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SOME RESULTS ON COHOMOLOGY MODULES WITH RESPECT TO AN IDEAL AND A NON-NEGATIVE INTEGER

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Abstract. Let R be a commutative Noetherian ring, J an ideal of R, M an R-module and let d be a non-negative integer. In this paper we introduce a generalization of the notion of a local cohomology module, which we call a local cohomology module with respect to non-negative integer d and ideal J, and study some of its properties. Also, we define $W(d,J)$ and $\widetilde{W}(d,J)$ and present the conditions under which $H^i_{d,J}(M)=0$. Finally, we investigate the concept of $\operatorname{depth}_{d,J}(M)$ and examine the top vanishing of the R-module $H^i_{d,J}(M)$.

Keywords: (d,J)-local cohomology; system of ideals; (d,J)-depth, $W(d,J)$

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1. Introduction

Local cohomology stands as a cornerstone in commutative algebra and algebraic geometry, offering deep insights into the local structure of modules and schemes. Rooted in Grothendieck's foundational work, this theory has been instrumental in solving problems ranging from duality theorems to vanishing results. A central theme in its evolution has been the pursuit of generalizations that adapt cohomological tools to broader contexts, often by incorporating additional geometric or algebraic constraints.

In recent decades, the study of local cohomology with respect to pairs of ideals has emerged as a powerful framework, capturing nuanced interactions between ideals and modules. These constructions, such as the generalized local cohomology modules $H^i_{(I,J)}(M)$ associated with a pair of ideals (I,J), have enriched our understanding of algebraic structures by encoding information about both localization and annihilation via ideals. However, many fundamental questions remain open: How can

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these tools be refined to incorporate intrinsic geometric invariants, such as dimension, into their definitions? What new properties emerge when ideals are replaced with dimension-based systems?

This paper addresses these questions by introducing a novel generalization of local cohomology that replaces one ideal in the pair with a non-negative integer d, tied to the Krull dimension of quotient rings. This shift from ideals to dimension-centric systems unlocks a fresh perspective, bridging classical local cohomology with the geometry of supports defined by ideals of bounded dimension.

Throughout this paper, R denotes a commutative Noetherian ring with a nonzero unity, $\mathcal{I}(R)$ is the set of all ideals of R, d is a non-negative integer and J is an ideal of R. Let

$$\Sigma = \{\mathfrak{a} \in \mathcal{I}(R) \mid \dim(R/\mathfrak{a}) \leqslant d\},$$

which is a system of ideals in R in the sense of [4], page 21. In [2], for an R-module M, the authors defined the set $L_d(M)$ which consists of all elements $m \in M$ such that $\mathfrak{a}m = 0$ for some $\mathfrak{a} \in \Sigma$. They showed that $L_d(M) \cong \varinjlim_{\mathfrak{a}\in\Sigma} \operatorname{Hom}(R/\mathfrak{a}, M)$. Moreover, they established that $H^i_d(M) \cong \varinjlim_{\mathfrak{a}\in\Sigma} \operatorname{Ext}^i_R(R/\mathfrak{a}, M)$, where $i \in \mathbb{N}_0$ and $H^i_d(-)$ denotes the ith right derived functor of $L_d(-)$. Also, in [11] and [12], many studies have been conducted on $H^i_d(M)$. In [1], [6], [9], [10], some properties of the generalized modules $H^i_{(I,J)}(M)$ with respect to a pair of ideals (I, J) are studied. Here, we replace the ideal I with the non-negative integer d in the pair of ideals (I, J) and define the new functor $\Gamma_{d,J}(-)$ and its right-derived functor. We also introduce the system of ideals $\widetilde{W}(d, J)$, describe some of its properties, and establish theorems regarding the functor $\Gamma_{d,J}(-)$ within this new system of ideals.

This paper is organized as follows. In Section 2 we introduce the functor $\Gamma_{d,J}(-)$ on the category of R-modules. For any R-module M, the functor $\Gamma_{d,J}(M)$ consists of all elements $x \in M$ for which there exists an $I \in \Sigma$ such that $Ix \subseteq Jx$. We show that $\Gamma_{d,J}(-)$ is an R-linear, left exact functor on the category of R-modules $\mathcal{C}(R)$. Furthermore, for any integer i, we define the ith local cohomology functor $H^i_{d,J}(-)$ with respect to the pair (d, J) as the ith right derived functor of $\Gamma_{d,J}(-)$. The module $H^i_{d,J}(M)$ is then called the ith (d, J)-local cohomology module of M. We explore several key properties of $H^i_{d,J}(M)$. Notably, if $J = 0$, the functor $H^i_{d,J}(-)$ coincides with the d-local cohomology functor $H^i_d(-)$, where the support is in the closed subset $V(d) := \bigcup_{\mathfrak{a}\in\Sigma} V(\mathfrak{a})$. On the other hand, when $J \in \Sigma$, $\Gamma_{d,J}(-)$ acts as the identity functor, and for $i > 0$, we have $H^i_{d,J}(-) = 0$.

In Section 3 we define the set $W(d, J)$, which consists of all prime ideals $\mathfrak{p}$ of R satisfying $I \subseteq \mathfrak{p} + J$ for some $I \in \Sigma$. It is easy to verify that when $J = 0$, $W(d, J)$

coincides with the Zariski closed set $\mathrm{V}(d)$. We also introduce the set

$$\widetilde{\mathrm{W}}(d, J) = \{\mathfrak{a} \mid \exists\, I \in \Sigma,\ I \subseteq \mathfrak{a} + J\},$$

which, with the reverse inclusion, forms a system of ideals in R as described in [4], page 21. We examine the properties of $W(d, J)$ and $\widetilde{W}(d, J)$, obtaining important results regarding $H^i_{d,J}(M)$. Specifically, we show that $H^i_{d,J}(M) = \varinjlim_{\mathfrak{a} \in \widetilde{W}(d,J)} H^i_{\mathfrak{a}}(M)$ and prove that for an injective R-module E, $\Gamma_{d,J}(E)$ remains injective. Additionally, for an R-module M, we establish that $H^i_{d,J}(M) \cong \varinjlim_{\mathfrak{a} \in \widetilde{W}(d,J)} \mathrm{Ext}^i_R(R/\mathfrak{a}, M)$, and under certain conditions, we demonstrate that $H^i_{d,J}(M) = 0$. Finally, in Corollary 3.18, we prove that if M is artinian, then $H^i_{d,J}(M)$ is artinian for all $i \geqslant 0$.

In Section 4 we introduce the concept of $\mathrm{depth}_{d,J}(M)$ and show that this invariant equals

$$\min\{\mathrm{depth}(\mathfrak{p}, M) \mid \mathfrak{p} \in W(d, J)\}.$$

We provide several formulas and inequalities for this invariant, and for a finitely generated R-module M, we investigate the top vanishing of the R-module $H^i_{d,J}(M)$.

In this paper, some of the results are inspired by [9], and we attempt to extend these ideas to special local cohomology with respect to the pair (d, J), where d is a non-negative integer and J is an ideal of R.

2. d-functor based on an ideal

In this section we introduce the functor $\Gamma_{d,J}(-)$ and explore some of its key properties, where J is an ideal of R and d is a non-negative integer.

