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A STUDY ON BEST APPROXIMATION AND BANACH ALGEBRA IN n-NORMED LINEAR SPACE

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Abstract. The idea of best approximation in linear n-normed space is presented and some examples showing various possibilities of best approximations in linear n-normed space is given. Also, we study strictly convex n-norm and enquire about the uniqueness of best approximations in n-normed linear space. Furthermore, best approximations in n-Hilbert space is discussed. Moreover, the notion of a Banach algebra in n-Banach space is presented and some examples are discussed. A set-theoretic property of invertible and noninvertible elements in an n-Banach algebra is explained and then topological divisor of zero in n-Banach algebra is defined. Finally, we introduce the notion of a complex homeomorphism in an n-Banach algebra and derive Gleason, Kahane, Zelazko type theorem with the help of complex b-homeomorphism in the case of n-Banach algebra.

Keywords: n-normed space; n-Banach space; b-linear functional; Banach algebra; complex homeomorphism

MSC 2020: 46A22, 46B07, 46B25

1. Introduction

We know that any real-valued continuous function in $[a, b]$ can be approximated by a set of polynomials. There are also another facts on approximations, for example any open interval in the real line can be approximated by a set of closed intervals. A natural setting for the problem of approximation is as follows.

Let X be a normed linear space and G be a subset of X. Let $x_0 \in X$ and $\delta = \inf\limits_{g\in G} \|x_0 - g\|$. Then δ is the distance of x_0 from G. The problem for best approximation of x_0 out of the element of G is to find an element $g_0 \in G$ such that $\delta = \|x_0 - g_0\|$. We see that a best approximation which occurs for g_0 is an element of minimum distance from the given x_0. Such a $g_0 \in G$ may or may not exist; which

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raises the problem of existence. The problem of uniqueness is of practical interest too, since for given x_0 and G there may be more than one best approximation.

Nagumo [14] introduced the notion of Banach algebra in 1936 and thereafter further development of the theory of Banach algebra was given by Gelfand et al. [4], [5]. In recent times, various Banach algebra's techniques are used to simplify the theories related to matrices, operators, integral equations and dynamical systems, etc.

The idea of linear 2-normed space was first introduced by Gähler [3] and thereafter the geometric structure of linear 2-normed spaces was developed by Cho and Freese [2]. The concept of 2-Banach space is briefly discussed in [20]. Mohammad and Siddiqui [13] introduced the concept of 2-Banach algebra and derived some known results of the usual Banach algebra in 2-Banach algebra. For more on 2-normed algebras one can go through the papers [17], [19]. Gunawan and Mashadi [11] developed a generalization of a linear 2-normed space for $n \geqslant 2$. Some fundamental results of classical normed space with respect to b-linear functional in linear n-normed space have been studied by Ghosh and Samanta [7], [8], [9]. Also they have studied few fixed point theorems in linear n-normed space [6]. The concept of 2-inner product space was first introduced by Diminnie et al. [1] in 1970's. In 1989, Misiak [12] developed a generalization of a 2-inner product space for $n \geqslant 2$.

In this paper, first we have studied best approximations theory in linear n-normed space and observed that finite dimensionality plays an important role for existence of best approximation. Strictly convex and uniformly convex cases in linear n-normed space are also discussed. The notion of a Banach algebra in n-Banach spaces is being discussed with a few examples. The consideration of Bergman space as a Banach algebra and the analyses made on Banach algebra structures in multidimensional spaces such as $C(n)$ reveal the richness of algebraic structures in functional spaces. In this context, these studies, which are closely related to the n-normed space framework, complement the discussions in this paper and provide natural and valuable contributions in terms of comparative analysis. We shall establish a set-theoretic property of invertible and noninvertible elements in an n-Banach algebra and then present topological divisor of zero in an n-Banach algebra. Finally, the idea of a complex homeomorphism in an n-Banach algebra is being given and few results with respect to complex b-homeomorphism in n-Banach algebra are established.

2. Preliminaries

In this section, we give some necessary definitions and theorems.

Definition 2.1 ([11])**.** Let X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field with $\dim X \geqslant n$, where n is a positive integer. A real valued function $\|\cdot,\dots,\cdot\|\colon X^n \to \mathbb{R}$ is called an n-norm on X if

(N1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent,
(N2) $\|x_1, x_2, \dots, x_n\|$ is invariant under permutations of $x_1, x_2, \dots, x_n$,
(N3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha|\, \|x_1, x_2, \dots, x_n\|$ for all $\alpha \in \mathbb{K}$,
(N4) $\|x + y, x_2, \dots, x_n\| \leqslant \|x, x_2, \dots, x_n\| + \|y, x_2, \dots, x_n\|$

hold for all $x, y, x_1, x_2, \dots, x_n \in X$. The pair $(X, \|\cdot,\dots,\cdot\|)$ is then called a linear n-normed space.

Definition 2.2 ([11])**.** A sequence $\{x_k\} \subseteq X$ is said to converge to $x \in X$ if

$$\lim_{k\to\infty} \|x_k - x, e_2, \dots, e_n\| = 0$$

for every $e_2, \dots, e_n \in X$ and it is called a Cauchy sequence if

$$\lim_{l,k\to\infty} \|x_l - x_k, e_2, \dots, e_n\| = 0$$

for every $e_2, \dots, e_n \in X$. The space X is said to be complete or n-Banach space if every Cauchy sequence in this space is convergent in X. If a sequence $\{x_k\}$ converges to x, then it will be denoted by $\lim\limits_{k\to\infty} x_k = x$.

Definition 2.3 ([16])**.** We define the following open and closed ball or sphere in X:

$$B_{\{e_2,\dots,e_n\}}(a, \delta) = \{x \in X\colon\ \|x - a, e_2, \dots, e_n\| < \delta\}$$

and

$$B_{\{e_2,\dots,e_n\}}[a, \delta] = \{x \in X\colon\ \|x - a, e_2, \dots, e_n\| \leqslant \delta\},$$

where $a, e_2, \dots, e_n \in X$ and δ is a positive number.

Definition 2.4 ([16])**.** A subset G of X is said to be open in X if for all $a \in G$, there exist $e_2, \dots, e_n \in X$ and $\delta > 0$ such that $B_{\{e_2,\dots,e_n\}}(a, \delta) \subseteq G$.

