ON SOME DIOPHANTINE EQUATIONS INVOLVING FACTORIALS

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Abstract. We study the Diophantine equations $(n!)^k - n^k = (k!)^n - k^n$ and $(n!)^k + n^k = (k!)^n + k^n$, where k and n are positive integers. According to H. Arzel, F. Luca (2017), only the first equation has nontrivial solutions. It is proved also that $(n!)^k - n^k > (k!)^n - k^n$ for $n > k \geqslant 3$. In the paper we prove that the stronger inequality $(n!)^k - n^k(n-1)^k > (k!)^n - k^n$ holds. We propose also an alternative proof of the results of H. Arzel, F. Luca (2017).

Keywords: Diophantine equations; factorial

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1. Introduction

Let us consider the following equations:

$$(n!)^k - n^k = (k!)^n - k^n, \tag{1.1}$$

and

$$(n!)^k + n^k = (k!)^n + k^n. \tag{1.2}$$

This note has been motivated by a result published by Arzel and Luca [1], in which the above equations were solved. It has been shown that except the trivial case $n = k$, the only solutions of (1.1) are the pairs $(n_0, k_0) \in \{(1,2); (2,1)\}$, while equation (1.2) has no nontrivial solutions. Arzel and Luca proved also that if $n > k \geqslant 3$, the left-hand side of equation (1.1) is always grater than the right one.

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2. Main results

Firstly, we provide another brief proof to the fact that the second equation has no solutions. We will use this result to obtain the inequality $((n!)^k - n^k > (k!)^n - k^n)$.

Proposition 2.1. *For all $n > k \geqslant 3$ we have*

$$(n!)^k + n^k > (k!)^n + k^n. \tag{2.1}$$

Proof. From the assumption we know that $n > k$, which implies $n - 1 \geqslant k$. Since the sequence $\{(n!)^{1/n}\}_{n\in\mathbb{N}}$ is strictly increasing, we obtain that

$$\begin{aligned}(n!)^{1/n} &> (k!)^{1/k},\\ (n!)^k &> (k!)^n\end{aligned}$$

and finally, $(n-1)!^k \geqslant (k!)^{n-1}$. It is easily seen that

$$\begin{aligned}(n!)^k + n^k &= n^k((n-1)!^k + 1)\\ &\geqslant n^k((k!)^{n-1} + 1)\\ &> k^k((k!)^{n-1} + 1).\end{aligned}$$

Since for $k \geqslant 3$, $k^k > k! + k$, we obtain

$$\begin{aligned}k^k((k!)^{n-1} + 1) &> (k! + k)((k!)^{n-1} + 1)\\ &> (k! + k)(k!)^{n-1}\\ &= (k!)^n + k(k!)^{n-1}\\ &= (k!)^n + k^n((k-1)!^{n-1})\\ &> (k!)^n + k^n.\end{aligned}$$

This ends the proof. □

The proposition proved above is obviously stronger than the subtractive variant of the main equation. Having proved (2.1), one can easily show the following result.

Proposition 2.2. *For $n > k \geqslant 3$, the following inequality holds:*

$$(n!)^k - n^k > (k!)^n - k^n. \tag{2.2}$$

P r o o f.

$$(n!)^k + n^k > (k!)^n + k^n,$$
$$(n!)^k + n^k - 2n^k > (k!)^n + k^n - 2n^k.$$

Since for $n \geqslant 3$, the sequence $\{n^{1/n}\}_{n\in\mathbb{N}}$ is strictly decreasing, we have

$$\begin{aligned} n^{1/n} &< k^{1/k}, \\ n^k &< k^n, \\ -n^k &> -k^n, \end{aligned}$$

and finally,

$$\begin{aligned} (n!)^k - n^k &= (n!)^k + n^k - 2n^k \\ &> (k!)^n + k^n - 2n^k \\ &> (k!)^n + k^n - 2k^n \\ &= (k!)^n - k^n. \end{aligned}$$

□

Lemma 2.3 ([1], Theorem 2.1)**.** *For $n > k \geqslant 3$, the following inequality holds:*

$$(n!)^k - n^k > (k!)^n - k^n. \tag{2.3}$$

Our main goal is to show the stronger inequality. For this purpose, we will refer to the lemma stated above.

Theorem 2.4. *The following inequality holds for $n > k \geqslant 3$:*

$$(n!)^k - n^k \cdot (n-1)^k > (k!)^n - k^n. \tag{2.4}$$

P r o o f. Factoring out n^k we obtain

$$(n!)^k - n^k \cdot (n-1)^k = n^k((n-1)!^k - (n-1)^k).$$

We use the inequality shown in Lemma 2.3. Moreover, substituting $n = k$, we obtain the identity. The fact that $n - 1 \geqslant k$ allows us to claim that

$$(n-1)!^k - (n-1)^k \geqslant (k!)^{n-1} - k^{n-1}.$$

After the transformation, we obtain the following comparison:

$$\begin{aligned} n^k((n-1)!^k-(n-1)^k) &\geqslant n^k((k!)^{n-1}-k^{n-1}) \\ &> k^k((k!)^{n-1}-k^{n-1}). \end{aligned}$$

We also know that $k^k > k + k!$ for $k \geqslant 3$, so

$$\begin{aligned} k^k((k!)^{n-1}-k^{n-1}) &> (k!+k)((k!)^{n-1}-k^{n-1}) \\ &= (k!)^n - k^n + (k\cdot(k!)^{n-1} - k!\cdot k^{n-1}). \end{aligned}$$

The expression $(k\cdot(k!)^{n-1} - k!\cdot k^{n-1})$ is greater than 0. In the other case, we have $k\cdot(k!)^{n-1} < k!\cdot k^{n-1}$, so $(k!)^{n-2} < k^{n-2}$, a contradiction. We obtain the desired inequality

$$(n!)^k - n^k\cdot(n-1)^k > (k!)^n - k^n$$

which also implies the fact that nontrivial solutions for $k, n \in \mathbb{N}$ do not exist. $\square$

References

[1] *H. Arzel, F. Luca*: Diophantine equations involving factorials. Math. Bohem. *142* (2017), 181–184. zbl MR doi

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