Definition 2.1. Let M be an R-module. We denote by $\Gamma_{d,J}(M)$ the set of elements x of M such that $Ix \subseteq Jx$ for some $I \in \Sigma$, i.e.,

$$\Gamma_{d,J}(M) = \{x \in M \mid \exists\, I \in \Sigma,\ Ix \subseteq Jx\}.$$

It is obvious that $x \in \Gamma_{d,J}(M)$ if and only if $I \subseteq \mathrm{Ann}(x) + J$ for some $I \in \Sigma$. Using this, we easily see that $\Gamma_{d,J}(M)$ is an R-submodule of M. It is clear that $\Gamma_{d,0}(M) = L_d(M)$ and if $J \in \Sigma$, then $\Gamma_{d,J}(M) = M$. It is easy to see that

$$\Gamma_{d,J}(M) = \Gamma_{d,J^n}(M) = \Gamma_{d,\sqrt{J}}(M)$$

for all $n \in \mathbb{N}$ and so if J is nilpotent, then $\Gamma_{d,J}(M) = L_d(M)$ and also,

$$\Gamma_{d,\sqrt{\mathrm{Ann}(M)}}(M) = L_d(M).$$

Lemma 2.2. *The functor $\Gamma_{d,J}$ is a left exact functor on the category of all R-modules.*

P r o o f. Let
$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$
be an exact sequence of R-modules. It is enough to prove that
$$0 \to \Gamma_{d,J}(A) \xrightarrow{\Gamma_{d,J}(\alpha)} \Gamma_{d,J}(B) \xrightarrow{\Gamma_{d,J}(\beta)} \Gamma_{d,J}(C)$$
is an exact sequence.

We easily see that $\Gamma_{d,J}(\alpha)$ is a monomorphism. On the other hand, $\mathrm{Im}(\Gamma_{d,J}(\alpha)) \subseteq \mathrm{Ker}(\Gamma_{d,J}(\beta))$, because $\beta\alpha = 0$. To prove $\mathrm{Ker}(\Gamma_{d,J}(\beta)) \subseteq \mathrm{Im}(\Gamma_{d,J}(\alpha))$, assume $x \in \mathrm{Ker}(\Gamma_{d,J}(\beta))$. Since $x \in \Gamma_{d,J}(B)$, then there exists $I \in \Sigma$ such that $Ix \subseteq Jx$. Also, since $\beta(x) = 0$, then $x \in \mathrm{Im}(\alpha)$ and so there exists $y \in A$ such that $\alpha(y) = x$. It suffices to show that $y \in \Gamma_{d,J}(A)$. For each $i \in I$ we have
$$\alpha(iy) = i\alpha(y) = ix \in Ix \subseteq Jx$$
and hence, there is $j \in J$ such that $ix = jx$. Thus, we have
$$\alpha((i-j)y) = ix - jx = 0.$$
Now, since α is a monomorphism, $(i-j)y = 0$. Hence, $iy = jy \in Jy$ and so $Iy \subseteq Jy$. Thus $y \in \Gamma_{d,J}(A)$. $\square$

Definition 2.3. For $i \in \mathbb{N}_0$, the ith right derived functor of $\Gamma_{d,J}$ is denoted by $H^i_{d,J}$ and will be referred to as the ith local cohomology functor with respect to (d,J). We say that M is a (d,J)-torsion (or (d,J)-torsion free) precisely when $\Gamma_{d,J}(M) = M$ (or $\Gamma_{d,J}(M) = 0$).

Note that M is a d-torsion (or d-torsion free) whenever $L_d(M) = M$ (or $L_d(M) = 0$).

We now collect some properties of the functor $\Gamma_{d,J}(-)$ in the following proposition.

Proposition 2.4. *Let M be an R-module. Then:*

(i) *if $J_1 \subseteq J_2$, then $\Gamma_{d,J_1}(M/\Gamma_{d,J_2}(M)) = 0$,*
(ii) $\Gamma_{d,J}(M/L_d(M)) = \Gamma_{d,J}(M)/L_d(M)$,
(iii) $L_d(\Gamma_{d,J}(M)) = \Gamma_{d,J}(L_d(M)) = L_d(M)$,
(iv) *if M is a d-torsion, then it is also a (d,J)-torsion,*
(v) *if M is a J-torsion, then $\Gamma_{d,J}(M) = L_d(M)$. Moreover, $H^i_{d,J}(H^j_J(M)) \cong H^i_d(H^j_J(M))$ for all $i \geqslant 0$ and all $j \geqslant 0$, where $H^j_J(M)$ is the jth local cohomology relative to J, introduced in* [4], Definition 1.2.1 *(note that $\Gamma_J(M) = H^0_J(M)$),*
(vi) $\Gamma_J(\Gamma_{d,J}(M)) = \Gamma_J(M)$.

P r o o f. (i) Let $m+\Gamma_{d,J_2}(M)\in\Gamma_{d,J_1}(M/\Gamma_{d,J_2}(M))$. Then there exists $I\in\Sigma$ such that $I(m+\Gamma_{d,J_2}(M))\subseteq J_1(m+\Gamma_{d,J_2}(M))$ and so $I\subseteq\operatorname{Ann}(m+\Gamma_{d,J_2}(M))+J_1$. Now, let $a\in I$. Hence, there exist $r\in\operatorname{Ann}(m+\Gamma_{d,J_2}(M))$ and $b\in J_1$ such that $a=r+b$. Thus $am=rm+bm$ and $rm\in\Gamma_{d,J_2}(M)$. Then

$$Krm\subseteq J_2rm\subseteq J_2m$$

for some $K\in\Sigma$. Also,

$$Kbm\subseteq KJ_1m\subseteq J_2m.$$

Hence, $Kam\subseteq J_2m$ and so $KIm\subseteq J_2m$. Since $KI\in\Sigma$, it follows that $m\in\Gamma_{d,J_2}(M)$. Then $m+\Gamma_{d,J_2}(M)=0$ and the result is complete.

(ii) Similarly to the proof of part (i), it is easy to see that $\Gamma_{d,J}(M/L_d(M))\subseteq\Gamma_{d,J}(M)/L(M)$. The converse inclusion is clear.

(iii) It is clear.

(iv) The result is obtained from $L_d(M)=M$ and part (iii).

(v) For the first part, clearly $L_d(M)\subseteq\Gamma_{d,J}(M)$. Now, it is enough to prove that $\Gamma_{d,J}(M)\subseteq L_d(M)$. Let $x\in\Gamma_{d,J}(M)$. Then there exists $I\in\Sigma$ such that $Ix\subseteq Jx$. On the other hand, since $\Gamma_J(M)=M$, then $J^nx=0$ for some $n\in\mathbb{N}_0$. Then $I^nx=0$ and so $x\in L_d(M)$, because $I^n\in\Sigma$. Hence $\Gamma_{d,J}(M)\subseteq L_d(M)$. For the second part, since $\Gamma_J(H^j_J(M))=H^j_J(M)$ for all $j\geqslant 0$, then using [4], Theorem 1.3.5 the result follows.

(vi) It is clear. □

Theorem 2.5. *Let d' be a non-negative integer and J' be an ideal of R. Also, let M be an R-module. Then:*

(i) $\Gamma_{d,J}(\Gamma_{d',J'}(M))=\Gamma_{d',J'}(\Gamma_{d,J}(M))$,
(ii) *if* $J\subseteq J'$ *then* $\Gamma_{d,J}(M)\subseteq\Gamma_{d,J'}(M)$,
(iii) *if* $d\leqslant d'$ *then* $\Gamma_{d,J}(M)\subseteq\Gamma_{d',J}(M)$,
(iv) $\Gamma_{d,J}(\Gamma_{d,J'}(M))=\Gamma_{d,JJ'}(M)=\Gamma_{d,J\cap J'}(M)$. *Moreover,* $\Gamma_{d,J}(\Gamma_{d,J}(M))=\Gamma_{d,J}(M)$ *and* $H^i_{d,JJ'}(M)=H^i_{d,J\cap J'}(M)$ *for all* $i\geqslant 0$,
(v) *if* $\sqrt{J}=\sqrt{J'}$, *then* $H^i_{d,J}(M)=H^i_{d,J'}(M)$ *for all* $i\geqslant 0$.