Definition 2.5 ([16])**.** Let $A \subseteq X$. Then the closure of A is defined as

$$\overline{A} = \{x \in X \mid \exists\, \{x_k\} \in A \text{ with } \lim_{k\to\infty} x_k = x\}.$$

The set A is said to be closed if $A = \overline{A}$.

Definition 2.6 ([16]). A point $x \in X$ is called a boundary point of a subset A of X if in every open ball with center at x there are points of A as well as points of $\overline{A}$. The boundary of A is the set of all boundary points of A.

Definition 2.7 ([7]). Let W be a subspace of X and $b_2, b_3, \ldots, b_n$ be fixed elements in X and $\langle b_i \rangle$ denote the subspaces of X generated by b_i, for $i = 2, 3, \ldots, n$. Then a map $T\colon W \times \langle b_2 \rangle \times \ldots \times \langle b_n \rangle \to \mathbb{K}$ is called a b-linear functional on $W \times \langle b_2 \rangle \times \ldots \times \langle b_n \rangle$ if for every $x, y \in W$ and $k \in \mathbb{K}$, the following conditions hold:

(I) $T(x + y, b_2, \ldots, b_n) = T(x, b_2, \ldots, b_n) + T(y, b_2, \ldots, b_n)$,
(II) $T(kx, b_2, \ldots, b_n) = kT(x, b_2, \ldots, b_n)$.

A b-linear functional is said to be bounded if there exists a real number $M > 0$ such that

$$|T(x, b_2, \ldots, b_n)| \leqslant M\|x, b_2, \ldots, b_n\| \quad \text{for all } x \in W.$$

The norm of the bounded b-linear functional T is defined by

$$\|T\| = \inf\{M > 0\colon\ |T(x, b_2, \ldots, b_n)| \leqslant M\|x, b_2, \ldots, b_n\| \text{ for all } x \in W\}.$$

If T is a bounded b-linear functional on $W \times \langle b_2 \rangle \times \ldots \times \langle b_n \rangle$, the norm of T can be expressed by any one of the following equivalent formula:

(I) $\|T\| = \sup\{|T(x, b_2, \ldots, b_n)|\colon\ \|x, b_2, \ldots, b_n\| \leqslant 1\}$,
(II) $\|T\| = \sup\{|T(x, b_2, \ldots, b_n)|\colon\ \|x, b_2, \ldots, b_n\| = 1\}$,
(III) $\|T\| = \sup\{|T(x, b_2, \ldots, b_n)|/\|x, b_2, \ldots, b_n\|\colon\ \|x, b_2, \ldots, b_n\| \neq 0\}$.

Also, we have $|T(x, b_2, \ldots, b_n)| \leqslant \|T\|\,\|x, b_2, \ldots, b_n\|$ for all $x \in W$.

In 1989, Misiak [12] introduced the concept of n-inner product for $n \geqslant 2$.

Definition 2.8 ([12]). Let $n \in \mathbb{N}$ and H be a linear space of dimension greater than or equal to n over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field. An n-inner product on H is a map

$$(x, y, x_2, \ldots, x_n) \mapsto \langle x, y \mid x_2, \ldots, x_n \rangle, x, y, x_2, \ldots, x_n \in H$$

from H^{n+1} to the set $\mathbb{K}$ such that for every $x, y, x_1, x_2, \ldots, x_n \in H$ and $\alpha \in \mathbb{K}$,

(I) $\langle x_1, x_1 \mid x_2, \ldots, x_n \rangle \geqslant 0$ and $\langle x_1, x_1 \mid x_2, \ldots, x_n \rangle = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,
(II) $\langle x, y \mid x_2, \ldots, x_n \rangle = \langle x, y \mid x_{i_2}, \ldots, x_{i_n} \rangle$ for every permutations $(i_2, \ldots, i_n)$ of $(2, \ldots, n)$,
(III) $\langle x, y \mid x_2, \ldots, x_n \rangle = \overline{\langle y, x \mid x_2, \ldots, x_n \rangle}$,
(IV) $\langle \alpha x, y \mid x_2, \ldots, x_n \rangle = \alpha \langle x, y \mid x_2, \ldots, x_n \rangle$,
(V) $\langle x + y, z \mid x_2, \ldots, x_n \rangle = \langle x, z \mid x_2, \ldots, x_n \rangle + \langle y, z \mid x_2, \ldots, x_n \rangle$.

A linear space H together with an n-inner product $\langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle$ is called an n-inner product space.

Theorem 2.9 ([12]). *For n-inner product space $(H, \langle \cdot, \cdot \mid \cdot, \dots, \cdot \rangle)$,*

$$|\langle x, y \mid x_2, \dots, x_n \rangle| \leqslant \|x, x_2, \dots, x_n\| \, \|y, x_2, \dots, x_n\| \tag{2.1}$$

holds for all $x, y, x_2, \dots, x_n \in H$, where

$$\|x_1, x_2, \dots, x_n\| = \sqrt{\langle x_1, x_1 \mid x_2, \dots, x_n \rangle}.$$

Inequality (2.1) *is called Cauchy-Schwarz inequality.*

An n-inner product space is called n-Hilbert space if it is complete with respect to its induced n-norm.

Definition 2.10 ([15]). A complex algebra is a vector space A over the complex field $\mathbb{C}$ in which a multiplication is defined that satisfies

$$x(yz) = (xy)z, \ (x + y)z = xz + yz, \ x(y + z) = xy + xz,$$

and $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for all x, y and z in A and for all scalars α.

3. Best approximation

Let X be an n-normed linear space, $b_2, b_3, \dots, b_n$ be fixed elements in X and G be a subset of X. Suppose $x_0 \in X$ with $\{x_0, b_2, b_3, \dots, b_n\}$ is linearly independent and

$$\delta_b(x_0, G) = \inf_{g \in G} \|x_0 - g, b_2, b_3, \dots, b_n\|.$$

Here $\delta_b(x_0, G)$ is the distance of x_0 from G with respect to $b_2, b_3, \dots, b_n$ and sometimes it will be denoted by δ_b. The problem for best approximation of x_0 out of the elements of G is to find an element $g_0 \in G$ such that

$$\delta_b = \|x_0 - g_0, b_2, b_3, \dots, b_n\|.$$

This element $g_0 \in G$ (if exists) has therefore minimum distance from x_0. Such an element g_0 may or may not exist. We shall see below that under certain sufficient conditions, an element g_0 with the above property may be found out. If g_0 exists, then next problem arises about the uniqueness which has practical importance too.

