$(\overline{R}(X,Y)\cdot\sigma)(U,V)=\lambda Q(g,\sigma)(U,V;X,Y)$$

for all $X,Y,U,V\in(TM)$. This leads to

$$\begin{aligned}R^{\perp}(X,Y)\sigma(U,V)-\sigma(R'(X,Y)U,V)-\sigma(U,R'(X,Y)V)\\ =\ -\lambda\{\sigma((X\wedge_g Y)U,V)+\sigma(U,(X\wedge_g Y)V)\},\end{aligned}$$

where R' denotes the Riemannian curvature tensor of submanifold with respect to semisymmetric metric connection $\overline{\nabla}$. This implies for $V=\xi$,

$$\sigma(R'(X,Y)\xi,U)=\lambda\sigma(U,\eta(Y)X-\eta(X)Y).$$

Taking into account (6), (12) and (29), we have

$$\begin{aligned}\sigma(U,(\alpha^2-\beta^2-1)[\eta(Y)X-\eta(X)Y]+2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y]\\ +Y(\alpha)\phi X-X(\alpha)\phi Y+Y(\beta)\phi^2X-X(\beta)\phi^2Y)\\ =\lambda\sigma(U,\eta(Y)X-\eta(X)Y),\end{aligned}$$

that is,

$$\begin{aligned}\eta(Y)(\alpha^2-\beta^2-\lambda-1)\sigma(U,X)-\eta(X)(\alpha^2-\beta^2-\lambda-1)\sigma(U,Y)\\ +2\alpha\beta\eta(Y)\phi\sigma(U,X)-2\alpha\beta\eta(X)\phi\sigma(U,Y)+Y(\alpha)\phi\sigma(U,X)\\ -X(\alpha)\phi\sigma(U,Y)-Y(\beta)\sigma(U,X)+X(\beta)\sigma(U,Y)=0.\end{aligned}\tag{32}$$

Taking $Y=\xi$ in (32) and after necessary revisions, we get

$$(\alpha^2-\beta^2-\xi(\beta)-\lambda-1)\sigma(U,X)=0,$$

which proves our assertion. $\square$

Theorem 6. *Let M be an invariant submanifold of a trans-Sasakian manifold $\widetilde{M}$. If M is a Ricci-generalized pseudoparallel submanifold with respect to semisymmetric metric connection $\overline{\nabla}$, then at least one of the following holds:*

(1) *M is totally geodesic,*
(2) *$\lambda=1/(2n)$,*
(3) *$\xi(\beta)=\alpha^2-\beta^2-1$.*

P r o o f. Since M is an invariant Ricci-generalized pseudoparallel submanifold, there exists a function λ on $\widetilde{M}$ such that

$$(\overline{R}(X,Y)\cdot\sigma)(U,V)=\lambda Q(\overline{S},\sigma)(U,V;X,Y).$$

This yields to

$$\begin{aligned}(33)\qquad R^{\perp}(X,Y)\sigma(U,V)&-\sigma(R'(X,Y)U,V)-\sigma(U,R'(X,Y)V)\\ &=-\lambda\{\sigma((X\wedge_{\overline{S}}Y)U,V)+\sigma(U,(X\wedge_{\overline{S}}Y)V)\}.\end{aligned}$$

Putting $V=\xi$ in (33) and making use of (12), (13) (20), (27), we have

$$\sigma(U,R'(X,Y)\xi)=\lambda\sigma(U,\overline{S}(Y,\xi)X-\overline{S}(X,\xi)Y),$$

or

$$\begin{aligned}(34)\qquad \sigma(U,(\alpha^2-\beta^2)&[\eta(Y)X-\eta(X)Y]+2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y]\\ &+Y(\alpha)\phi X-X(\alpha)\phi Y+Y(\beta)\phi^2X-X(\beta)\phi^2Y)\\ &=\lambda\{S(\xi,Y)\sigma(U,X)-S(X,\xi)\sigma(U,Y)\}.\end{aligned}$$