P r o o f. It is easy to prove all parts. Here, we will only prove part (iv).

Let $x\in\Gamma_{d,J}(\Gamma_{d,J'}(M))$. Then there exist $I_1,I_2\in\Sigma$ such that $I_1x\subseteq Jx$ and $I_2x\subseteq J'x$. Thus, $I_1\subseteq\operatorname{Ann}x+J$ and $I_2\subseteq\operatorname{Ann}x+J'$. Now we have:

$$I_1I_2\subseteq(\operatorname{Ann}x+J)(\operatorname{Ann}x+J')\subseteq\operatorname{Ann}x+JJ'\subseteq\operatorname{Ann}x+J\cap J'.$$

Hence

$$\Gamma_{d,J}(\Gamma_{d,J'}(M))\subseteq\Gamma_{d,JJ'}(M)\subseteq\Gamma_{d,J\cap J'}(M).$$

To prove the converse inclusion, let $x \in \Gamma_{d,J\cap J'}(M)$. Then there exists $I \in \Sigma$ such that $I \subseteq \operatorname{Ann} x + J \cap J'$. Since $J \cap J' \subseteq J$ and $J \cap J' \subseteq J'$, then $I \subseteq \operatorname{Ann} x + J$ and $I \subseteq \operatorname{Ann} x + J'$ and so $x \in \Gamma_{d,J}(\Gamma_{d,J'}(M))$ and the result is complete. □

3. $W(d, J)$ and $\widetilde{W}(d, J)$

In this section we introduce the set of ideals $W(d, J)$ and $\widetilde{W}(d, J)$ and present some of its properties and their relationships with Supp and Ass of module. Also, by using these two sets of ideals, we do more study on $H^i_{d,J}(M)$ and present some relations of $H^i_{d,J}(M)$ with $H^i_J(M)$ and prove some results.

Definition 3.1. Let d be a non-negative integer. For an ideal J of R, we denote by $W(d, J)$ the set

$$W(d, J) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \exists\, I \in \Sigma;\ I \subseteq \mathfrak{p} + J\}.$$

Proposition 3.2. *Let d' be a non-negative integer and J' be an ideal of R. Then:*

(i) *if $d \leqslant d'$, then $W(d, J) \subseteq W(d', J)$,*

(ii) *if $J \subseteq J'$, then $W(d, J) \subseteq W(d, J')$,*

(iii) *$W(d, JJ') = W(d, J \cap J') = W(d, J) \cap W(d, J')$,*

(iv) *$W(d, J) = W(d, \sqrt{J})$,*

(v) *$V(d) = \bigcap\limits_{I\in\mathcal{I}(R)} W(d, I) = \bigcap\limits_{\mathfrak{q}\in\operatorname{Spec}(R)\setminus V(d)} W(d, \mathfrak{q})$, where $V(d) = \bigcup\limits_{\mathfrak{a}\in\Sigma} V(\mathfrak{a})$, which is studied in* [11],

(vi) *if $J \in \Sigma$, then $W(d, J) = \operatorname{Spec}(R)$.*

P r o o f. It is easy to prove all parts. Here, we will only prove parts (iii) and (v).

(iii) Since

$$JJ' \subseteq J \cap J' \subseteq J, J',$$

it holds that

$$W(d, JJ') \subseteq W(d, J \cap J') \subseteq W(d, J) \cap W(d, J').$$

Now, let $\mathfrak{p} \in W(d, J) \cap W(d, J')$. Then there exist $I, I' \in \Sigma$ such that $I \subseteq \mathfrak{p} + J$ and $I' \subseteq \mathfrak{p} + J'$. Thus

$$II' \subseteq (\mathfrak{p} + J)(\mathfrak{p} + J') \subseteq \mathfrak{p} + JJ'.$$

Hence, we have $\mathfrak{p} \in W(d, JJ')$ and so $W(d, J) \cap W(d, J') \subseteq W(d, JJ')$.

(v) Let $\mathfrak{p} \in \mathrm{V}(d)$. Then there exists $\mathfrak{a} \in \Sigma$ such that $\mathfrak{a} \subseteq \mathfrak{p}$. Hence, $\mathfrak{a} \subseteq \mathfrak{p} + I$ for all $I \in \mathcal{I}(R)$ and so $\mathfrak{p} \in \bigcap\limits_{I\in\mathcal{I}(R)} W(d, I)$. Then $V(d) \subseteq \bigcap\limits_{I\in\mathcal{I}(R)} W(d, I)$.

On the other hand, clearly $\bigcap\limits_{I\in\mathcal{I}(R)} W(d,I) \subseteq \bigcap\limits_{\mathfrak{q}\in \mathrm{Spec}(R)\setminus V(d)} W(d,\mathfrak{q})$. Now, let $\mathfrak{p} \in \bigcap\limits_{\mathfrak{q}\in \mathrm{Spec(R)}\setminus V(d)} W(d,\mathfrak{q}) \setminus V(d)$. Then $\mathfrak{p} \notin \Sigma$ and also $\mathfrak{p} \in W(d,\mathfrak{p})$. Therefore there exists $I \in \Sigma$ such that $I \subseteq \mathfrak{p}+\mathfrak{p}$ and so $\mathfrak{p} \in \Sigma$, which is a contradiction. Then $\bigcap\limits_{\mathfrak{q}\in \mathrm{Spec(R)}\setminus V(d)} W(d,\mathfrak{q}) \subseteq V(d)$ and the result is complete. □

Proposition 3.3. *Let M be an R-module. Then the following are equivalent:*

(i) *M is (d,J)-torsion R-module,*
(ii) $\mathrm{Min}(M) \subseteq W(d,J)$,
(iii) $\mathrm{Ass}(M) \subseteq W(d,J)$,
(iv) $\mathrm{Supp}(M) \subseteq W(d,J)$.

P r o o f. The implications (iv) ⇒ (iii) ⇒ (ii) are straightforward.

(ii) ⇒ (iv) Let $\mathfrak{p} \in \mathrm{Supp}(M)$. Then there exists $\mathfrak{q} \in \mathrm{Min}(M)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Since $\mathfrak{q} \in W(d,J)$, there exists $I \in \Sigma$ such that

$$I \subseteq \mathfrak{q} + J \subseteq \mathfrak{p} + J.$$

Thus, $\mathfrak{p} \in W(d,J)$, and we conclude that $\mathrm{Supp}(M) \subseteq W(d,J)$.

(i) ⇒ (iii) Let $\mathfrak{p} \in \mathrm{Ass}(M)$. Then there exists a nonzero element $x \in M$ such that $\mathfrak{p} = \mathrm{Ann}(x)$. Since $M = \Gamma_{d,J}(M)$, there exists $I \in \Sigma$ such that $I \subseteq \mathrm{Ann}(x) + J$, which implies $I \subseteq \mathfrak{p} + J$. Therefore, $\mathfrak{p} \in W(d,J)$.

(iv) ⇒ (i) We aim to show that $M \subseteq \Gamma_{d,J}(M)$. Let $x \in M$ and suppose $\mathrm{Min}(Rx) = \{\mathfrak{p}_1,\ldots,\mathfrak{p}_t\}$. Since

$$\mathrm{Min}(Rx) \subseteq \mathrm{Supp}(M) \subseteq W(d,J),$$

there exists $I_i \in \Sigma$ such that $I_i \subseteq \mathfrak{p}_i + J$ for all $i = 1,2,\ldots,t$. Put $I := \prod \mathfrak{p}_i$. Then $I \subseteq \prod \mathfrak{p}_i + J$. Since

$$\sqrt{\mathrm{Ann}(x)} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_t,$$

there exists $n \in \mathbb{N}$ such that $\left(\prod \mathfrak{p}_i\right)^n \subseteq \mathrm{Ann}(x)$. Thus

$$I^n \subseteq \left(\prod \mathfrak{p}_i\right)^n + J \subseteq \mathrm{Ann}(x) + J.$$

This implies that $I^n x \subseteq Jx$, which shows that $x \in \Gamma_{d,J}(M)$. Hence, the desired inclusion follows. □

Corollary 3.4.