In this section, we shall deal with some of these problems. In the following theorem we observe that the finite dimensionality plays an important role for the existence of best approximation.

Theorem 3.1. *If G is a finite dimensional subspace of an n-normed linear space X, then for each x of X with $\{x, b_2, b_3, \ldots, b_n\}$ linearly independent, there exists a best approximation to x out of the elements of G.*

Proof. For $x \in X$, consider the closed sphere

$$\overline{B} = \{y \in G\colon\ \|y, b_2, b_3, \ldots, b_n\| \leqslant 2\|x, b_2, b_3, \ldots, b_n\|\}.$$

Then $\theta \in \overline{B}$ and for the distance of x from $\overline{B}$, we see that

$$\delta_b(x, \overline{B}) = \inf_{g \in \overline{B}} \|x - g, b_2, b_3, \ldots, b_n\| \leqslant \|x - \theta, b_2, b_3, \ldots, b_n\| = \|x, b_2, b_3, \ldots, b_n\|.$$

If $y \notin \overline{B}$, then

$$\|y, b_2, b_3, \ldots, b_n\| > 2\|x, b_2, b_3, \ldots, b_n\|$$

and using the above

$$\begin{aligned}\|x - y, b_2, b_3, \ldots, b_n\| &\geqslant \|y, b_2, b_3, \ldots, b_n\| - \|x, b_2, b_3, \ldots, b_n\| \\ &> \|x, b_2, b_3, \ldots, b_n\| \geqslant \delta_b(x, \overline{B}).\end{aligned}$$

From this we see that $\delta_b(x, \overline{B}) = \delta_b(x, G)$ and this value cannot be assumed by any $y \in G - \overline{B}$ because of the strict inequality in the above relation. Therefore if a best approximation to x exists, it must be an element of $\overline{B}$. Since $\overline{B}$ is a closed and bounded subset of a finite dimensional subspace G, $\overline{B}$ is compact. Consider the map

$$f\colon\ y \mapsto \|x - y, b_2, b_3, \ldots, b_n\|, \quad y \in \overline{B}.$$

Since the n-norm function $\|x - y, b_2, b_3, \ldots, b_n\|$ (y variable) is continuous, f is a real valued continuous function and therefore f attains its least upper bound. So, there is some $y_0 \in \overline{B}$ such that $f(y_0) = \inf_{y \in \overline{B}} f(y)$. From this it follows that

$$\|x - y_0, b_2, b_3, \ldots, b_n\| = \inf_{y \in \overline{B}} \|x - y, b_2, b_3, \ldots, b_n\|.$$

By definition, y_0 is a best approximation to x out of the elements G. This proves the theorem. $\square$

In the next theorem, we see that the set of best approximations has certain geometric property.

Theorem 3.2. *Let G be a subspace of an n-normed linear space X and $x \in X$ with $\{x, b_2, b_3, \ldots, b_n\}$ is linearly independent. Then the set M of best approximations to x out of G is convex.*

Proof. Let $\delta_b = \delta_b(x, G)$, i.e., δ_b is the distance of x_0 from G with respect to $b_2, b_3, \ldots, b_n$. If M is void or M contains only one element, then there is nothing to prove. We therefore assume that M contains more than one element. Let $y, z \in M$, then from definition

$$\|x - y, b_2, b_3, \ldots, b_n\| = \delta_b, \quad \|x - z, b_2, b_3, \ldots, b_n\| = \delta_b.$$

Let w be any element of the form $w = \lambda y + (1-\lambda)z$, where $0 \leqslant \lambda \leqslant 1$. We should show that $w \in M$. Since $w \in G$, we see that $\|x - y, b_2, b_3, \ldots, b_n\| \geqslant \delta_b$ and

$$\begin{aligned}
\|x - w, b_2, b_3, \ldots, b_n\| &= \|\lambda(x-y) + (1-\lambda)(x-z), b_2, b_3, \ldots, b_n\| \\
&\leqslant \lambda\|x - y, b_2, b_3, \ldots, b_n\| + (1-\lambda)\|x - z, b_2, b_3, \ldots, b_n\| \\
&= \lambda\delta_b + (1-\lambda)\delta_b = \delta_b,
\end{aligned}$$

so $\|x - w, b_2, b_3, \ldots, b_n\| = \delta_b$ and $w \in M$. Since y and z are arbitrary elements of M, this proves the theorem. □

Now, we exhibit some examples showing various possibilities of best approximations.

Example 3.3. (i) Consider the space $C[a,b]$. For $x_1 = x_1(t)$, $x_2 = x_2(t), \ldots$, $x_n = x_n(t) \in C[a,b]$, define

$$(3.1)\quad \|x_1, \ldots, x_n\| = \begin{cases} \sup\limits_{a\leqslant t\leqslant b} |x_1(t)| \times \ldots \times \sup\limits_{a\leqslant t\leqslant b} |x_n(t)| & \text{if } x_1, \ldots, x_n \text{ are linearly independent,} \\ 0 & \text{if } x_1, \ldots, x_n \text{ are linearly dependent.} \end{cases}$$

