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KILLING REEB VECTOR FIELDS ON SOME ALMOST CONTACT MANIFOLDS

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Abstract. We discuss the geometric impact of a Killing vector field on certain almost contact manifolds with constant sectional curvatures. We prove that the additional structures given in A. De Nicola, G. Dileo, I. Yudin (2018) can be naturally constructed in the case of nearly Sasakian and nearly cosymplectic manifolds with constant sectional curvatures.

Keywords: nearly Sasakian manifold; nearly cosymplectic manifold; Killing Reeb vector field

MSC 2020: 53C15, 53C25

1. Introduction

A Killing vector field on a smooth manifold is a vector field for which the Lie derivative of the metric tensor vanishes, i.e., a vector field that generates a 1-parameter group of isometries (see [8] and [12] for more details and references therein). Killing vector fields plays an important role in the geometry and topology of smooth manifolds. In [1], for instance, the authors studied nontrivial Killing vector fields of constant length and their flows on complete smooth Riemannian manifolds. They obtained some curvature constraints on the Riemannian manifolds admitting nontrivial Killing fields of constant length. By nontrivial Killing vector field we mean a nonparallel Killing vector field. This does not allow a Riemannian manifold to have negative Ricci curvature, and if a Riemannian manifold has positive sectional curvature, then its fundamental group contains a cyclic subgroup with a constant index that depends only on the dimension of the underlying manifold (see [13] for

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more details). As Killing vector fields are incompressible, they have applications in physics (see [14]).

In this paper, we investigate the geometric impact of the presence of a Killing vector field on certain almost contact manifolds. We focus on the Killing vector fields that appear naturally on nearly Sasakian and nearly cosymplectic manifolds. Several results closely related to a Killing vector field are presented in [2]. For instance, it is proven that, on all K-contact manifolds, there is a unit Killing vector field called the Reeb vector field. In particular, the characteristic vector field ξ is Killing on some almost contact manifolds such as Sasakian, almost cosymplectic, cosymplectic, nearly Sasakian, and nearly cosymplectic manifolds.

The present paper builds on [6] and [7], in which the authors proved that there are additional structures on the 5-dimensional sphere $S^5(2)$ of constant sectional curvature 2 endowed with a nearly Sasakian structure, as well as on a unit sphere endowed with a nearly cosymplectic structure.

The paper is organized as follows. In Section 2, we present some basic background on almost contact metric structures and Killing vector fields, as well as related properties. Section 3 is devoted to nearly Sasakian manifolds of constant sectional curvature c. We prove that there are no nearly Sasakian manifolds of constant sectional curvature $c = 0$. When $c \neq 0$, we show that a nearly Sasakian manifold of constant sectional curvature c is either Sasakian $c = 1$ or it is of dimension 5 and naturally admits two additional structures homothetic to Sasakian and nearly cosymplectic structures, carrying a Sasaki-Einstein structure. Finally, in Section 4, we discuss nearly cosymplectic manifolds of constant sectional curvature κ. We prove that such manifolds are either locally isometric to the Riemannian product of a real line and a flat nearly Kähler manifold immersed as a submanifold, or they naturally admit two additional structures homothetic to Sasakian and nearly cosymplectic structures. If $\kappa \neq 0$, then we show that $\kappa > 0$, the dimension of the underlying manifold is 5, and there exists a $(1,1)$-tensor field with the spectrum of its square given by $0, -\kappa$, $-\kappa$, $-\kappa$, $-\kappa$, where 0 is a simple eigenvalue and $-\kappa$ is an eigenvalue of multiplicity 4. Additionally, there are integrable distributions with totally geodesic leaves, some of which are endowed with a nearly Kähler structure. A nearly cosymplectic manifold also carries a Sasaki-Einstein structure.

2. Preliminaries

Let M be a $(2n+1)$-dimensional manifold endowed with an almost contact structure (ϕ, ξ, η), i.e., ϕ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying [2]:

$$\phi^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{2.1}$$

It follows that $\phi\xi = 0$, $\eta \circ \phi = 0$ and $\operatorname{rank}(\phi) = 2n$. Then (ϕ, ξ, η, g) is called an almost contact metric structure on M if (ϕ, ξ, η) is an almost contact structure on M and g is a Riemannian metric on M such that [2]

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector field X, Y on M. It follows that the $(1,1)$-tensor field ϕ is skew-symmetric and $\eta(X) = g(\xi, X)$.