(1) *Let $x \in M$. Then the following statements are equivalent:*
   (i) $x \in \Gamma_{d,J}(M)$,
   (ii) $\mathrm{Supp}(Rx) \subseteq W(d,J)$.

(2) *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence of* R*-module. Then* M *is a* (d,J)*-torsion module if and only if* L *and* N *are* (d,J)*-torsion modules.*

Proof. (1) For (i) $\Rightarrow$ (ii), since $x \in \Gamma_{d,J}(M)$, thus $\Gamma_{d,J}(Rx) = Rx$. Hence, by Proposition 3.3, we have $\mathrm{Supp}(Rx) \subseteq W(d,J)$.

(ii) $\Rightarrow$ (i) By Proposition 3.3, we have $x \in Rx = \Gamma_{d,J}(Rx) \subseteq \Gamma_{d,J}(M)$.

(2) This follows from Proposition 3.3 and $\mathrm{Supp}(M) = \mathrm{Supp}(L) \cup \mathrm{Supp}(N)$. □

Corollary 3.5. *Let* N *be an* R*-submodule of* M*. Then* $N \subseteq \Gamma_{d,J}(M)$ *if and only if* $\mathrm{Supp}(N) \subseteq W(d,J)$*. In particular,* $\mathrm{Supp}(\Gamma_{d,J}(M)) \subseteq W(d,J)$*.*

Proof. First, let $\mathrm{Supp}(N) \subseteq W(d,J)$. Then by Proposition 3.3, $N = \Gamma_{d,J}(N)$ and so $N \subseteq \Gamma_{d,J}(M)$.

Now, let $N \subseteq \Gamma_{d,J}(M)$. By Proposition 3.3, it suffices to prove that $\mathrm{Ass}(N) \subseteq W(d,J)$. Let $\mathfrak{p} \in \mathrm{Ass}(N)$. Then there exists $0 \neq x \in N$ such that $\mathfrak{p} = \mathrm{Ann}(x)$. Since $x \in \Gamma_{d,J}(M)$, then by Corollary 3.4 (1), $\mathrm{Ass}(Rx) \subseteq W(d,J)$. Then $\mathfrak{p} \in W(d,J)$ and so $\mathrm{Ass}(N) \subseteq W(d,J)$. □

Corollary 3.6. *If* M *is a* (d,J)*-torsion* R*-module, then* M/JM *is a* d*-torsion* R*-module. The converse holds if* M *is a finitely generated* R*-module.*

Proof. Since $M = \Gamma_{d,J}(M)$, thus $\mathrm{Supp}(M) \subseteq W(d,J)$ and we have

$$\mathrm{Supp}(M/JM)) \subseteq \mathrm{Supp}(M) \cap \mathrm{V}(J) \subseteq W(d,J) \cap \mathrm{V}(J) \subseteq \mathrm{V}(d).$$

Then M/JM is a d-torsion R-module.

Conversely: Let M be a finitely generated R-module, and let $x \in M$. We aim to show that $x \in \Gamma_{d,J}(M)$. By the Artin-Rees lemma, there exists $n \in \mathbb{N}_0$ such that $J^nM \cap Rx \subseteq Jx$. Since, by assumption, M/JM is a d-torsion R-module, it follows that $L_d(M/JM) = M/JM$. Then $L_d(M/J^nM) = M/J^nM$, which implies the existence of $I \in \Sigma$ such that $I(x + J^nM) = 0$. This means that $Ix \subseteq J^nM$ and so $Ix \subseteq Jx$. Consequently, we obtain $x \in \Gamma_{d,J}(M)$. □

Theorem 3.7. *Let* M *be an* R*-module. Then*

$$\mathrm{Ass}(\Gamma_{d,J}(M)) = \mathrm{Ass}(M) \cap W(d,J).$$

In particular, $\Gamma_{d,J}(M) \neq 0$ *if and only if* $\mathrm{Ass}(M) \cap W(d,J) \neq \emptyset$*.*

Proof. Since $\Gamma_{d,J}(M)$ is a (d,J)-torsion R-module, we have $\mathrm{Ass}(\Gamma_{d,J}(M)) \subseteq W(d,J)$ by Proposition 3.3 and so $\mathrm{Ass}(\Gamma_{d,J}(M)) \subseteq W(d,J) \cap \mathrm{Ass}(M)$. Now, let $\mathfrak{q} \in \mathrm{Ass}(M) \cap W(d,J)$. Then $\mathfrak{q} = \mathrm{Ann}(x)$ for some $0 \neq x \in M$ and there exists $I \in \Sigma$ such that $I \subseteq \mathfrak{q} + J$. Then $Ix \subseteq Jx$ and so $x \in \Gamma_{d,J}(M)$. Hence $\mathfrak{q} \in \mathrm{Ass}(\Gamma_{d,J}(M))$. □

For an R-module M, let $E(M)$ denote the injective hull of M, as introduced in [7], page 281.

Theorem 3.8. *If $\mathfrak{p} \in W(d, J)$, then $E(R/\mathfrak{p})$ is a (d, J)-torsion R-module. On the other hand, if $\mathfrak{p} \notin W(d, J)$, then $E(R/\mathfrak{p})$ is a (d, J)-torsion-free R-module.*

Proof. Assume that $\mathfrak{p} \in W(d, J)$. Since $\operatorname{Ass}_R(E(R/\mathfrak{p})) = \{\mathfrak{p}\}$ (see [7], page 146 (v)), it follows that $\operatorname{Ass}_R(E(R/\mathfrak{p})) \subseteq W(d, J)$. Thus, by Proposition 3.3, $\Gamma_{d,J}(E(R/\mathfrak{p})) = E(R/\mathfrak{p})$.

For the second part, let $\mathfrak{p} \notin W(d, J)$. Then

$$\operatorname{Ass}_R(E(R/\mathfrak{p})) \cap W(d, J) = \{\mathfrak{p}\} \cap W(d, J) = \emptyset.$$

Thus, by Theorem 3.7, we have $\Gamma_{d,J}(E(R/\mathfrak{p})) = 0$. $\square$

Theorem 3.9. *Let M be a (d, J)-torsion R-module. Then there exists an injective resolution of M in which each term is a (d, J)-torsion R-module.*

Proof. First, note that the injective hull E^0 of M is also a (d, J)-torsion module. In fact, since M is (d, J)-torsion, we have $\operatorname{Ass}(E^0) = \operatorname{Ass}(M) \subseteq W(d, J)$ by Proposition 3.3. Hence, E^0 is (d, J)-torsion. Thus, we see that M can be embedded in an (d, J)-torsion injective R-module E^0.

Suppose, inductively, we have constructed an exact sequence

$$0 \to M \to E^0 \to \ldots \to E^{n-1} \xrightarrow{d^{n-1}} E^n$$

of R-modules in which $E^0, \ldots, E^{n-1}, E^n$ are (d, J)-torsion injective R-modules. Let C be the cokernel of the map d^{n-1}. Since E^n is a (d, J)-torsion module, C is a (d, J)-torsion as well by Corollary 3.4 (2). Applying the argument in the first paragraph to C, we can embed C into a (d, J)-torsion injective R-module E^{n+1}. This completes the proof by induction. $\square$

Corollary 3.10. *Let M be an R-module.*

(i) *If M is a (d, J)-torsion R-module, then $H^i_{d,J}(M) = 0$ for all $i > 0$.*
(ii) *$H^i_{d,J}(\Gamma_{d,J}(M)) = 0$ for all $i > 0$.*
(iii) *$M/\Gamma_{d,j}(M)$ is a (d, J)-torsion free R-module.*
(iv) *$H^i_{d,J}(M) \cong H^i_{d,J}(M/\Gamma_{d,j}(M))$ for all $i > 0$.*

Proof. (i) Follows from Theorem 3.9.