Putting (13) into (34), we have

$$\begin{aligned}(35)\quad (\alpha^2-\beta^2)&\sigma(U,\eta(Y)X-\eta(X)Y)+2\alpha\beta\phi\sigma(U,\eta(Y)X-\eta(X)Y)\\ &+\phi\sigma(Y(\alpha)X-X(\alpha)Y,U)-\sigma(Y(\beta)X-X(\beta)Y,U)\\ &=\lambda\{[2n(\alpha^2-\beta^2-1)-\xi(\beta)]\eta(Y)-(2n-1)Y(\beta)-(\phi Y)(\alpha)\}\sigma(U,X)\\ &\quad-\lambda\{[2n(\alpha^2-\beta^2-1)-\xi(\beta)]\eta(X)-(2n-1)X(\beta)-(\phi X)(\alpha)\}\sigma(U,Y).\end{aligned}$$

Setting $Y=\xi$ in (35) and by using (29) we reach at

$$[\alpha^2-\beta^2-\xi(\beta)-1]\sigma(U,X)+[2\alpha\beta+\xi(\alpha)]\phi\sigma(U,X)=2n\lambda[\alpha^2-\beta^2-\xi(\beta)-1]\sigma(U,X).$$

From (9), we conclude that

$$[\alpha^2-\beta^2-\xi(\beta)-1][2n\lambda-1]\sigma(U,X)=0.$$

This proves our assertion. □

Theorem 7. *Let M be an invariant 2-pseudoparallel submanifold of a trans-Sasakian manifold $\widetilde{M}$ with respect to semisymmetric metric connection $\overline{\nabla}$. Then at least one of the following holds:*

(1) *M is a totally geodesic submanifold,*
(2) *$\lambda=\alpha^2-\beta^2-\xi(\beta)-1$,*
(3) *$\widetilde{M}$ is a $\beta=-1$-Kenmotsu manifold.*

P r o o f. If M is a 2-pseudo parallel submanifold, then (24) implies that

$$(\overline{R}(X,Y)\cdot\overline{\nabla}\sigma)(U,V,Z)=\lambda Q(g,\overline{\nabla}\cdot\sigma)(U,V,Z:X,Y).$$

That means

$$\begin{aligned}R^{\perp}(X,Y)(\overline{\nabla}_U\sigma)(V,Z)&-(\overline{\nabla}_{R'(X,Y)U}\sigma)(V,Z)\\&-(\overline{\nabla}_U\sigma)(R'(X,Y)V,Z)-(\overline{\nabla}_U\sigma)(V,R'(X,Y)Z)\\=&-\lambda\big\{(\overline{\nabla}_{(X\wedge_g Y)U}\sigma)(V,Z)+(\overline{\nabla}_U\sigma)((X\wedge_g Y)V,Z)\\&+(\overline{\nabla}_U\sigma)(V,(X\wedge_g Y)Z)\big\}\end{aligned}$$

for all $X,Y,U,V,Z\in\Gamma(TM)$. Taking $X=Z=\xi$ in the last equality, we have

$$\begin{aligned}(36)\quad &R^{\perp}(\xi,Y)(\overline{\nabla}_U\sigma)(V,\xi)-(\overline{\nabla}_{R'(\xi,Y)U}\sigma)(V,\xi)\\&\quad-(\overline{\nabla}_U\sigma)(R'(\xi,Y)V,\xi)-(\overline{\nabla}_U\sigma)(V,R'(\xi,Y)\xi)\\&=-\lambda\{(\overline{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi)+(\overline{\nabla}_U\sigma)((\xi\wedge_g Y)V,\xi)+(\overline{\nabla}_U\sigma)(V,(\xi\wedge_g Y)\xi)\}.\end{aligned}$$