Now, for every $x = x(t)$, $y = y(t)$, $x_2 = x_2(t)$, $\ldots, x_n = x_n(t) \in C[a,b]$, and $\alpha \in \mathbb{R}$, we get

$$\begin{aligned}
\|x + y, x_2, \ldots, x_n\| &= \sup_{a\leqslant t\leqslant b} |x(t) + y(t)| \times \sup_{a\leqslant t\leqslant b} |x_2(t)| \times \ldots \times \sup_{a\leqslant t\leqslant b} |x_n(t)| \\
&\leqslant \Big(\sup_{a\leqslant t\leqslant b} |x(t)| \times \sup_{a\leqslant t\leqslant b} |x_2(t)| \times \ldots \times \sup_{a\leqslant t\leqslant b} |x_n(t)|\Big) \\
&\quad + \Big(\sup_{a\leqslant t\leqslant b} |y(t)| \times \sup_{a\leqslant t\leqslant b} |x_2(t)| \times \ldots \times \sup_{a\leqslant t\leqslant b} |x_n(t)|\Big) \\
&= \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|
\end{aligned}$$

and

$$\begin{aligned}\|\alpha x, x_2, \dots, x_n\| &= \sup_{a\leqslant t\leqslant b} |\alpha x(t)| \times \sup_{a\leqslant t\leqslant b} |x_2(t)| \times \dots \times \sup_{a\leqslant t\leqslant b} |x_n(t)| \\ &= |\alpha| \sup_{a\leqslant t\leqslant b} |x(t)| \times \sup_{a\leqslant t\leqslant b} |x_2(t)| \times \dots \times \sup_{a\leqslant t\leqslant b} |x_n(t)| \\ &= |\alpha| \, \|x, x_2, \dots, x_n\|.\end{aligned}$$

Therefore $C[a,b]$ becomes a linear n-normed space with respect to the n-norm defined by (3.1). Let $x_k = x_k(t) \in C[a,b]$ be a Cauchy sequence, i.e., for every $e_2 = e_2(t)$, $\dots, e_n = e_n(t) \in C[a,b]$, $\|x_k - x_l, e_2, \dots, e_n\| \to 0$ as $k, l \to \infty$. Thus, for $\varepsilon > 0$, there exists a positive integer N such that $\|x_k - x_l, e_2, \dots, e_n\| < \varepsilon$ if $k, l \geqslant N$. This implies that if $k, l \geqslant N$,

$$\begin{aligned}&\sup_{a\leqslant t\leqslant b} |x_k(t) - x_l(t)| \times \sup_{a\leqslant t\leqslant b} |e_2(t)| \times \dots \times \sup_{a\leqslant t\leqslant b} |e_n(t)| < \varepsilon \\ &\qquad\qquad \Rightarrow \sup_{a\leqslant t\leqslant b} |x_k(t) - x_l(t)| < \frac{\varepsilon}{M^{n-1}} = \varepsilon',\end{aligned}$$

where $M = \max\{\sup_{a\leqslant t\leqslant b} |e_2(t)|, \dots, \sup_{a\leqslant t\leqslant b} |e_n(t)|\}$. Thus, we get that $|x_k(t) - x_l(t)| < \varepsilon'$ for $k, l \geqslant N$ and for all $t \in [a,b]$. This shows that the sequence $\{x_k(t)\}$ converges uniformly to a function, say $x(t)$, in $[a,b]$. Since $x_k(t)$ are continuous, $x(t)$ is also continuous on $[a,b]$ and so $x \in C[a,b]$. Letting $l \to \infty$ in the above inequality, we get

$$\sup_{a\leqslant t\leqslant b} |x_k(t) - x(t)| \times \sup_{a\leqslant t\leqslant b} |e_2(t)| \times \dots \times \sup_{a\leqslant t\leqslant b} |e_n(t)| < \varepsilon$$

for $k \geqslant N$. This implies that

$$\|x_k - x, e_2, \dots, e_n\| < \varepsilon \quad \text{for } k \geqslant N.$$

So, $x_k \to x$ as $k \to \infty$. This shows that $C[a,b]$ is complete with respect to the above n-norm and hence, it is an n-Banach space.

Now, let $x_k(t) = t^k$ and Y be the subspace of $C[a,b]$ generated by $x_0, x_1, \dots, x_m$, where $m \geqslant n$ is fixed. Then Y is finite dimensional and Y consists of all polynomials of degree at most m. If x is any element of $C[a,b]$, then by Theorem 3.1, there exists a polynomial, say p_m, of degree at most m such that

$$\begin{aligned}&\sup_{a\leqslant t\leqslant b} |x(t) - p_m(t)| \times \sup_{a\leqslant t\leqslant b} |e_2(t)| \times \dots \times \sup_{a\leqslant t\leqslant b} |e_n(t)| \\ &\qquad \leqslant \sup_{a\leqslant t\leqslant b} |x(t) - p(t)| \times \sup_{a\leqslant t\leqslant b} |e_2(t)| \times \dots \times \sup_{a\leqslant t\leqslant b} |e_n(t)|,\end{aligned}$$

for every $p(t) \in Y$.

(ii) In this example, we shall see that the finite dimensionality of G in Theorem 3.1 is essential. Consider the space $C[0,a]$, where $0 < a < 1$ and let G be the set of polynomials of any degree on $[0,a]$. Then the dimension of G is infinite. Let $x(t) = (1-t)^{-1}$ and $e_j(t) = (-1)^j$ for $j = 2, \ldots, n$. Then for arbitrary $\varepsilon > 0$ there is N such that $\|x - y_k, e_2, \ldots, e_n\| < \varepsilon$ for all $k \geqslant N$, where $y_k(t) = 1 + t + t^2 + \ldots + t^k \in G$. Since $\varepsilon > 0$ is arbitrary, this shows that $\delta_b(x, G) = 0$. However, since x is not a polynomial, we see that there is no $y_0 \in G$ such that

$$\delta_b(x, G) = \|x - y_0, e_2, \ldots, e_n\| = 0.$$

(iii) In this example we show that the uniqueness of best approximations may not hold. Consider the space $\mathbb{R}^n$ and for $x_1 = (a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)})$, $x_2 = (a_1^{(2)}, a_2^{(2)}, \ldots, a_n^{(2)}), \ldots, x_n = (a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)}) \in \mathbb{R}^n$ define

$$(3.2) \quad \|x_1, x_2, \ldots, x_n\| = \begin{cases} \max_i |a_i^{(1)}| \times \max_i |a_i^{(2)}| \times \ldots \times \max_i |a_i^{(n)}| & \\ & \text{if } x_1, \ldots, x_n \text{ are linearly independent,} \\ 0 & \text{if } x_1, \ldots, x_n \text{ are linearly dependent.} \end{cases}$$