(ii) Since $\Gamma_{d,J}(M)$ is a (d,J)-torsion R-module, then the results follow from part (i).

(iii) By Proposition 2.4 (i), the result follows.

(iv) From the short exact sequence

$$(\sharp) \qquad 0 \to \Gamma_{d,J}(M) \to M \to M/\Gamma_{d,j}(M) \to 0$$

and part (ii), we obtain the short exact sequence

$$0 \to \Gamma_{d,J}(\Gamma_{d,J}(M)) \to \Gamma_{d,J}(M) \to \Gamma_{d,J}(M/\Gamma_{d,J}(M)) \to 0$$

and isomorphism $H^i_{d,J}(M) \cong H^i_{d,J}(M/\Gamma_{d,J}(M))$ for $i \geqslant 1$ and therefore (iv) is true. $\square$

Definition 3.11. Let d be a non-negative integer and J be an ideal of R. We define the set $\widetilde{W}(d,J)$ as

$$\widetilde{W}(d,J) = \{\mathfrak{a} \in \mathcal{I}(R) \mid \exists\, I \in \Sigma,\ I \subseteq \mathfrak{a} + J\}.$$

Theorem 3.12. *Let M be an R-module. Then there is a natural isomorphism*

$$H^i_{d,J}(M) = \varinjlim_{\mathfrak{a} \in \widetilde{W}(d,J)} H^i_{\mathfrak{a}}(M)$$

for any integer $i \geqslant 0$.

Proof. First, we show that $\Gamma_{d,J}(M) = \bigcup\limits_{\mathfrak{a} \in \widetilde{W}(d,J)} \Gamma_{\mathfrak{a}}(M)$. To do this, suppose $x \in \Gamma_{d,J}(M)$. Then there exists $I \in \Sigma$ such that $I \subseteq \operatorname{Ann}(x) + J$. Let $\mathfrak{a} = \operatorname{Ann}(x)$, thus $\mathfrak{a} \in \widetilde{W}(d,J)$ and $x \in \Gamma_{\mathfrak{a}}(M)$.

Conversely, let $x \in \bigcup\limits_{\mathfrak{a} \in \widetilde{W}(d,J)} \Gamma_{\mathfrak{a}}(M)$. Then there is an ideal $\mathfrak{a} \in \widetilde{W}(d,J)$ with $x \in \Gamma_{\mathfrak{a}}(M)$. Thus, $I \subseteq \mathfrak{a} + J$ and $\mathfrak{a}^n x = 0$ for some integers $n \geqslant 0$ and some $I \in \Sigma$. Since $I^n \subseteq (\mathfrak{a}+J)^n \subseteq \mathfrak{a}^n + J$, then $I^n x \subseteq Jx$ and so $x \in \Gamma_{d,J}(M)$. Now, since $\widetilde{W}(d,J)$ is a system of ideals, it follows that

$$\Gamma_{d,J}(M) = \bigcup_{\mathfrak{a} \in \widetilde{W}(d,J)} \Gamma_{\mathfrak{a}}(M) = \varinjlim_{\mathfrak{a} \in \widetilde{W}(d,J)} \Gamma_{\mathfrak{a}}(M).$$

Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Then it implies a long exact sequence

$$0 \to H^0_{\mathfrak{a}}(L) \to H^0_{\mathfrak{a}}(M) \to H^0_{\mathfrak{a}}(N) \to H^1_{\mathfrak{a}}(L) \to H^1_{\mathfrak{a}}(M) \to H^1_{\mathfrak{a}}(N) \to \dots$$

for each $\mathfrak{a} \in \widetilde{W}(d,J)$. Then we obtain the long exact sequence

$$\begin{aligned} 0 \to \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} H^0_{\mathfrak{a}}(L) \to \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} H^0_{\mathfrak{a}}(M) \to \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} H^0_{\mathfrak{a}}(N) \\ \to \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} H^1_{\mathfrak{a}}(L) \to \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} H^1_{\mathfrak{a}}(M) \to \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} H^1_{\mathfrak{a}}(N) \to \ldots \end{aligned}$$

On the other hand, for any injective R-module E and any positive integer i, we have $H^i_{\mathfrak{a}}(E) = 0$ for each $\mathfrak{a} \in \widetilde{W}(d,J)$. Thus, we have $\varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} H^i_{\mathfrak{a}}(E) = 0$.

These arguments imply that $\{\varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} H^i_{\mathfrak{a}} \mid i = 0,1,2,\ldots\}$ is a system of right derived functors of $\Gamma_{d,J}$ and the proof is completed. □

Corollary 3.13. *If M is an injective R-module, then $\Gamma_{d,J}(M)$ is also injective.*

P r o o f. Since M is injective then by [4], Theorem 2.1.4, $\Gamma_{\mathfrak{a}}(M)$ is also injective for any ideal $\mathfrak{a}$ of R. Now, since R is Noetherian, then by [8], page 255 (Exercise 5.23) and Theorem 3.12 the result is complete. □

Corollary 3.14. *Let M be an R-module. Then for all $i \geqslant 0$ we have*

$$H^i_{d,J}(M) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} \operatorname{Ext}^i_R(R/\mathfrak{a}, M).$$

P r o o f. Using Theorem 3.12, we have

$$\begin{aligned} \Gamma_{d,J}(M) &= \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} \Gamma_{\mathfrak{a}}(M) \\ &\cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} \Big(\varinjlim_{i\in\mathbb{N}} \operatorname{Hom}_R(R/\mathfrak{a}^i, M)\Big) \\ &\cong \varinjlim_{i\in\mathbb{N}} \Big(\varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} \operatorname{Hom}_R(R/\mathfrak{a}^i, M)\Big). \end{aligned}$$

Now, since $\varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} \operatorname{Hom}_R(R/\mathfrak{a}^i, M) = \bigcup_{\mathfrak{a}\in\widetilde{W}(d,J)} (M :_R \mathfrak{a}^i)$ and also, since $\mathfrak{a} \in \widetilde{W}(d,J)$ iff $\mathfrak{a}^i \in \widetilde{W}(d,J)$ for all $i \geqslant 0$, it follows that

$$\bigcup_{\mathfrak{a}\in\widetilde{W}(d,J)} (M :_R \mathfrak{a}^i) = \bigcup_{\mathfrak{a}\in\widetilde{W}(d,J)} (M :_R \mathfrak{a})$$

and so

$$\varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} \operatorname{Hom}_R(R/\mathfrak{a}^i, M) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} \operatorname{Hom}_R(R/\mathfrak{a}, M).$$