Here, the expansion of the first term gives

$$\begin{aligned}(37)\quad R^{\perp}(\xi,Y)(\overline{\nabla}_U\sigma)(V,\xi)&=R^{\perp}(\xi,Y)\{\nabla_U^{\perp}\sigma(\xi,Y)-\sigma(\nabla_UV,\xi)-\sigma(V,\nabla_U\xi)\}\\&=R^{\perp}(\xi,Y)\{\alpha\phi\sigma(U,V)+(\beta+1)\phi^2\sigma(U,V)\}.\end{aligned}$$

As for the second term, in view of (5) and (7),

$$\begin{aligned}(38)\qquad(\overline{\nabla}_{R'(\xi,Y)U}\sigma)(V,\xi)&=-\sigma(V,\nabla'_{R'(\xi,Y)U}\xi)\\&=\alpha\sigma(\phi R'(\xi,Y)U,V)+(\beta+1)\phi^2\sigma(R'(\xi,Y)U,V)\\&=\alpha\phi\sigma(R'(\xi,Y)U,V)-(\beta+1)\sigma(R'(\xi,Y)U,V).\end{aligned}$$

In the same way, nonzero components of the third term are

$$\begin{aligned}(39)\qquad(\overline{\nabla}_U\sigma)(R'(\xi,Y)V,\xi)&=-\sigma(R'(\xi,Y)V,\nabla'_U\xi)\\&=\sigma(R'(\xi,Y)V,\alpha\phi U+(\beta+1)\phi^2U)\\&=\alpha\phi\sigma(R'(\xi,Y)V,U)-(\beta+1)\sigma(R'(\xi,Y)V,U).\end{aligned}$$

For the last term of the left-hand side of the equality, making use of (12), (14) and (18), we obtain

$$\begin{aligned}(40)\quad(\overline{\nabla}_U\sigma)(V,R'(\xi,Y)\xi)&=(\widetilde{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta)-1)\phi^2Y)\\&=(\overline{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta)-1)\eta(Y)\xi)\\&\quad-(\overline{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta)-1)Y)\end{aligned}$$

$$
\begin{aligned}
&= -\sigma(\nabla_U(\alpha^2-\beta^2-\xi(\beta)-1)\eta(Y)\xi, V)\\
&\quad -(\overline{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta)-1)Y)\\
&= \eta(Y)(\alpha^2-\beta^2-\xi(\beta)-1)\sigma(\alpha\phi U+(\beta+1)\phi^2U, V)\\
&\quad -(\overline{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta)-1)Y)\\
&= \eta(Y)\alpha(\alpha^2-\beta^2-\xi(\beta)-1)\phi\sigma(U,V)\\
&\quad -\eta(Y)(\beta+1)(\alpha^2-\beta^2-\xi(\beta)-1)\sigma(U,V)\\
&\quad -(\overline{\nabla}_U\sigma)(V,(\alpha^2-\beta^2-\xi(\beta)-1)Y).
\end{aligned}
$$

The first term of the right-hand side of the equality gives us

$$
\begin{aligned}
(\overline{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi) &= -\sigma(\nabla'_{(\xi\wedge_g Y)U}\xi, V)\\
&= -\alpha\eta(U)\phi\sigma(Y,V)+\eta(U)(\beta+1)\sigma(Y,V).
\end{aligned}
\tag{41}
$$

For the nonzero components of the second term,

$$
\begin{aligned}
(\overline{\nabla}_U\sigma)(\xi,(\xi\wedge_g Y)V) &= -\sigma(\nabla'_U\xi,(\xi\wedge_g Y)V)\\
&= \sigma(\alpha\phi U+(\beta+1)\phi^2U, g(Y,V)\xi-\eta(V)Y)\\
&= -\eta(V)\alpha\phi\sigma(U,Y)+(\beta+1)\eta(V)\sigma(U,Y).
\end{aligned}
\tag{42}
$$