Now, for every $x_1 = (a_1^{(1)}, a_2^{(1)}, \ldots, a_n^{(1)})$, $y_1 = (b_1^{(1)}, b_2^{(1)}, \ldots, b_n^{(1)})$, $x_2 = (a_1^{(2)}, a_2^{(2)}, \ldots, a_n^{(2)}), \ldots, x_n = (a_1^{(n)}, a_2^{(n)}, \ldots, a_n^{(n)}) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \|x_1 + y_1, x_2, \ldots, x_n\| &= \max_i |a_i^{(1)} + b_i^{(1)}| \times \max_i |a_i^{(2)}| \times \ldots \times \max_i |a_i^{(n)}| \\ &\leqslant \Big(\max_i |a_i^{(1)}| \times \max_i |a_i^{(2)}| \times \ldots \times \max_i |a_i^{(n)}|\Big) \\ &\quad + \Big(\max_i |b_i^{(1)}| \times \max_i |a_i^{(2)}| \times \ldots \times \max_i |a_i^{(n)}|\Big) \\ &= \|x_1, x_2, \ldots, x_n\| + \|y_1, x_2, \ldots, x_n\|, \end{aligned}$$

and

$$\|\alpha x_1, x_2, \ldots, x_n\| = \max_i |\alpha a_i^{(1)}| \times \max_i |a_i^{(2)}| \times \ldots \times \max_i |a_i^{(n)}| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|.$$

Thus, $\mathbb{R}^n$ is a linear n-normed space with respect to the above n-norm defined by (3.2). Let $G = \{y = (g, g, \ldots, g) \colon g \in \mathbb{R}\}$. Then G is a subspace of $\mathbb{R}^n$. Let $x = (1, 1, \ldots, 1)$ and $e_2 = (1, -1, \ldots, 1), \ldots, e_n = (1, 1, \ldots, -1) \in \mathbb{R}^n$. Then for all $y \in G$ we have

$$\|x - y, e_2, \ldots, e_n\| = |1 - g| \geqslant 1.$$

The distance from x to G with respect to $e_2, \ldots, e_n$ is $\delta_b(x, G) = 1$, and all $y = (g, g, \ldots, g)$ with $|g| \leqslant 1$ are best approximations to x out of G.

4. Strictly convex n-norm

We observe from the above examples that there are spaces where the uniqueness of best approximations in n-normed space may not hold. To enquire about the uniqueness, the following definition becomes helpful in some cases.

Definition 4.1. An n-normed linear space X is called strictly convex if for all $x, y, e_2, \ldots, e_n \in X$ with $\|x, e_2, \ldots, e_n\| = 1$, $\|y, e_2, \ldots, e_n\| = 1$ we have $\|x + y, e_2, \ldots, e_n\| < 2$, unless $x = y$. An n-normed linear space with such an n-norm is called a strictly convex n-normed linear space.

If $\|x, e_2, \ldots, e_n\| = 1$, $\|y, e_2, \ldots, e_n\| = 1$, then

$$\|x + y, e_2, \ldots, e_n\| \leqslant \|x, e_2, \ldots, e_n\| + \|y, e_2, \ldots, e_n\| = 2$$

and the strict convexity excludes the equality sign, unless $x = y$.

In the following theorem, we observe that any n-Hilbert space is strictly convex.

Theorem 4.2. *Any n-Hilbert space H is strictly convex.*

Proof. Let $x, y \in H$, $x \neq y$ and

$$\|x, e_2, \ldots, e_n\| = 1, \; \|y, e_2, \ldots, e_n\| = 1.$$

Let $\|x - y, e_2, \ldots, e_n\| = \alpha$, then $\alpha > 0$. By the parallelogram law

$$\begin{aligned} \|x + y, e_2, \ldots, e_n\|^2 &= 2(\|x, e_2, \ldots, e_n\|^2 + \|y, e_2, \ldots, e_n\|^2) - \|x - y, e_2, \ldots, e_n\|^2 \\ &= 2(1 + 1) - \alpha^2 = 4 - \alpha^2 < 4 \end{aligned}$$

and this gives $\|x + y, e_2, \ldots, e_n\| < 2$. $\square$

Example 4.3. We consider the space $C[a, b]$ with respect to the n-norm given by (3.1). Let $x, y, e_2, \ldots, e_n$ be defined by

$$x(t) = 1, y(t) = \frac{t - a}{b - a}, \quad e_k(t) = (-1)^k, \quad k = 2, \ldots, n,$$

where $t \in [a, b]$. Then $x \neq y$ and $x, y, e_2, \ldots, e_n \in C[a, b]$. Also,

$$\|x, e_2, \ldots, e_n\| = 1, \quad \|y, e_2, \ldots, e_n\| = 1.$$

Now,

$$\|x + y, e_2, \ldots, e_n\| = \sup_{a \leqslant t \leqslant b} \left|1 + \frac{t - a}{b - a}\right| \times \sup_{a \leqslant t \leqslant b} |e_2(t)| \times \ldots \times \sup_{a \leqslant t \leqslant b} |e_n(t)| = 2.$$

Therefore $C[a, b]$ is not strictly convex.

In the next theorem, we see that strict convexity implies the uniqueness of best approximations.

Theorem 4.4. *Let X be an n-normed linear space with a strictly convex n-norm and G be a subspace of X. If $x \in X$, then there is at most one best approximation to x out of the elements of G.*

Proof. Let $\delta_b = \operatorname{dist}(x, G)$ and suppose if possible, that there are two distinct best approximations g_1 and g_2 to x. Then

$$\|x - g_1, b_2, \ldots, b_n\| = \delta_b, \ \|x - g_2, b_2, \ldots, b_n\| = \delta_b$$

and $x - g_1$ and $x - g_2$ are distinct. Therefore by strict convexity

$$\|x - g_1 + x - g_2, b_2, \ldots, b_n\| < 2\delta_b$$

and this implies that

$$\left\| x - \frac{(g_1 + g_2)}{2}, b_2, \ldots, b_n \right\| < \delta_b.$$

Because $(g_1 + g_2)/2 \in G$, this implies a contradiction and the theorem is proved. □

Theorem 4.5. *Let C be a closed convex subset of an n-Hilbert space H. Then C contains a unique element of smallest n-norm.*