Therefore, we have

$$\Gamma_{d,J}(M) \cong \varinjlim_{i\in\mathbb{N}}\Big(\varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} \operatorname{Hom}_R(R/\mathfrak{a}, M)\Big) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} \operatorname{Hom}_R(R/\mathfrak{a}, M),$$

where the second isomorphism follows from [8], Corollary 5.38. Now, since for any injective R-module E and all $i > 0$ we have

$$H^i_{d,J}(E) = 0 = \varinjlim_{\mathfrak{a}\in\widetilde{W}(d,J)} \operatorname{Ext}^i_R(R/\mathfrak{a}, E),$$

then using [4], Theorem 1.3.5 the result follows. □

Corollary 3.15. *Let M be an R-module and $i \in \mathbb{N}_0$. Then:*

(i) *$H^i_{d,J}(M)$ is a (d,J)-torsion R-module,*

(ii) *$H^i_d(M)$ is a (d,J)-torsion R-module.*

Proof. (i) It suffices to prove that $H^i_{d,J}(M) \subseteq \Gamma_{d,J}(H^i_{d,J}(M))$. Let $x \in H^i_{d,J}(M)$. Then by Theorem 3.12, there exist $\mathfrak{a} \in \widetilde{W}(d,J)$ and an R-homomorphism $\varphi_{\mathfrak{a}}\colon H^i_{\mathfrak{a}}(M) \to H^i_{d,J}(M)$ such that $x = \varphi_{\mathfrak{a}}(y)$ for some $y \in H^i_{\mathfrak{a}}(M)$. Thus, there exists $n \in \mathbb{N}$ such that $\mathfrak{a}^n y = 0$ and so $\mathfrak{a}^n x = 0$. Let $I \subseteq \mathfrak{a} + J$ for some $I \in \Sigma$. Hence $I^n x \subseteq Jx$. Now since $I^n \in \Sigma$, then $x \in \Gamma_{d,J}(H^i_{d,J}(M))$ and the result is complete.

(ii) It suffices to prove that $H^i_d(M) \subseteq \Gamma_{d,J}(H^i_d(M))$. Assume that $x \in H^i_d(M)$. Since by [11], Remark 1, $L_d(H^i_d(M)) = H^i_d(M)$, then there exists $I \in \Sigma$ such that $Ix = 0$. Hence, $Ix \subseteq Jx$ and so $x \in \Gamma_{d,J}(H^i_d(M))$ and the proof is complete. □

Theorem 3.16. *Let $t \in \mathbb{N}$, $(R,\mathfrak{m})$ be a local ring and M be a finitely generated R-module. If $H^i_{d,J}(M)$ is finitely generated for all $i \geqslant t$, then $H^i_{d,J}(M) = 0$ for all $i \geqslant t$.*

Proof. We use induction on $n = \dim(M)$. When $n = 0$, we have $H^i_{d,J}(M) = 0$ for all $i \geqslant t$. Assume inductively that $n > 0$ and that the result has been proved for finitely generated R-module of dimension $n - 1$. The exact sequence

$$0 \to \Gamma_{d,J}(M) \to M \to M/\Gamma_{d,J}(M) \to 0$$

yields the long exact sequence

$$\ldots \to H^i_{d,J}(\Gamma_{d,J}(M)) \to H^i_{d,J}(M) \to H^i_{d,J}(M/\Gamma_{d,J}(M)) \to H^{i+1}_{d,J}(\Gamma_{d,J}(M)) \to \ldots$$

Now, by Corollary 3.10 (i), $H^i_{d,J}(\Gamma_{d,J}(M)) = 0$ for all $i \geqslant 1$. Then $H^i_{d,J}(M) \cong H^i_{d,J}(M/\Gamma_{d,J}(M))$ for all $i \geqslant 1$. Thus, we may assume, by replacing M with

$M/\Gamma_{d,J}(M)$, that $\Gamma_{d,J}(M) = 0$. Hence $\mathfrak{m} \nsubseteq Z_R(M)$. So there exists $x \in \mathfrak{m} \setminus Z_R(M)$, where $Z_R(M)$ is the set of all zero-divisors on M over R (i.e., $Z_R(M) = \{r \in R \mid \exists\, m \in M \setminus \{0\},\ rm = 0\}$). Consider the R-homomorphism $x\cdot\colon M \to M$ defined by $x\cdot(m) = xm$ for all $m \in M$. The exact sequence

$$0 \to M \xrightarrow{x\cdot} M \to M/xM \to 0$$

induces the long exact sequence

$$\ldots \to H^i_{d,J}(M) \to H^i_{d,J}(M) \to H^i_{d,J}(M/xM) \to H^{i+1}_{d,J}(M) \to \ldots,$$

which implies that $H^i_{d,J}(M/xM)$ is finitely generated for all $i \geqslant t$. Since M/xM is finitely generated with $\dim(M/xM) = n - 1$, then by the inductive hypothesis, $H^i_{d,J}(M/xM) = 0$ for all $i \geqslant t$. Therefore $H^i_{d,J}(M) \cong xH^i_{d,J}(M)$ for all $i \geqslant t + 1$. Then $H^i_{d,J}(M) = 0$ for all $i \geqslant t + 1$, by Nakayama Lemma. Also, $H^t_{d,J}(M) = 0$ because $H^t_{d,J}(M) \to H^t_{d,J}(M)$ is an epimorphism. Now, the result is complete. $\square$

Theorem 3.17. *Let M be an R-module and $\overline{M} = M/\Gamma_{d,J}(M)$. If E is the injective hull of $\overline{M}$ and $L := E/\overline{M}$, then $H^i_{d,J}(L) \cong H^{i+1}_{d,J}(M)$ for all $i \geqslant 0$.*

Proof. Let $\mathfrak{a} \in \widetilde{W}(d,J)$. Then $\Gamma_{\mathfrak{a}}(M) \subseteq \Gamma_{d,J}(M)$. By Corollary 3.10 (iii), $\Gamma_{d,J}(\overline{M}) = 0$ and so $\Gamma_{d,J}(E) = 0$. Then $\Gamma_{\mathfrak{a}}(\overline{M}) = 0 = \Gamma_{\mathfrak{a}}(E)$. Let $\psi \in \operatorname{Hom}_R(R/\mathfrak{a}, E)$. Thus $\psi(1+\mathfrak{a}) \in E$. Since $\mathfrak{a}\cdot\psi(1+\mathfrak{a}) = 0$, it follows that $\psi(1+\mathfrak{a}) \in \Gamma_{\mathfrak{a}}(E)$. Consequently, $\psi(1+\mathfrak{a}) = 0$, implying that $\operatorname{Hom}_R(R/\mathfrak{a}, E) = 0$. Furthermore, since E is an injective R-module, it follows that $\operatorname{Ext}^i_R(R/\mathfrak{a}, E) = 0$ for all $i \geqslant 0$. Now, the short exact sequence

$$0 \to \overline{M} \to E \to L \to 0$$

induces the long exact sequence

$$\ldots \to \operatorname{Ext}^i_R(R/\mathfrak{a}, E) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, L) \to \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, \overline{M}) \to \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, E) \to \ldots$$

Then for each $i \geqslant 0$ we have

$$\operatorname{Ext}^i_R(R/\mathfrak{a}, L) \cong \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, \overline{M}) \Rightarrow H^i_{d,J}(L) \cong H^{i+1}_{d,J}(\overline{M}).$$

Now, by Corollary 3.10 (iv), $H^i_{d,J}(L) \cong H^{i+1}_{d,J}(M)$ for all $i \geqslant 0$. $\square$

Corollary 3.18. *Let M be an artinian R-module. Then $H^i_{d,J}(M)$ is also artinian for $i \geqslant 0$.*

P r o o f. Since $\Gamma_{d,J}(M) \subseteq M$, then $\Gamma_{d,J}(M)$ is artinian. Now, we show that $H^i_{d,J}(M)$ is artinian for all $i \geqslant 1$. By Corollary 3.10 (iv), it can be assumed that $\Gamma_{d,J}(M) = 0$. Denote by $E := E(M)$ the injective hull of M. Let $L = E/M$. Note that if $M \subseteq K$ is an essential submodule, then M is artinian if and only if K is artinian. Hence, E is artinian and so L and also $\Gamma_{d,J}(L)$ are artinian. By Theorem 3.17, $H^1_{d,J}(M) \cong \Gamma_{d,J}(L)$. Therefore $H^1_{d,J}(M)$ is artinian and so with this method, $H^1_{d,J}(L)$ is also artinian. Again, by Theorem 3.17, $H^2_{d,J}(M) \cong H^1_{d,J}(L)$. Hence $H^2_{d,J}(M)$ is artinian. The continuation of this process implies that $H^i_{d,J}(M)$ is artinian for all $i \geqslant 0$. $\square$