Finally,

$$
\begin{aligned}
(\overline{\nabla}_U\sigma)(V,(\xi\wedge_g Y)\xi) &= (\overline{\nabla}_U\sigma)(V,\eta(Y)\xi-Y)\\
&= (\overline{\nabla}_U\sigma)(\eta(Y)\xi,Y)-(\overline{\nabla}_U\sigma)(V,Y)\\
&= -\sigma(\nabla'_U\eta(Y)\xi,V)-(\overline{\nabla}_U\sigma)(V,Y)\\
&= -\sigma(U[\eta(Y)]\xi+\eta(Y)\nabla'_U\xi,V)-(\overline{\nabla}_U\sigma)(V,Y)\\
&= \eta(Y)\sigma(\alpha\phi U+(\beta+1)\phi^2U,V)-(\overline{\nabla}_U\sigma)(V,Y)\\
&= \eta(Y)\alpha\phi\sigma(U,V)-\eta(Y)(\beta+1)\sigma(U,V)-(\overline{\nabla}_U\sigma)(V,Y).
\end{aligned}
\tag{43}
$$

Thus, putting (37)–(43) in (36), taking $V=\xi$, we obtain

$$
\begin{aligned}
&-\{\alpha\phi\sigma(R(\xi,Y)\xi,U)-(\beta+1)\sigma(R(\xi,Y)\xi,U)\}\\
&+(\overline{\nabla}_U\sigma)(\xi,(\alpha^2-\beta^2-\xi(\beta)-1)Y)\\
=& -\lambda\{-\alpha\phi\sigma(Y,U)+(\beta+1)\sigma(U,Y)-(\overline{\nabla}_U\sigma)(\xi,Y)\}.
\end{aligned}
$$

Taking account of (12), (14) and (18), this equality is reduced to

$$
\alpha[\alpha^2-\beta^2-1-\lambda-\xi(\beta)]\phi\sigma(U,Y)-(\beta+1)[\alpha^2-\beta^2-1-\lambda-\xi(\beta)]\sigma(U,Y)=0.
$$

Since σ and $\phi\sigma$ are orthogonal vector fields, we conclude that $\alpha=0$, $\beta=-1$, $\lambda=\alpha^2-\beta^2-\xi(\beta)-1$ or $\sigma=0$, which proves our assertions. □

Theorem 8. *Let M be an invariant 2-Ricci-generalized pseudoparallel submanifold of a trans-Sasakian manifold $\widetilde{M}$ with respect to semisymmetric metric connection $\overline{\nabla}$. Then at least one of the following holds:*

(1) *M is a totally geodesic submanifold,*
(2) *$\widetilde{M}$ is a $(0,-1)$-type Kenmotsu manifold,*
(3) *$\lambda = (\alpha^2 - \beta^2 - \xi(\beta) - 1)$,*
(4) *$\lambda = 1/2n$.*

Proof. Since M is an invariant 2-Ricci-generalized pseudoparallel submanifold of a trans-Sasakian manifold $\widetilde{M}$ with respect to semisymmetric connection $\overline{\nabla}$, then there exist a function λ such that

$$(\overline{R}(X,Y)\cdot\overline{\nabla}\sigma)(U,V,Z) = \lambda Q(\overline{S},\overline{\nabla}\cdot\sigma)(U,V,Z;X,Y).$$

This means that

$$\begin{aligned}
R^{\perp}(X,Y)&(\overline{\nabla}_U\sigma)(V,Z) - (\overline{\nabla}_{R'(X,Y)U}\sigma)(V,Z)\\
&- (\overline{\nabla}_U\sigma)(R'(X,Y)V,Z) - (\overline{\nabla}_U\sigma)(V,R'(X,Y)Z)\\
&= -\lambda\{(\overline{\nabla}_{(X\wedge_{\overline{S}}Y)U}\sigma)(V,Z) + (\overline{\nabla}_U\sigma)((X\wedge_{\overline{S}}Y)V,Z) + (\overline{\nabla}_U\sigma)(V,(X\wedge_{\overline{S}}Y)Z)\}
\end{aligned}$$