Let (M, ϕ, ξ, η, g) be an almost contact manifold. If the characteristic vector field ξ is Killing, then for any vector fields X and Y on M,

$$(L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0 \tag{2.3}$$

and the differential of η gives

$$2\,\mathrm{d}\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) = 2g(\nabla_X \xi, Y). \tag{2.4}$$

This means that there exists a $(1,1)$-tensor field ψ on M such that

$$\mathrm{d}\eta(X, Y) = g(\psi X, Y). \tag{2.5}$$

This means that

$$\nabla_X \xi = \psi X \quad \text{for all } X \in \Gamma(TM). \tag{2.6}$$

Let R be the curvature tensor of the almost contact manifold M. Using (2.6), we have

$$R(X, Y)\xi = (\nabla_X \psi)Y - (\nabla_Y \psi)X. \tag{2.7}$$

Note that the operator $\nabla\psi$ is skew-symmetric, i.e.,

$$g((\nabla_X \psi)Y, Z) = -g(Y, (\nabla_X \psi)Z). \tag{2.8}$$

Define the smooth 2-form of M by

$$\Omega(X, Y) = g(\psi X, Y) \quad \text{for all } X, Y \in \Gamma(TM). \tag{2.9}$$

Then, using (2.5), Ω is closed, and using (2.8), we have for any X, Y, $Z \in \Gamma(TM)$,

$$\begin{aligned} 0 &= X(\Omega(Y, Z)) + Y(\Omega(Z, X)) + Z(\Omega(X, Y)) - \Omega([X, Y], Z) \\ &\quad - \Omega([Z, X], Y) - \Omega([Y, Z], X) \\ &= g((\nabla_Z \psi)X, Y) - g((\nabla_X \psi)Z, Y) + g((\nabla_Y \psi)Z, X) \\ &= g((\nabla_Z \psi)X, Y) + g((\nabla_X \psi)Y, Z) + g((\nabla_Y \psi)Z, X). \end{aligned} \tag{2.10}$$

Therefore, we have the following.

Lemma 2.1. *Let (M, ϕ, ξ, η, g) be an almost contact manifold. If the characteristic vector field ξ is Killing, then*

$$R(X, \xi)Y = (\nabla_X \psi)Y \quad \textit{for all } X \in \Gamma(TM). \tag{2.11}$$

Proof. From (2.10) and using the skew-symmetry of $\nabla\psi$, for any X, Y, $Z \in \Gamma(TM)$ we have

$$\begin{aligned} g(R(X,\xi)Y, Z) &= -g(R(Y,Z)\xi, X) \\ &= -g((\nabla_Z\psi)Y, X) - g((\nabla_X\psi)Z, Y) + g((\nabla_Z\psi)Y, X)) \\ &= g((\nabla_X\psi)Y, Z), \end{aligned}$$

which completes the proof. □

3. Nearly Sasakian manifolds

The manifold M is a *nearly Sasakian manifold* if it admits an almost contact metric structure (ϕ, ξ, η, g) that satisfies (see [3], [4] or [10] for more details)

$$(\nabla_X\phi)Y + (\nabla_Y\phi)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X, \tag{3.1}$$

where ∇ is the Levi-Civita connection for the Riemannian metric g. It is easy to see that the characteristic vector field ξ is Killing, and hence there exists a skew-symmetric tensor field ϕ_1 on M satisfying

$$\nabla_X\xi = \phi_1 X \quad \text{for all } X \in \Gamma(TM). \tag{3.2}$$

Assume that the nearly Sasakian manifold M is of constant sectional curvature c. Then the curvature tensor R of M satisfies equation [15]:

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\} \tag{3.3}$$

for any vector fields X, Y and Z on M.

By equation (2.11), we have

$$(\nabla_X\phi_1)Y = R(X,\xi)Y = c\{\eta(Y)X - g(X,Y)\xi\} \quad \text{for all } X, Y \in \Gamma(TM). \tag{3.4}$$

Taking $Y=\xi$, we have

$$\phi_1^2 X = c\{-X+\eta(X)\xi\}, \tag{3.5}$$

as $\phi_1\xi = \nabla_\xi\xi = 0$, since ξ is a unit vector field. Note that if we consider $X\in\Gamma(TM)$ such that $\eta(X)=0$, then using relation (3.5), one has $g(\phi_1X,\phi_1X)=cg(X,X)$, that is, $c\geqslant 0$.

Note that the operator $\nabla_\xi\phi$ is skew-symmetric, i.e., for any vector fields X and Y on M,

$$g((\nabla_\xi\phi)X,Y) = -g(X,(\nabla_\xi\phi)Y). \tag{3.6}$$

In [6], it is showed that every nearly Sasakian manifold of dimension $2n+1>5$ is Sasakian, which cannot be flat. The case of 5-dimensional nearly Sasakian manifolds is treated in [11], where it is showed that in the non-Sasakian case, the manifold is Einstein with scalar curvature strictly greater than 20, which obviously means that the manifold cannot be flat. It means that the constant c is nonzero, i.e., $c\neq 0$. This implies that $(M(c),\phi,\xi,\eta,g)$ is homothetic to a Sasakian manifold. This new Sasakian structure (ϕ_1',ξ',η',g') on M is given by

$$\phi_1' = \frac{1}{\sqrt{c}}\phi_1,\quad \xi' = \frac{1}{\sqrt{c}}\xi,\quad \eta' = \sqrt{c}\eta,\quad g' = cg,\quad c\neq 0. \tag{3.7}$$