Proof. Let $d = \inf\{\|x, b_2, \ldots, b_n\| \colon x \in C\}$. Then there exists a sequence $\{x_k\}$ of elements of C such that $\|x_k, b_2, \ldots, b_n\| \to d$. Now, $x_k, x_l \in C$ and C is convex, so $(x_k + x_l)/2 \in C$ and therefore

$$\left\| \frac{x_k + x_l}{2}, b_2, \ldots, b_n \right\| \geqslant d.$$

This implies that

$$\|x_k + x_l, b_2, \ldots, b_n\| \geqslant 2d.$$

Using parallelogram law, we get

$$\begin{aligned} (4.1) \qquad \|x_k - x_l, b_2, \ldots, b_n\|^2 &= 2\|x_k, b_2, \ldots, b_n\|^2 \\ &\quad + 2\|x_l, b_2, \ldots, b_n\|^2 - \|x_k + x_l, b_2, \ldots, b_n\|^2 \\ &\leqslant 2\|x_k, b_2, \ldots, b_n\|^2 + 2\|x_l, b_2, \ldots, b_n\|^2 - 4d^2. \end{aligned}$$

Since $\|x_k, b_2, \ldots, b_n\| \to d$ as $k \to \infty$ and $\|x_l, b_2, \ldots, b_n\| \to d$ as $l \to \infty$, the right-hand side of (4.1) tends to $2d^2 + 2d^2 - 4d^2 = 0$ as $k, l \to \infty$. So, $\{x_k\}$ is a Cauchy sequence. Since H is complete, there exists x such that $x_k \to x$. As C is closed, we have $x \in C$. So,

$$\|x, b_2, \dots, b_n\| = \lim_{k\to\infty} \|x_k, b_2, \dots, b_n\| = d.$$

Therefore, x is an element in C with the smallest n-norm (i.e., d). Now, we prove the uniqueness. Let y be an element of C such that $\|y, b_2, \dots, b_n\|$ also equals d. Then C being convex, we have $(x+y)/2 \in C$ and also

$$\left\| \frac{x+y}{2}, b_2, \dots, b_n \right\| \geqslant d. \tag{4.2}$$

Applying the parallelogram law, we obtain

$$\begin{aligned} \left\| \frac{x+y}{2}, b_2, \dots, b_n \right\|^2 &= \frac{\|x, b_2, \dots, b_n\|^2}{2} + \frac{\|y, b_2, \dots, b_n\|^2}{2} - \frac{\|x-y, b_2, \dots, b_n\|^2}{2} \\ &< \frac{\|x, b_2, \dots, b_n\|^2}{2} + \frac{\|y, b_2, \dots, b_n\|^2}{2} = d^2 \quad \text{if } x \neq y \end{aligned}$$

and this gives that

$$\left\| \frac{x+y}{2}, b_2, \dots, b_n \right\| < d.$$

This contradicts (4.2) and so $x = y$. This proves the theorem. □

Theorem 4.6. *Let M be a closed subspace of an n-Hilbert space H and $x \in H - M$. Let d be the distance from x to M with respect to $b_2, \dots, b_n$, i.e.,*

$$d = \inf_{y \in M} \|x - y, b_2, b_3, \dots, b_n\|.$$

Then there exists a unique element $y_0 \in M$ such that

$$\|x - y_0, b_2, b_3, \dots, b_n\| = d.$$

P r o o f. Consider the set C defined by

$$C = \{x + y \colon \ y \in M\}.$$

We will now see that C is a closed convex set and that d is the distance from the origin θ to C.

(1) C is closed: Let $x + y_k \to x + y$, where $y_k \in M$. So,

$$\|x + y_k - x - y, b_2, b_3, \dots, b_n\| = \|y_k - y, b_2, b_3, \dots, b_n\| \to 0,$$

i.e., $y_k \to y$. Since M is closed, $y \in M$ and this implies that $x + y \in C$.

(2) C is convex: Let $z_1, z_2 \in C$, then $z_1 = x + y_1$, $z_2 = x + y_2$, where $y_1, y_2 \in M$. So, if $0 \leqslant \lambda \leqslant 1$, then

$$\lambda z_1 + (1-\lambda)z_2 = x + \lambda y_1 + (1-\lambda)y_2 = x + y,$$

where $y = \lambda y_1 + (1-\lambda)y_2 \in M$, because M is a subspace. So, $\lambda z_1 + (1-\lambda)z_2 \in C$ and C is convex.

(3) d is the distance from the origin to C: We have

$$\begin{aligned} d = \inf_{y' \in M} \|x - y', b_2, b_3, \ldots, b_n\| &= \inf_{y=-y' \in M} \|x + y, b_2, b_3, \ldots, b_n\| \\ &= \inf_{z=x+y \in C} \|z, b_2, b_3, \ldots, b_n\|. \end{aligned}$$

By Theorem 4.5, there exists a unique element $z \in C$ such that

$$\|z, b_2, b_3, \ldots, b_n\| = d.$$

Let $y_0 = x - z$, then clearly $y_0 \in M$ and

$$\|x - y_0, b_2, b_3, \ldots, b_n\| = \|z, b_2, b_3, \ldots, b_n\| = d.$$

We now prove the uniqueness of y_0.

Let $y_1 \in M$ and $y_1 \neq y_0$ and if possible, suppose that

$$\|x - y_1, b_2, b_3, \ldots, b_n\| = d.$$

Then $z_1 = x - y_1 \in C$ such that $z_1 \neq z_0$ and moreover,

$$\|z_1, b_2, b_3, \ldots, b_n\| = d.$$

This contradicts the uniqueness of z_0. Hence, the theorem is proved. □

In Theorem 4.6, if instead of assuming M to be a closed subspace, we assume M to be a closed convex subset of H, we still obtain the best approximation as the next theorem shows.