for all $X,Y,U,V,Z \in \Gamma(TM)$. Taking $X = V = \xi$ in the last equality, we have

$$\begin{aligned}
R^{\perp}(\xi,Y)&(\overline{\nabla}_U\sigma)(\xi,Z) - (\overline{\nabla}_{R'(\xi,Y)U}\sigma)(\xi,Z)\\
&- (\overline{\nabla}_U\sigma)(R'(\xi,Y)\xi,Z) - (\overline{\nabla}_U\sigma)(\xi,R'(\xi,Y)Z)\\
&= -\lambda\{(\overline{\nabla}_{(\xi\wedge_{\overline{S}}Y)U}\sigma)(\xi,Z) + (\overline{\nabla}_U\sigma)((\xi\wedge_{\overline{S}}Y)\xi,Z) + (\overline{\nabla}_U\sigma)(\xi,(\xi\wedge_{\overline{S}}Y)Z)\}.
\end{aligned} \tag{44}$$

Now, nonzero components of the first term are

$$\begin{aligned}
R^{\perp}(\xi,Y)(\overline{\nabla}_U\sigma)(\xi,Z) &= R^{\perp}(\xi,Y)\{\overline{\nabla}_U\sigma(\xi,Z) - \sigma(\nabla'_U\xi,Z) - \sigma(\xi,\nabla'_UZ)\}\\
&= -R^{\perp}(\xi,Y)\sigma(\nabla'_U\xi,Z)\\
&= R^{\perp}(\xi,Y)\sigma(\alpha\phi U + (\beta+1)\phi^2U,Z)\\
&= R^{\perp}(\xi,Y)(\alpha\phi - (\beta+1))\sigma(U,Z).
\end{aligned} \tag{45}$$

In the same way, after the necessary revisions, the second term gives us

$$\begin{aligned}
(\overline{\nabla}_{R'(\xi,Y)U}\sigma)(\xi,Z) &= -\sigma(\nabla_{R'(\xi,Y)U}\xi,Z)\\
&= \sigma(\alpha\phi R'(\xi,Y)U + (\beta+1)\phi^2R'(\xi,Y)U,Z)\\
&= (\alpha\phi - (\beta+1))\sigma(R'(\xi,Y)U,Z).
\end{aligned} \tag{46}$$

Also, the nonzero component of the third term calculations is

$$
\begin{aligned}
(47)\quad (\overline{\nabla}_U\sigma)(R(\xi,Y)\xi,Z) &= (\overline{\nabla}_U\sigma)((\alpha^2-\beta^2-\xi(\beta)-1)\phi^2 Y,Z)\\
&= (\overline{\nabla}_U\sigma)((\alpha^2-\beta^2-\xi(\beta)-1)\eta(Y)\xi,Z)\\
&\quad -(\overline{\nabla}_U\sigma)((\alpha^2-\beta^2-\xi(\beta)-1)Y,Z)\\
&= -\sigma((\alpha^2-\beta^2-\xi(\beta)-1)\eta(Y)\nabla'_U\xi,Z)\\
&\quad -(\overline{\nabla}_U\sigma)((\alpha^2-\beta^2-\xi(\beta)-1)Y,Z)\\
&= (\alpha^2-\beta^2-\xi(\beta)-1)\eta(Y)\sigma(\alpha\phi U+(\beta+1)\phi^2 U,Z)\\
&\quad -(\overline{\nabla}_U\sigma)((\alpha^2-\beta^2-\xi(\beta)-1)Y,Z)\\
&= (\alpha^2-\beta^2-\xi(\beta))\eta(Y)(\alpha\phi-(\beta+1))\sigma(U,Z)\\
&\quad -(\overline{\nabla}_U\sigma)((\alpha^2-\beta^2-\xi(\beta)-1)Y,Z).
\end{aligned}
$$