Theorem 3.19. *Let M be a finitely generated R-module and $t \in \mathbb{N}_0$ such that $H^t_{d,J}(R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Supp}(M)$. Then $H^t_{d,J}(N) = 0$ for any finitely generated R-module N with $\operatorname{Supp}(N) \subseteq \operatorname{Supp}(M)$. In particular, $H^t_{d,J}(M) = 0$.*

P r o o f. Let N be a finitely generated R-module with $\operatorname{Supp}(N) \subseteq \operatorname{Supp}(M)$. There exists a chain of submodules of N in the form of

$$0 = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N_l = N$$

such that for each $1 \leqslant i \leqslant l$, $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Supp}(N)$. For each $1 \leqslant i \leqslant l$, the sequence

$$0 \to N_{i-1} \to N_i \to R/\mathfrak{p}_i \to 0$$

is exact. From $N_1 \cong R/\mathfrak{p}_1$ we have $H^t_{d,J}(N_1) \cong H^t_{d,J}(R/\mathfrak{p}_1)$. Since $\mathfrak{p}_1 \in \operatorname{Supp}(M)$, then $H^t_d(R/\mathfrak{p}_1) = 0$ and so $H^t_d(N_1) = 0$. Again, from the short exact sequence

$$0 \to N_1 \to N_2 \to R/\mathfrak{p}_2 \to 0$$

we have $H^t_{d,J}(N_2) = 0$. Now, by continuing this process, we conclude that $H^t_d(N) = 0$. $\square$

4. $\operatorname{depth}_{d,J}(-)$ and top vanishing theorem

In this section, for the finitely generated R-module M, we define $\operatorname{depth}_{d,J}(M)$ and will examine the top vanishing of the R-module $H^i_{d,J}(M)$.

Definition 4.1. Let M be an R-module. We define the depth of (d, J) on M by

$$\begin{aligned} \operatorname{depth}_{d,J}(M) &= \min\{\operatorname{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \widetilde{W}(d, J)\} \\ &= \min\{\operatorname{depth}(\mathfrak{a}, M) \mid \exists\, I \in \Sigma;\ I \subseteq \mathfrak{a} + J\}. \end{aligned}$$

Clearly, we have

$$\operatorname{depth}_{d,0}(M) = \min\{\operatorname{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \Sigma\} = \operatorname{depth}_d(M),$$

which is the second equality studied in [12].

Remark 4.2. Let M be a finitely generated R-module such that $\mathfrak{a}M = M$ for any $\mathfrak{a} \in \widetilde{W}(d, J)$. Then by [4], Exercise 6.2.6 and Theorem 3.12, we get $H^i_{d,J}(M) = 0$ for all $i \in \mathbb{N}_0$. If there exists $\mathfrak{a} \in \widetilde{W}(d, J)$ such that $\mathfrak{a}M \neq M$, then $\operatorname{depth}_{d,J}(M) < \infty$.

Theorem 4.3. *Let M be an R-module. Then*

$$\min\{i \in \mathbb{N}_0 \mid H^i_{d,J}(M) \neq 0\} = \min\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in W(d, J)\}.$$

Proof. Let $n = \min\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in W(d, J)\}$, and let

$$\begin{aligned} E^\bullet(M)\colon\ 0 \to M \to E^0(M) &\overset{d^0}{\to} E^1(M) \overset{d^1}{\to} \dots \\ &\to E^{n-1}(M) \overset{d^{n-1}}{\to} E^n(M) \overset{d^n}{\to} E^{n+1}(M) \to \dots \end{aligned}$$

be a minimal injective resolution of M. That is, each $E^i(M)$ is an injective module, and the differentials satisfy $\operatorname{Ker} d^n = \operatorname{Im} d^{n-1}$ for all $n \geqslant 0$. If $\mathfrak{p} \in W(d, J)$, then

$$n \leqslant \operatorname{depth}(M_{\mathfrak{p}}) = \min\{i \mid \mu_i(\mathfrak{p}, M) \neq 0\},$$

where $\mu_i(\mathfrak{p}, M)$ denotes the ith Bass number of M at $\mathfrak{p}$, first introduced by Hyman Bass in [3], page 11. Thus, we have

$$(*) \qquad \Gamma_{d,J}(E^i(M)) = \bigoplus_{\mathfrak{p} \in W(d,J)} E(R/\mathfrak{p})^{\mu_i(\mathfrak{p},M)} = 0$$

for any integer $i < n$. Hence $H^i_{d,J}(M) = 0$ if $i < n$. It suffices to show that $H^n_{d,J}(M) \neq 0$. By using $(*)$, the complex $\Gamma_{d,J}(E^\bullet(M))$ starts from its nth term. Now, consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & H^n_{d,J}(M) \arrow[r] & \Gamma_{d,J}(E^n(M)) \arrow[r] \arrow[d] & \Gamma_{d,J}(E^{n+1}(M)) \arrow[d] \\
 & & E^{n-1}(M) \arrow[r, "d^{n-1}"] & E^n(M) \arrow[r, "d^n"] & E^{n+1}(M)
\end{tikzcd}$$

with exact rows. Since

$$\operatorname{Ker} d^n = \operatorname{Im} d^{n-1} \subseteq E^n(M)$$

is an essential extension, it follows that

$$H^n_{d,J}(M) = \Gamma_{d,J}(E^n(M)) \cap \operatorname{Ker} d^n \neq 0.$$

□

Corollary 4.4. *For a finitely generated R-module M, we have the following equalities:*

$$\begin{aligned}\operatorname{depth}_{d,J}(M) &= \min\{i \in \mathbb{N}_0 \mid H^i_{d,J}(M) \neq 0\}\\ &= \min\{\operatorname{depth}(\mathfrak{p}, M) \mid \mathfrak{p} \in W(d,J)\}\\ &= \min\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in W(d,J)\}.\end{aligned}$$

Proof. Let $t := \min\{i \in \mathbb{N}_0 \mid H^i_{d,J}(M) \neq 0\}$. By Definition 4.1, there exists $\mathfrak{b} \in \widetilde{W}(d,J)$ such that $\operatorname{depth}_{d,J}(M) = \operatorname{depth}(\mathfrak{b}, M)$. Since $\mathrm{V}(\mathfrak{b}) \subseteq W(d,J)$, by [5], Proposition 1.2.10, we have

$$\operatorname{depth}(\mathfrak{b}, M) = \min\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{b})\} \geqslant \min\{\operatorname{depth}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in W(d,J)\}.$$

Then by Theorem 4.3, $\operatorname{depth}_{d,J}(M) \geqslant t$. Now, if $t < \operatorname{depth}_{d,J}(M)$, then $H^t_{\mathfrak{a}}(M) = 0$ for all $\mathfrak{a} \in \widetilde{W}(d,J)$. Thus, by Theorem 3.12, we get $H^t_{d,J}(M) = 0$, which is a contradiction. Then

$$\operatorname{depth}_{d,J}(M) = \min\{i \in \mathbb{N}_0 \mid H^i_{d,J}(M) \neq 0\}.$$

Finally, for the second equality, since

$$\operatorname{depth}(\mathfrak{a}, M) = \min\{\operatorname{depth}(\mathfrak{p}, M) \mid p \in \mathrm{V}(\mathfrak{a})\},$$

the equality follows from the definition. □

Corollary 4.5. *Let J_1, J_2 be two ideals of R, M be a finitely generated R-module and d be a non-negative integer.*