Indeed,

$$2\,\mathrm{d}\eta(X,Y) = g(\nabla_X\xi,Y) - g(X,\nabla_Y\xi) = 2g(\phi_1X,Y)\quad \text{for all } X,Y\in\Gamma(TM), \tag{3.8}$$

and we have $\mathrm{d}\eta'(X,Y) = g'(\phi_1'X,Y)$ implying that (ϕ_1',ξ',η',g') is a contact metric structure. In virtue of (2.7), the curvature tensor R_1' associated to the structure given in (3.7) is

$$R_1'(X,Y)\xi' = \frac{1}{\sqrt{c}}R_1(X,Y)\xi = \eta'(Y)X - \eta'(X)Y,$$

which proves by Proposition 7.6 in [2], that the structure (ϕ_1',ξ',η',g') is Sasakian. Here R_1 is the curvature tensor associated to the almost structure (ϕ_1,ξ,η,g).

Define a tensor field ϕ_2 of type (1,1) on the nearly Sasakian manifold $(M(c),\phi,\xi,\eta,g)$ by

$$\phi_2X = (\nabla_\xi\phi)X\quad \text{for all } X\in\Gamma(TM). \tag{3.9}$$

By (3.6), the operator ϕ_2 is skew-symmetric, and using (3.1), it is easy to see that $\phi_2\xi = 0$.

Next, we investigate the relationships between the three tensors ϕ, ϕ_1 and ϕ_2. Putting $Y = \xi$ into (3.1), we obtain

$$\phi\phi_1 X = X - \eta(X)\xi + \phi_2 X \quad \text{for all } X \in \Gamma(TM). \tag{3.10}$$

Replacing X by ϕX in this equation and then applying ϕ, one has

$$\phi\phi_1\phi X = \phi X + \phi_2\phi X, \tag{3.11}$$

which implies that

$$\phi_1\phi X = X - \eta(X)\xi - \phi\phi_2\phi X. \tag{3.12}$$

Note that

$$\begin{aligned} \phi\phi_2\phi X &= \phi(\nabla_\xi\phi)\phi X = \phi(\nabla_\xi\phi^2 X) - \phi^2(\nabla_\xi\phi X) \\ &= -\phi\nabla_\xi X + \nabla_\xi\phi X - \eta(\nabla_\xi\phi X)\xi = \phi_2 X, \end{aligned} \tag{3.13}$$

that is,

$$\phi_2\phi X = -\phi\phi_2 X. \tag{3.14}$$

This means that the operators ϕ and ϕ_2 anti-commute. Using this in (3.12), we have

$$\phi_1\phi X = X - \eta(X)\xi - \phi_2 X. \tag{3.15}$$

This relation together with (3.10) gives

$$\phi\phi_1 X = 2\{X - \eta(X)\xi\} - \phi_1\phi X. \tag{3.16}$$

Now, from (3.10) and (3.15), one has

$$\phi_2 X = \frac{1}{2}\{\phi\phi_1 X - \phi_1\phi X\} \quad \text{and} \quad X - \eta(X)\xi = \frac{1}{2}\{\phi\phi_1 X + \phi_1\phi X\}. \tag{3.17}$$

Also, using (3.10) and (3.14), we have

$$\phi_1 X = \phi\phi_1\phi X - 2\phi X. \tag{3.18}$$

Therefore, we have the following.

Lemma 3.1. *The operators ϕ_1 and ϕ_2 on the nearly Sasakian space form $(M(c), \phi, \xi, \eta, g)$ satisfy, for any vector field X on M,*

$$\phi_1 X = \phi\phi_1\phi X - 2\phi X, \tag{3.19}$$

$$X - \eta(X)\xi = \frac{1}{2}\{\phi\phi_1 X + \phi_1\phi X\}, \tag{3.20}$$

$$\phi_2 X = \frac{1}{2}\{\phi\phi_1 X - \phi_1\phi X\}. \tag{3.21}$$

Using (3.1), we obtain

$$\phi_2^2 X = \{\phi\phi_1\phi - 2\phi\}\phi_1 X - \eta(X)\xi + X. \tag{3.22}$$

By Lemma 3.1 and using (3.5), relation (3.22) becomes

$$\phi_2^2 X = \{\phi\phi_1\phi - 2\phi\}\phi_1 X - \eta(X)\xi + X = (c-1)\{-X + \eta(X)\xi\}. \tag{3.23}$$