After the necessary abbreviations, the fourth term is of the form

$$
\begin{aligned}
(48)\qquad (\overline{\nabla}_U\sigma)(R(\xi,Y)Z,\xi) &= -\sigma(\nabla'_U\xi,R'(\xi,Y)Z)\\
&= \sigma(\alpha\phi U+(\beta+1)\phi^2 U,R'(\xi,Y)Z)\\
&= (\alpha\phi-(\beta+1))\sigma(U,R'(\xi,Y)Z).
\end{aligned}
$$

For nonzero components of the first term of the right-hand side of the equality, by using (18) and (29), we have

$$
\begin{aligned}
(49)\qquad (\overline{\nabla}_{(\xi\wedge_{\overline{S}}Y)U}\sigma)(\xi,Z) &= -\sigma(\nabla'_{(\xi\wedge_{\overline{S}}Y)U}\xi,Z)\\
&= \sigma(\alpha\phi(\xi\wedge_{\overline{S}}Y)U+(\beta+1)\phi^2(\xi\wedge_{\overline{S}}Y)U,Z)\\
&= -[\alpha\phi-(\beta+1)]\overline{S}(U,\xi)\sigma(Y,Z).
\end{aligned}
$$

With similar thought, the second term gives us

$$
\begin{aligned}
(50)\quad (\overline{\nabla}_U\sigma)(\overline{S}(Y,\xi)\xi-\overline{S}(\xi,\xi)Y,Z) &= -\sigma(\nabla_U\overline{S}(Y,\xi)\xi,Z)-(\overline{\nabla}_U\sigma)(\overline{S}(\xi,\xi)Y,Z)\\
&= \overline{S}(Y,\xi)\sigma(\alpha\phi U+(\beta+1)\phi^2 U,Z)\\
&\quad -(\overline{\nabla}_U\sigma)(2n[\alpha^2-\beta^2-\xi(\beta)]Y,Z)\\
&= \overline{S}(Y,\xi)(\alpha\phi-(\beta+1))\sigma(U,Z)\\
&\quad -(\overline{\nabla}_U\sigma)(2n[\alpha^2-\beta^2-\xi(\beta)]Y,Z).
\end{aligned}
$$

We obtain the nonzero components of the last term:

$$
\begin{aligned}
(51)\qquad (\overline{\nabla}_U\sigma)(\xi,\overline{S}(Y,Z)\xi-\overline{S}(\xi,Z)Y) &= \sigma(\nabla_U\xi,\overline{S}(\xi,Z)Y)\\
&= -\overline{S}(\xi,Z)\sigma(\alpha\phi U+(\beta+1)\phi^2 U,Y)\\
&= -\overline{S}(\xi,Z)(\alpha\phi-(\beta+1))\sigma(U,Y).
\end{aligned}
$$

Consequently, (45)–(51) are put in (44) for $Z = \xi$ and after the necessary revisions are made, we reach at

$$\begin{aligned}
&-(\alpha\phi - (\beta+1))\sigma(U, (\alpha^2 - \beta^2 - \xi(\beta) - 1)\phi^2 Y) \\
&+(\alpha^2 - \beta^2 - \xi(\beta) - 1)\sigma(\nabla'_U \xi, Y) \\
= &-\lambda\{-2n(\alpha\phi - (\beta+1))(\alpha^2 - \beta^2 - \xi(\beta))\sigma(U, Y) \\
&+ 2n(\alpha^2 - \beta^2 - \xi(\beta))\sigma(\nabla'_U \xi, Y)\}.
\end{aligned}$$

Also, this implies that

$$[2n\lambda - 1][\alpha^2 - \beta^2 - \xi(\beta) - 1][\alpha\phi\sigma(U, Y) - (\beta+1)\sigma(U, Y)] = 0.$$