Theorem 4.7. *Let M be a nonempty closed convex subset of H and $x \in H$. Let d be the distance from x to M with respect to $b_2, \ldots, b_n$, i.e.,*

$$d = \inf_{y \in M} \|x - y, b_2, b_3, \ldots, b_n\|.$$

Then there exists a unique element $y_0 \in M$ such that

$$\|x - y_0, b_2, b_3, \ldots, b_n\| = d.$$

P r o o f. It follows from the definition d that there exists a sequence $\{y_k\}$ in M such that

$$d = \lim_{k\to\infty} \|x - y_k, b_2, b_3, \ldots, b_n\|. \tag{4.3}$$

Using parallelogram law, we see that

$$\begin{aligned}\|y_k - y_l, b_2, b_3, \ldots, b_n\|^2 &= 2\|y_k - x, b_2, b_3, \ldots, b_n\|^2 + 2\|y_l - x, b_2, b_3, \ldots, b_n\|^2 \\ &\quad - \|2x - y_k - y_l, b_2, b_3, \ldots, b_n\|^2 \\ &= 2\|y_k - x, b_2, b_3, \ldots, b_n\|^2 + 2\|y_l - x, b_2, b_3, \ldots, b_n\|^2 \\ &\quad - 4\left\|x - \frac{1}{2}(y_k + y_l), b_2, b_3, \ldots, b_n\right\|^2.\end{aligned}$$

As M is convex, $\frac{1}{2}(y_k + y_l) \in M$ and therefore

$$\left\|x - \frac{1}{2}(y_k + y_l), b_2, b_3, \ldots, b_n\right\|^2 \geqslant d^2.$$

Using this we obtain from the above

$$\|y_k - y_l, b_2, b_3, \ldots, b_n\|^2 \leqslant 2\|y_k - x, b_2, b_3, \ldots, b_n\|^2 + 2\|y_l - x, b_2, b_3, \ldots, b_n\|^2 - 4d^2$$

and now from (4.3) it follows that

$$\|y_k - y_l, b_2, b_3, \ldots, b_n\| \to \infty \quad \text{as } k, l \to \infty,$$

i.e., $\{y_k\}$ is a Cauchy sequence. Consequently, there exists $y_0 \in H$ such that $\|y_k - y_0, b_2, b_3, \ldots, b_n\| \to \infty$ as $k \to \infty$. Since M is closed, y_0 also belongs to M. Now,

$$\|x - y_0, b_2, b_3, \ldots, b_n\| \leqslant \|x - y_k, b_2, b_3, \ldots, b_n\| + \|y_k - y_0, b_2, b_3, \ldots, b_n\|$$

and because

$$\|x - y_k, b_2, b_3, \ldots, b_n\| \to d, \quad \|y_k - y_0, b_2, b_3, \ldots, b_n\| \to 0,$$

it follows that $\|x - y_0, b_2, b_3, \ldots, b_n\| \leqslant d$. Also, it is easy to clear that

$$\|x - y_0, b_2, b_3, \ldots, b_n\| \geqslant d.$$

Thus, $\|x - y_0, b_2, b_3, \ldots, b_n\| = d$.

We now prove the uniqueness part. Assume $y_1 \in M$ be such that

$$\|x - y_1, b_2, b_3, \ldots, b_n\| = d.$$

The convexity of M implies that $\frac{1}{2}(y_0 + y_1) \in M$. From these we obtain that

$$\begin{aligned} d &\leqslant \left\| x - \frac{1}{2}(y_0 + y_1), b_2, b_3, \ldots, b_n \right\| \\ &= \left\| \frac{1}{2}x - \frac{1}{2}y_0 + \frac{1}{2}x - \frac{1}{2}y_1, b_2, b_3, \ldots, b_n \right\| \\ &\leqslant \frac{1}{2}\|x - y_0, b_2, b_3, \ldots, b_n\| + \frac{1}{2}\|x - y_1, b_2, b_3, \ldots, b_n\| = d \end{aligned}$$

and therefore

$$\left\| x - \frac{1}{2}(y_0 + y_1), b_2, b_3, \ldots, b_n \right\| = d.$$

Applying parallelogram law we now see that

$$\begin{aligned} \|y_0 - y_1, b_2, b_3, \ldots, b_n\|^2 &= 2\|x - y_0, b_2, b_3, \ldots, b_n\|^2 + 2\|x - y_1, b_2, b_3, \ldots, b_n\|^2 \\ &\quad - 4\left\| x - \frac{1}{2}(y_0 + y_1), b_2, b_3, \ldots, b_n \right\|^2 \\ &= 2d^2 + 2d^2 - 4d^2 = 0, \end{aligned}$$

i.e., $y_0 = y_1$. This proves the theorem. □

Definition 4.8. The n-Banach space X is said to be uniformly convex if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$\|x, b_2, b_3, \ldots, b_n\| \leqslant 1, \quad \|y, b_2, b_3, \ldots, b_n\| \leqslant 1$$

and

$$\|x - y, b_2, b_3, \ldots, b_n\| \geqslant \varepsilon,$$

then

$$\left\| \frac{1}{2}(x + y), b_2, b_3, \ldots, b_n \right\| \leqslant 1 - \delta.$$

Theorem 4.9. *Every n-inner product space H is uniformly convex with respect to the n-norm defined by the n-inner product. In particular, every n-Hilbert space is uniformly convex.*

P r o o f. Let $\varepsilon > 0$ be arbitrary and

$$\|x, b_2, b_3, \ldots, b_n\| \leqslant 1, \quad \|y, b_2, b_3, \ldots, b_n\| \leqslant 1,$$
$$\|x - y, b_2, b_3, \ldots, b_n\| \geqslant \varepsilon.$$

By the parallelogram law we obtain

$$\begin{aligned}
&\left\|\frac{1}{2}(x+y), b_2, b_3, \ldots, b_n\right\|^2 \\
&\quad = 2\left(\left\|\frac{x}{2}, b_2, \ldots, b_n\right\|^2 + \left\|\frac{y}{2}, b_2, \ldots, b_n\|^2\right) - \left\|\frac{x-y}{2}, b_2, \ldots, b_n\right\|^2 \\
&\quad \leqslant 2\left(\frac{1}{4} + \frac{1}{4}\right) - \left(\frac{\varepsilon}{2}\right)^2 = 1 - \frac{\varepsilon^2}{4} \\
&\quad \Rightarrow \left\|\frac{1}{2}(x+y), b_2, b_3, \ldots, b_n\right\| \leqslant \left(1 - \frac{\varepsilon^2}{4}\right)^{1/2}
\end{aligned}$$

and this gives that $\delta(\varepsilon) \geqslant 1 - (1 - \varepsilon^2/4)^{1/2}$, which is positive for all positive $\varepsilon \leqslant 2$. Hence, H is uniformly convex. This proves the theorem. $\square$

Theorem 4.10. *Every uniformly convex n-Banach space is strictly convex.*

P r o o f. This follows from the definition. $\square$

Theorem 4.11. *Let X be a uniformly convex n-Banach space and $\{x_k\}$ be a sequence in X such that $\|x_k, b_2, b_3, \ldots, b_n\| \to 1$ as $k \to \infty$ and*

$$\|x_k + x_l, b_2, b_3, \ldots, b_n\| \to 2 \quad \text{as } k, l \to \infty.$$

Then $\{x_k\}$ is a Cauchy sequence.