Also, as ϕ_2 is skew-symmetric, we have for any vector fields X and Y on $M(c)$,

$$\begin{aligned} g(\phi_2 X, \phi_2 Y) &= -g(X, \phi_2^2 Y) = -g(X, (c-1)\{-Y + \eta(Y)\xi\}) \\ &= (c-1)\{g(X,Y) - \eta(X)\eta(Y)\}. \end{aligned} \tag{3.24}$$

Now, using (3.1), (3.2), (3.16) and Lemma 3.1 and for any vector fields X and Y on $M(c)$, we have

$$(\nabla_X \phi_2)Y = -(\nabla_X \phi_1)\phi Y - \phi_1(\nabla_X \phi)Y - g(\phi_1 X, Y)\xi - \eta(Y)\phi_1 X. \tag{3.25}$$

By (3.4), we have

$$(\nabla_X \phi_2)Y = cg(X, \phi Y)\xi - \phi_1(\nabla_X \phi)Y - g(\phi_1 X, Y)\xi - \eta(Y)\phi_1 X. \tag{3.26}$$

This leads to

$$(\nabla_X \phi_2)Y + (\nabla_Y \phi_2)X = 0. \tag{3.27}$$

If the sectional curvature $c = 1$, then by (3.5) and (3.23), we have

$$\phi_1^2 X = -X + \eta(X)\xi \quad \text{and} \quad \phi_2 X = 0 \quad \text{for all } X \in \Gamma(TM). \tag{3.28}$$

From identities (3.17) we have

$$\phi_1\phi = \phi\phi_1 = \mathbb{I} - \eta \otimes \xi. \tag{3.29}$$

This implies that $\phi_1 = -\phi\phi_1\phi$ and together with (3.18) gives

$$\phi_1 = -\phi. \tag{3.30}$$

This means that the structure (ϕ, ξ, η, g) is Sasakian.

Now, assume that the sectional curvature $c \neq 1$. From (3.23) and (3.27), the nearly Sasakian manifold $(M(c), \phi, \xi, \eta, g)$ is homothetic to a nearly cosympletic manifold. The new nearly cosymplectic structure $(\phi_2'', \xi'', \eta'', g'')$ on M is given by

$$\phi_2'' = \frac{1}{\sqrt{c-1}}\phi_2, \quad \xi'' = \frac{1}{\sqrt{c-1}}\xi, \quad \eta'' = \sqrt{c-1}\eta, \quad g'' = (c-1)g. \tag{3.31}$$

In the same case and using the notation $R\xi$ to denote the $(1,2)$-tensor on M given by $(R\xi)(X,Y) = R(X,Y)\xi$, a direct calculation of $R\xi$ gives

$$R\xi = \mathbb{I} \otimes \eta - \eta \otimes \mathbb{I} + \eta \otimes H^2 - H^2 \otimes \eta, \tag{3.32}$$

where

$$H = \phi(\nabla_\xi \phi). \tag{3.33}$$

Comparing this relation with (3.3), we have

$$\{(c-1)\mathbb{I} + H^2\} \otimes \eta = \eta \otimes \{(c-1)\mathbb{I} + H^2\}. \tag{3.34}$$

This implies that

$$H^2 = -(c-1)\{\mathbb{I} - \eta \otimes \xi\}.$$

This means that M is non-Sasakian satisfying

$$H^2 = -\lambda^2\{\mathbb{I} - \eta \otimes \xi\} \tag{3.35}$$

with $c = 1 + \lambda^2$ and $\lambda \neq 0$. The operator in (3.35) is similar to the symmetric operator H^2 given in [11] in which the author proved that it is equivalent to the dimension of M, which is 5. Therefore, by [11], Theorem 4.1, we have the following.

Theorem 3.1. *Let $(M(c), \phi, \xi, \eta, g)$ be a nearly Sasakian manifold of constant sectional curvature c. We have:*

1. *If $c = 1$, the structure (ϕ, ξ, η, g) is Sasakian.*
2. *If $c \neq 1$, then $(M(c), \phi, \xi, \eta, g)$ is a 5-dimensional proper nearly Sasakian manifold in which there exist two structures (ϕ_i, ξ, η, g), $i = 1, 2$, associated to the nearly Sasakian (ϕ, ξ, η, g) on $M(c)$ such that $(M(c), \phi_1, \xi, \eta, g)$ is homothetic to a Sasakian manifold and $(M(c), \phi_2, \xi, \eta, g)$ is homothetic to a nearly cosymplectic manifold.*

As an example, we have the following. In [2], the authors considered a nearly Sasakian structure on the 5-dimensional sphere $S^5(2)$ of constant sectional curvature 2. This was introduced as a totally umbilical hypersurface of S^6 with no Sasakian structure. Being of constant sectional curvature 2, its curvature tensor R satisfies

$$R(X,Y)\xi = 2\{\eta(Y)X - \eta(X)Y\} \quad \text{for all } X,Y \in \Gamma(TS^5(2)).$$

From (3.5), (3.23) and (3.24) we have the following, which is similar to Theorem 3.1 in [7].