This completes the proof. □

Example 9. Let $\widetilde{M}^5 = \{(x_1, x_2, x_3, x_4, t) \in \mathbf{E}^5 \colon t \neq 0\}$ be a 5-dimensional differentiable manifold with the standart coordinate system (x_1, x_2, x_3, x_4, t). Then vector fields

$$E_1 = \mathrm{e}^t \frac{\partial}{\partial x_1}, \quad E_2 = \mathrm{e}^t \frac{\partial}{\partial x_2}, \quad E_3 = \mathrm{e}^t \frac{\partial}{\partial x_3}, \quad E_4 = \mathrm{e}^t \frac{\partial}{\partial x_4}, \quad E_5 = \xi = \mathrm{e}^t \frac{\partial}{\partial t}$$

are linear independent at each points of $\widetilde{M}$, that is, these vector fields are a basis of the tangent space of $\widetilde{M}$. Now, we define the contact structure and metric tensor ϕ and g by

$$\phi E_1 = -E_2, \quad \phi E_2 = E_1, \quad \phi E_3 = -E_4, \quad \phi E_4 = E_3, \quad \phi E_5 = 0,$$

and

$$g(E_i, E_j) = \delta_{ij}, \quad 1 \leqslant i,\ j \leqslant 5.$$

Then we can easily verify that

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi).$$

So, $\widetilde{M}^5(\phi, \xi, \eta, g)$ is a contact metric manifold. By direct calculations, we can get nonzero components of Lie-bracket as

$$[E_i, E_5] = -\mathrm{e}^t E_i, \quad 1 \leqslant i \leqslant 4,$$
$$[E_1, E_2] = 0,\ [E_1, E_3] = 0,\ [E_1, E_4] = 0,\ [E_2, E_3] = 0,\ [E_2, E_4] = 0,\ [E_3, E_4] = 0.$$

In view of Kozsul formulae, we can find the following nonzero components of connections as

$$\widetilde{\nabla}_{E_1} E_5 = -\mathrm{e}^t E_1, \quad \widetilde{\nabla}_{E_2} E_5 = -\mathrm{e}^t E_2, \quad \widetilde{\nabla}_{E_3} E_5 = -\mathrm{e}^t E_3, \quad \widetilde{\nabla}_{E_4} E_5 = -\mathrm{e}^t E_4.$$

By the straightforward calculations, by using (5), we can observe $\alpha = 0$ and $\beta = -\mathrm{e}^t$. This tells us that $\widetilde{M}^5(\phi, \xi, \eta, g)$ is a 5-dimensional trans-Sasakain manifold of type $(0, -\mathrm{e}^t)$ with respect to Levi-Civita connection $\widetilde{\nabla}$. Whereas, $\widetilde{M}^5(\phi, \xi, \eta, g)$ is also a trans-Sasakian manifold of type $(0, 1-\mathrm{e}^t)$ with semisymmetric metric connection $\overline{\nabla}$.

Now, we consider vector fields

$$\mathrm{e}_1 = \mathrm{e}^t\Big(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}\Big), \quad \mathrm{e}_2 = \mathrm{e}^t\Big(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\Big), \quad \mathrm{e}_3 = \mathrm{e}^t\frac{\partial}{\partial t}.$$

These vector fields are linear independent. By $\mathbf{D}$, let us denote the distribution spanned by these vectors. One can observe $\mathbf{D}$ is integrable and involutive. So its integral manifold M is a submanifold of $\widetilde{M}^5(\phi, \xi, \eta, g)$. On can easily see that

$$\phi\mathrm{e}_1 = -\mathrm{e}_2, \quad \phi\mathrm{e}_2 = \mathrm{e}_1 \quad \text{and} \quad \phi\mathrm{e}_3 = 0. \tag{52}$$

This tells us that M is an invariant submanifold of a trans-Sasakaim manifold $\widetilde{M}^5(\phi, \xi, \eta, g)$. On the other hand, by a direct calculation we get that $\sigma(e_1, e_2) = 0$, that is, M is a totally geodesic submanifold.

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