(i) *If $\widetilde{W}(d,J_1) = \widetilde{W}(d,J_2)$, then $\operatorname{depth}_{d,J_1}(M) = \operatorname{depth}_{d,J_2}(M)$.*
(ii) *If $W(d,J_1) = W(d,J_2)$, then $\operatorname{depth}_{d,J_1}(M) = \operatorname{depth}_{d,J_2}(M)$.*
(iii) *If $n, m \in \mathbb{N}$ such that ${J_1}^n \subseteq {J_2}^m$, then $\operatorname{depth}_{d,J_2}(M) \leqslant \operatorname{depth}_{d,J_1}(M)$.*
(iv) *If $\sqrt{J_1} = \sqrt{J_2}$, then $\operatorname{depth}_{d,J_1}(M) = \operatorname{depth}_{d,J_2}(M)$. In particular, $\operatorname{depth}_{d,J}(M) = \operatorname{depth}_{d,\sqrt{J}}(M)$.*
(v) $\operatorname{depth}_{d,J_1J_2}(M) = \operatorname{depth}_{d,J_1\cap J_2}(M)$.

Proof. (i) Let $n = \operatorname{depth}_{d,J_1}(M)$ and $m = \operatorname{depth}_{d,J_2}(M)$. Therefore $\mathfrak{b} \in W(d,J_1)$ such that $n = \operatorname{depth}(\mathfrak{b}, M)$. We have

$$\begin{aligned}n &= \operatorname{depth}(\mathfrak{b}, M)\\ &\geqslant \min\{\operatorname{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \widetilde{W}(d,J_1)\}\\ &= \min\{\operatorname{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \widetilde{W}(d,J_2)\}\\ &= \operatorname{depth}_{d,J_2}(M) = m.\end{aligned}$$

Then $n \leqslant m$. Similarly, we have $m \leqslant n$ and so $n = m$.

(ii) Using Corollary 4.4 and similarly to the method used in the proof of statement (i), it is clear.

(iii) Let ${J_1}^n \subseteq {J_2}^m$ for $n, m \in \mathbb{N}$. We show that $W(d, J_1) \subseteq W(d, J_2)$. Let $\mathfrak{p} \in W(d, J_1)$. Hence, there exists $I \in \Sigma$ such that $I \subseteq \mathfrak{p} + J_1$ and so

$$I^n \subseteq (\mathfrak{p} + J_1)^n \subseteq \mathfrak{p} + {J_1}^n \subseteq \mathfrak{p} + {J_2}^m \subseteq \mathfrak{p} + J_2.$$

Since $I^n \in \Sigma$, then $\mathfrak{p} \in W(d, J_2)$. Hence $W(d, J_1) \subseteq W(d, J_2)$ and so by Corollary 4.4, we have

$$\begin{aligned}\operatorname{depth}_{d,J_2}(M) &= \min\{\operatorname{depth}(\mathfrak{p}, M) \mid \mathfrak{p} \in W(d, J_2)\}\\ &\leqslant \min\{\operatorname{depth}(\mathfrak{q}, M) \mid \mathfrak{q} \in W(d, J_1)\}\\ &= \operatorname{depth}_{d,J_1}(M).\end{aligned}$$

(iv) By Proposition 3.2 (iv) and part (ii), we have

$$\operatorname{depth}_{d,J_1}(M) = \operatorname{depth}_{d,\sqrt{J_1}}(M) = \operatorname{depth}_{d,\sqrt{J_2}}(M) = \operatorname{depth}_{d,J_2}(M).$$

The second part is the result of $\sqrt{J} = \sqrt{\sqrt{J}}$ and the first part.

(v) The assertion follows from Proposition 3.2 (iii) and part (ii). $\square$

Corollary 4.6. *For a finitely generated R-module M, and all $\mathfrak{a} \in \widetilde{W}(d, J)$, we have*

$$\operatorname{depth}_{d,J}(M) \leqslant \operatorname{depth}_{\mathfrak{a},J}(M) \leqslant \operatorname{depth}(\mathfrak{a}, M).$$

In particular, $\operatorname{depth}_{d,J}(M) \leqslant \operatorname{depth}_d(M)$.

Proof. It is easy to see that $\operatorname{depth}_{\mathfrak{a},J}(M) \leqslant \operatorname{depth}(\mathfrak{a}, M)$. Now since $\mathfrak{a} \in \widetilde{W}(d, J)$, then $\widetilde{W}(\mathfrak{a}, J) \subseteq \widetilde{W}(d, J)$ and so $\operatorname{depth}_{d,J}(M) \leqslant \operatorname{depth}_{\mathfrak{a},J}(M)$.

For the second part, since $\Sigma \subseteq \widetilde{W}(d, J)$, the inequality follows from the definition. $\square$

Proposition 4.7. *Let $0 \to U \to M \to N \to 0$ be an exact sequence of finitely generated R-modules. Let $r := \operatorname{depth}_{d,J}(U)$, $t := \operatorname{depth}_{d,J}(M)$, and $s := \operatorname{depth}_{d,J}(N)$. Then:*

(i) $t \geqslant \min\{r, s\}$,

(ii) $r \geqslant \min\{t, s + 1\}$,

(iii) $s \geqslant \min\{r - 1, t\}$,

(iv) *one of the following equalities holds:*

$$t = r; \quad t = s; \quad t = r = s; \quad s = r - 1.$$

P r o o f. For (i), (ii) and (iii), apply Definition 4.1 and Theorem 4.3. To prove (iv), suppose that none of the equalities holds. Then only one of the following cases must occur:

$$r < s < t; \quad s < r < t; \quad t < r < s;$$
$$t < s < r; \quad s < t < r; \quad r < t < s.$$

Let $r < s < t$. Then $s + 1 \leqslant t$, and by (ii), $s + 1 \leqslant r < s$, which is a contradiction.

If $s < r < t$, then $s \leqslant r - 1 < t$. Thus by (iii), $r - 1 \leqslant s$. Hence $s = r - 1$, which is a contradiction.

If $t < r < s$: Thus by (i), $r < t$ and so $t < t$, which is a contradiction. The same method can be applied to the other cases. $\square$

Theorem 4.8. *Let M be a finitely generated R-module and $t := \operatorname{depth}_{d,J}(M)$. Then:*

(i) *$H_d^t(M) \neq 0$ if and only if $t = \operatorname{depth}_d(M)$,*
(ii) *if $H_d^t(M) \neq 0$, then $\operatorname{depth}_d(M) \leqslant \operatorname{depth}(\mathfrak{a}, M)$ for all $\mathfrak{a} \in \widetilde{W}(d, J)$.*

P r o o f. (i) First, assume that $H_d^t(M) \neq 0$. Let $\operatorname{depth}_d(M) > t$. Hence, by [2], Vanishing Theorem, $H_d^t(M) = 0$, which is a contradiction. Thus $\operatorname{depth}_d(M) \leqslant t$. Let $\operatorname{depth}_d(M) \leqslant t - 1$. Since $\Sigma \subseteq \widetilde{W}(d, J)$, then

$$\begin{aligned} t = \operatorname{depth}_{d,J}(M) &= \min\{\operatorname{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \widetilde{W}(d, J)\} \\ &\leqslant \min\{\operatorname{depth}(\mathfrak{a}, M) \mid \mathfrak{a} \in \Sigma\} \\ &= \operatorname{depth}_d(M). \end{aligned}$$

Hence, in this case, $t \leqslant t - 1$, which is again a contradiction. Thus $\operatorname{depth}_d(M) = t$.

The converse part is clear by [2], Vanishing Theorem.

(ii) Using part (i), the result is obtained. $\square$

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