Corollary 3.1. *There exist two structures (ϕ_i,ξ,η,g), $i=1,2$, associated to the nearly Sasakian (ϕ,ξ,η,g) on $S^5(2)$ of constant sectional curvature 2 such that $(S^5(2),\phi_1,\xi,\eta,g)$ is homothetic to a Sasakian manifold and $(S^5(2),\phi_2,\xi,\eta,g)$ is a nearly cosymplectic manifold.*

From (3.7), (3.33) and (3.35) we have

$$\phi_1^2 = \phi^2 + H^2 = (1+\lambda^2)\{-\mathbb{I} + \eta\otimes\xi\}, \tag{3.36}$$

and the structures (ϕ_1',ξ',η',g') in (3.7) and $(\phi_1'',\xi'',\eta'',g'')$ in (3.31) become respectively

$$\phi_1' = \frac{1}{\sqrt{1+\lambda^2}}(-\phi - H),\ \xi' = \frac{1}{\sqrt{1+\lambda^2}}\xi,\ \eta' = \sqrt{1+\lambda^2}\eta,\ g' = (1+\lambda^2)g, \tag{3.37}$$

$$\phi_2'' = \frac{1}{|\lambda|}\phi_2,\ \xi'' = \frac{1}{|\lambda|}\xi,\ \eta'' = |\lambda|\eta,\ g'' = \lambda^2 g, \tag{3.38}$$

which define Sasaki-Einstein structure associated with any 5-dimensional nearly Sasakian manifold $(M(c),\phi,\xi,\eta,g)$ of constant sectional curvature $1+\lambda^2$ with $\lambda \neq 0$ as $c \neq \{0,1\}$ (see [5] for more details). Therefore, we have the following result.

Theorem 3.2. *Each structure (ϕ_i,ξ,η,g), $i=1,2$, and (ϕ,ξ,η,g) on a 5-dimensional nearly Sasakian manifold $M(c)$ with $c\neq\{0,1\}$ carries a Sasaki-Einstein structure.*

As a converse to Theorem 3.2, each Sasaki-Einstein 5-manifold carries a 1-parameter family of nearly Sasakian structures (see [5] for more details and reference therein). Through this, new examples of nearly Sasakian manifolds can be provided, and in particular, each Sasaki-Einstein metric of the infinite family of Sasakian structures on $S^2\times S^3$ discussed by Martelli and Sparks in [9] gives examples of nearly Sasakian structures.

4. Nearly cosymplectic manifolds

Let $(M, \varphi_1, \xi, \eta, g)$ be an almost contact manifold. Then M is said to be nearly cosymplectic manifold if (see [6])

$$(\nabla_X \varphi_1)Y + (\nabla_Y \varphi_1)X = 0 \tag{4.1}$$

for any vector fields X and Y on M. Also, it is known that the Reeb vector field ξ in a nearly cosymplectic manifold M is Killing [2]. Therefore, there exits a skew-symmetric tensor φ_2 of type $(1,1)$ on the nearly cosymplectic manifold $(M, \varphi_1, \xi, \eta, g)$ such that

$$\mathrm{d}\eta(X,Y) = g(\varphi_2 X, Y) \quad \text{for all } X, Y \in \Gamma(TM),$$

and then we obtain

$$\nabla_X \xi = \varphi_2 X \quad \text{for all } X \in \Gamma(TM). \tag{4.2}$$

The vector field ξ being Killing and φ_2 skew-symmetric, (4.2) gives $\varphi_2\xi = \nabla_\xi \xi = 0$. Now, if the smooth 2-form $\mathrm{d}\eta(X,Y) = g(\varphi_2 X, Y)$ is closed and if the nearly cosymplectic manifold M is of constant sectional curvature κ, then its curvature tensor R is given by

$$R(X,Y)Z = \kappa\{g(Y,Z)X - g(X,Z)Y\} \tag{4.3}$$

for any vector fields X, Y and Z on M.

We obtain

$$(\nabla_X \varphi_2)Y = R(X,\xi)Y = \kappa\{\eta(Y)X - g(X,Y)\xi\} \quad \text{for all } X, Y \in \Gamma(TM). \tag{4.4}$$

If $\kappa = 0$, then $\nabla\varphi_2 = 0$. This implies that $\nabla\xi = 0$. Then, for any vector fields X and Y on M,

$$2\,\mathrm{d}\eta(X,Y) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) = 0. \tag{4.5}$$

Thus, the distribution $D = \ker\eta$ is integrable, and for any X, $Y \in \Gamma(D)$,

$$g(\nabla_X Y, \xi) = X(\eta(Y)) - g(Y, \nabla_X \xi) = 0. \tag{4.6}$$

Let us denote by M' an integral manifold of D. Equation (4.6) means that M' is totally geodesic hypersurfaces of $(M, \varphi_1, \xi, \eta, g)$. Therefore, $(M, \varphi_1, \xi, \eta, g)$ is locally isometric to the Riemannian product $\mathbb{R} \times M'$. The almost contact metric structure induces on M' an almost Hermitian structure which is nearly Kähler. Note that if we denote byR' the curvature tensor of M', then for any vector fields X, Y and Z on M',

$$R(X,Y)Z = R'(X,Y)Z + A_{\sigma'(X,Z)}Z - A_{\sigma'(Y,Z)}X + (\nabla_X \sigma')(Y,Z) - (\nabla_Y \sigma')(X,Z), \tag{4.7}$$

where σ' and A_V are the second fundamental form of M' and shape operator in the direction of V, respectively. Since M' is totally geodesic, i.e., $\sigma' = 0$ and $\kappa = 0$, then $R' = 0$. This means that the leaf M' is flat.

Now, assume that the sectional curvature $\kappa \neq 0$. From (4.4) and taking $Y = \xi$, we have

$$\varphi_2^2 X = \kappa\{-X + \eta(X)\xi\} \quad \text{for all } X \in \Gamma(TM). \tag{4.8}$$

This leads to $g(\varphi_2 X, \varphi_2 Y) = \kappa\{g(X,Y) - \eta(X)\eta(Y)\}$ and $\kappa > 0$. Therefore, the structure $(\varphi_2, \xi, \eta, g)$ is homothetic to an almost contact metric structure on $(M(\kappa), \varphi_1, \xi, \eta, g)$, and (4.2) and (4.8) show that $(\varphi_2, \xi, \eta, g)$ is homothetic to a Sasakian structure.

Let now define another operator φ_3 on the nearly cosymplectic manifold $(M(\kappa), \varphi_1, \xi, \eta, g)$ by

$$\varphi_3 X = (\nabla_\xi \varphi_1)X \quad \text{for all } X \in \Gamma(TM), \tag{4.9}$$

which by (4.1) shows that φ_3 is skew-symmetric tensor of type $(1,1)$ and satisfies $\varphi_3 \xi = 0$. Likewise, by (4.1) and (4.2), we get

$$\varphi_3 X = (\nabla_\xi \varphi_1)X = -(\nabla_X \varphi_1)\xi = \varphi_1 \nabla_X \xi = \varphi_1 \varphi_2 X \quad \text{for all } X \in \Gamma(TM), \tag{4.10}$$

which gives

$$\varphi_3 X = (\nabla_\xi \varphi_1)X = -(\nabla_X \varphi_1)\xi = \varphi_1 \nabla_X \xi = \varphi_1 \varphi_2 X \quad \text{for all } X \in \Gamma(TM), \tag{4.11}$$

which gives

$$\varphi_2 X = -\varphi_1 \varphi_3 X \quad \text{for all } X \in \Gamma(TM). \tag{4.12}$$

Replacing X by $\varphi_1 X$ in (4.12) one has

$$\varphi_2 \varphi_1 X = -\varphi_1 \varphi_3 \varphi_1 X. \tag{4.13}$$

Also, we have

$$\varphi_1\varphi_3\varphi_1 X = \varphi_3 X, \tag{4.14}$$

which by virtue of (4.13) gives

$$\varphi_2\varphi_1 X = -\varphi_3 X. \tag{4.15}$$

Thus, equations (4.10) and (4.15) lead to

$$\varphi_3 = \varphi_1\varphi_2 = -\varphi_2\varphi_1. \tag{4.16}$$

This induces

$$\varphi_3^2 X = \varphi_1\varphi_2\varphi_1\varphi_2 X = \kappa\varphi_1^2 X = \kappa\{-X + \eta(X)\xi\} \tag{4.17}$$

and

$$g(\varphi_3 X, \varphi_3 Y) = \kappa\{g(X,Y) - \eta(X)\eta(Y)\} \quad \text{for all } X, Y \in \Gamma(TM). \tag{4.18}$$

Therefore, the quadruplet $(\varphi_3, \xi, \eta, g)$ is also homothetic to an almost contact metric structure on $(M(\kappa), \varphi_1, \xi, \eta, g)$ and the three structures satisfy the following:

$$\varphi_1\varphi_2 = -\varphi_2\varphi_1 = \varphi_3, \quad \varphi_2\varphi_3 = -\varphi_3\varphi_2 = \kappa\varphi_1, \quad \varphi_3\varphi_1 = -\varphi_1\varphi_3 = \varphi_2. \tag{4.19}$$

For any X, $Y \in \Gamma(TM)$, one has

$$(\nabla_X\varphi_3)Y = \kappa g(X, \varphi_1 Y)\xi - \varphi_2(\nabla_X\varphi_1)Y, \tag{4.20}$$

which, using (4.1), gives

$$(\nabla_X\varphi_3)Y + (\nabla_Y\varphi_3)X = 0. \tag{4.21}$$

This means that the structure $(\varphi_3, \xi, \eta, g)$ is homothetic to nearly cosymplectic structure. This new nearly cosymplectic $(\varphi_3', \xi', \eta', g')$ on $M(\kappa)$ with $\kappa > 0$ is given by

$$\varphi_3' = \frac{1}{\sqrt{\kappa}}\varphi_3, \quad \xi' = \frac{1}{\sqrt{\kappa}}\xi, \quad \eta' = \sqrt{\kappa}\eta, \quad g' = \kappa g.$$

The relations in (4.19) can be rewritten as $\varphi_1\varphi_2 = -\varphi_2\varphi_1 = \sqrt{\kappa}\varphi_3'$, $\varphi_2\varphi_3' = -\varphi_3'\varphi_2 = \sqrt{\kappa}\varphi_1$, $\varphi_3'\varphi_1 = -\varphi_1\varphi_3' = \kappa^{-1/2}\varphi_2$. This means that the nearly cosymplectic structures $(\varphi_1, \xi, \eta, g)$ and $(\varphi_3', \xi', \eta', g')$ are independent.

Therefore, we have the following result.

Theorem 4.1. *Let $(M(\kappa), \varphi_1, \xi, \eta, g)$ be a nearly cosymplectic manifold with a constant sectional curvature κ. Then:*

(1) *If $\kappa = 0$, then M is locally isometric to the Riemannian product $\mathbb{R} \times M'$, where M' is a flat nearly Kähler manifold immersed as a submanifold of M.*
(2) *If $\kappa \neq 0$, then there exist three structures $(\varphi_i, \xi, \eta, g)$, $i = 1, 2, 3$ on the almost contact manifold M of nonzero constant sectional curvature κ satisfying*

$$\varphi_1\varphi_2 = -\varphi_2\varphi_1 = \varphi_3, \quad \varphi_2\varphi_3 = -\varphi_3\varphi_2 = \kappa\varphi_1, \quad \varphi_3\varphi_1 = -\varphi_1\varphi_3 = \varphi_2, \tag{4.22}$$

such that $(M(\kappa), \varphi_1, \xi, \eta, g)$ is a nearly cosymplectic manifold, $(M(\kappa), \varphi_2, \xi, \eta, g)$ is homothetic to a Sasakian manifold and $(M(\kappa), \varphi_3, \xi, \eta, g)$ is homothetic to a nearly cosymplectic manifold.

A typical example of this theorem is the unit sphere S^5, which is known to have a nearly cosymplectic structure (see [2] for more details). As shown in [8], there are two additional structures on the unit sphere S^5 endowed with the nearly cosymplectic structure (ψ_1, ξ, η, g), namely, (ψ_2, ξ, η, g) and (ψ_3, ξ, η, g), such that $(S^5, \psi_2, \xi, \eta, g)$ is a Sasakian manifold and $(S^5, \psi_3, \xi, \eta, g)$ is a nearly cosymplectic manifold. The three structures satisfy $\psi_1\psi_2 = -\psi_2\psi_1 = \psi_3$, $\psi_2\psi_3 = -\psi_3\psi_2 = \psi_1$ and $\psi_3\psi_1 = -\psi_1\psi_3 = \psi_2$. These identities are particular cases of (4.22) with $\kappa = 1$, the sectional curvature of the unit sphere S^5, and $\psi_i = \varphi_i$, $i = 1, 2, 3$.

In [4] and [5], the authors define a skew-symmetric tensor h of type $(1, 1)$ which anti-commutes with φ_1 as

$$\nabla_X \xi = hX \quad \text{for all } X \in \Gamma(TM). \tag{4.23}$$

The operator h also satisfies $h\xi = 0$, $\eta \circ h = 0$, $\nabla_\xi \varphi_1 = \varphi_1 h$, and is related to the Lie derivative of φ_1 in the direction of ξ, i.e., $\mathcal{L}_\xi \varphi_1 = 3\varphi_1 h$. Comparing (4.1) and (4.23), also using (4.9) and (4.19), we have

$$h = -\varphi_1\varphi_3. \tag{4.24}$$

On a nearly cosymplectic manifold, the following formulas hold (see [4] and [5] for more details and references therein):

$$g((\nabla_X \varphi_1)Y, hZ) = \eta(Y) g(h^2 X, \varphi_1 Z) - \eta(X) g(h^2 Y, \varphi_1 Z), \tag{4.25}$$

$$(\nabla_X h)Y = g(h^2 X, Y)\xi - \eta(Y) h^2 X. \tag{4.26}$$

Using (4.26), one has

$$(\nabla_X h^2)Y = g(X, h^3 Y)\xi - \eta(Y)h^3 X. \tag{4.27}$$

Using the anti-commutativity of structures φ_1 and φ_3, the square of the operator h gives

$$h^2 = \varphi_1\varphi_3\varphi_1\varphi_3 = -\varphi_1^2\varphi_3^2 = \varphi_3^2 = \kappa\{-\mathbb{I} + \eta \otimes \xi\}. \tag{4.28}$$

Note that $h\xi = 0$, i.e., 0 is an eigenvalue of h^2. The operator h being skew-symmetric, the nonvanishing eigenvalues of h^2 are negative. So, the spectrum of h^2 is of type

$$\mathrm{Spec}(h^2) = \{0, -\kappa, \ldots, -\kappa\}.$$

It is easy to see that the nonzero eigenvalues of h^2 are constant (see Proposition 4.2 in [6] for more details). Since $\kappa > 0$, 0 is a simple eigenvalue. By Theorem 4.4 in [6], the nearly cosymplectic manifold $(M(\kappa), \varphi_1, \xi, \eta, g)$ is of dimension 5 and multiplicity of the eigenvalue $-\kappa$ is 4. Therefore, we have the following result.

Theorem 4.2. *Let $(M(\kappa), \varphi_1, \xi, \eta, g)$ be a nearly cosymplectic manifold with a nonzero constant sectional curvature κ. Then $\kappa > 0$, M is a 5-dimensional manifold and the spectrum of h^2 has the form*

$$\mathrm{Spec}(h^2) = \{0, -\kappa, -\kappa, -\kappa, -\kappa\},$$

where 0 is a simple eigenvalue and $-\kappa$ is an eigenvalue of multiplicity 4.

Moreover, the distributions $D(0)$ and $\mathbb{R}\xi \oplus D(-\kappa)$ are integrable with totally geodesic leaves, and each leaf of $D(0)$ is equipped with a nearly Kähler structure.

Proof. Let us denote by $\mathbb{R}\xi$, $D(0)$ and $D(-\kappa)$ the distribution of dimension 1 generated by ξ, the distributions of the eigenvectors corresponding to eigenvalues 0 and $-\kappa$, respectively.

Applying (4.27), we have $(\nabla_X h^2)Y = 0$ for any X, $Y \in D(0)$, which implies $h^2\nabla_X Y = 0$, i.e, $\nabla_X Y \in D(0)$. Also,

$$g(\nabla_X Y, \xi) = -g(Y, \nabla_X \xi) = -g(Y, hX) = 0.$$

Thus, $D(0)$ is integrable with totally geodesic leaves. Here the leaves of $D(0)$ are φ_1-invariant, so the nearly cosymplectic structure induces a nearly Kähler structure on each integral manifold of $D(0)$.

If X is an eigenvector of h^2 with corresponding eigenvalue $-\kappa$, then from (4.27) we have

$$h^2\nabla_X\xi = -(\nabla_X h^2)\xi = h^3X = -\kappa hX = -\kappa\nabla_X\xi. \tag{4.29}$$

This means that $\nabla_X\xi$ is an eigenvector corresponding to the eigenvalue $-\kappa$. As (4.27) leads to $\nabla_\xi h^2 = 0$, we have

$$h^2\nabla_\xi X = \nabla_\xi h^2 X = -\xi(\kappa)X - \kappa\nabla_\xi X = -\kappa\nabla_\xi X. \tag{4.30}$$

So, $\nabla_\xi X$ is also an eigenvector corresponding to the eigenvalue $-\kappa$. If vector fields X and Y are both eigenvectors with eigenvalue $-\kappa$ and orthogonal to ξ, then from (4.27) one obtains

$$h^2(\nabla_X Y) = \nabla_X h^2 Y - (\nabla_X h^2)Y = -\kappa\nabla_X Y + \kappa g(X, hY)\xi. \tag{4.31}$$

This leads to

$$h^2(\phi^2\nabla_X Y) = \phi^2(h^2\nabla_X Y) = -\kappa\phi^2(\nabla_X Y), \tag{4.32}$$

i.e., $\phi^2\nabla_X Y$ belongs to $D(-\kappa)$, and thus,

$$\nabla_X Y = -\phi^2\nabla_X Y + \eta(\nabla_X Y)\xi \in \mathbb{R}\xi \oplus D(-\kappa),$$

which completes the proof. $\square$

In virtue of Theorem 4.2, we have the following result.

Theorem 4.3. *A nearly cosymplectic manifold $(M, \varphi_1, \xi, \eta, g)$ with constant sectional curvature κ carries a Sasaki-Einstein structure.*

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