P r o o f. If $\{x_k\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ and a sequence of positive integers $k_1 < k_2 < \ldots$ such that

$$\|x_{k_j}, b_2, b_3, \ldots, b_n\| < 1 + \frac{1}{j}, \quad \|x_{k_{j+1}} - x_{k_j}, b_2, b_3, \ldots, b_n\| \geqslant 2\varepsilon$$

for $j = 1, 2, \ldots$. Since X is uniformly convex, corresponding to this ε, there exists $\delta > 0$ such that

$$\left\|\frac{1}{2}(x+y), b_2, b_3, \ldots, b_n\right\| \leqslant 1 - \delta$$

whenever

$$\|x, b_2, b_3, \ldots, b_n\| \leqslant 1, \quad \|y, b_2, b_3, \ldots, b_n\| \leqslant 1$$

and

$$\|x-y,b_2,b_3,\ldots,b_n\| \geqslant \varepsilon.$$

Now, it is easily seen that

$$\left\|\frac{x_{k_j}}{1+1/j},b_2,b_3,\ldots,b_n\right\| \leqslant 1, \quad \left\|\frac{x_{k_{j+1}}}{1+1/j},b_2,b_3,\ldots,b_n\right\| \leqslant 1$$

and

$$\left\|\frac{x_{k_{j+1}}-x_{k_j}}{1+1/j},b_2,b_3,\ldots,b_n\right\| \geqslant \frac{2\varepsilon}{1+1/j} \geqslant \varepsilon.$$

Therefore

$$\left\|\frac{x_{k_{j+1}}+x_{k_j}}{2(1+1/j)},b_2,b_3,\ldots,b_n\right\| \leqslant 1-\delta$$

and this implies that

$$\lim_{j\to\infty} \|x_{k_{j+1}}+x_{k_j},b_2,b_3,\ldots,b_n\| \leqslant 2(1-\delta) < 2.$$

This contradicts the assumption that

$$\|x_k+x_l,b_2,b_3,\ldots,b_n\| \to 2 \quad \text{as } k,l\to\infty.$$

Therefore, $\{x_k\}$ is a Cauchy sequence and this proves the theorem. □

5. Preliminaries of Banach algebra in n-Banach space

In this section, we shall see that if a linear space is equipped with an algebra, many interesting results are obtained. We give the formal definition of a Banach algebra in n-Banach space.

Definition 5.1. Let X be a complex algebra of $\dim X \geqslant n$ with the n-norm $\|\cdot,\ldots,\cdot\|$. Then X is said to be an n-normed algebra if the n-norm satisfies the inequality, known as multiplicative inequality,

$$\|xy,a_2,\ldots,a_n\| \leqslant \|x,a_2,\ldots,a_n\|\,\|y,a_2,\ldots,a_n\| \tag{5.1}$$

for all $x,y,a_2,a_3,\ldots,a_n \in X$ and if X contains a unit element e, i.e., $xe=ex=x$, $x\in X$, and $\|e,a_2,\ldots,a_n\| = 1$, for all $a_2,a_3,\ldots,a_n \in X$. If, in addition, X is an n-Banach space with respect to the n-norm $\|\cdot,\ldots,\cdot\|$, then X is called an n-Banach algebra.

In the definition of an n-Banach algebra, we do not assume that $xy = yx$ for all $x, y \in X$. But if this happens, then X is called a commutative n-Banach algebra. An n-Banach subalgebra of the n-Banach algebra X is a subset of X which is itself an n-Banach algebra with respect to the algebraic operations defined on X and with the same identity and with the same n-norm.

Theorem 5.2. *Let $\{x_k\}$ and $\{y_k\}$ be two sequences in X such that $x_k \to x$ and $y_k \to y$ as $k \to \infty$ for some $x, y \in X$. Then $x_k y_k \to xy$ as $k \to \infty$.*

Proof. Let $x, y, a_2, \ldots, a_n \in X$. Then

$$\begin{aligned}\|x_k y_k - xy, a_2, \ldots, a_n\| & \|(x_k - x)y_k + x(y_k - y), a_2, \ldots, a_n\| \\ & \leqslant \|x_k - x, a_2, \ldots, a_n\| \, \|y_k, a_2, \ldots, a_n\| \\ & \quad + \|y_k - y, a_2, \ldots, a_n\| \, \|x, a_2, \ldots, a_n\| \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Thus, $x_k y_k \to xy$ as $k \to \infty$. This means that the multiplication is a continuous operation in X. □

Example 5.3. Let X denote the set of all polynomials

$$x(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_m t^m \ (c_0, \ldots, c_m \text{ are complex}),$$

where $m \geqslant n$ is not a fixed positive integer. Then X is a complex linear space with respect to the addition of polynomials and scalar multiplication of a polynomial. For each $i = 1, 2, \ldots, n$, let $x_i(t) = a_0^i + a_1^i t + a_2^i t^2 + \ldots + a_m^i t^m$. Now, define

$$\begin{aligned}&\|x_1, \ldots, x_n\| \\ &= \begin{cases} \max\limits_j |a_j^1| \times \ldots \times \max\limits_j |a_j^n| & \text{if } x_1, \ldots, x_n \text{ are linearly independent,} \\ 0 & \text{if } x_1, \ldots, x_n \text{ are linearly dependent.} \end{cases}\end{aligned} \tag{5.2}$$

Let

$$y_1(t) = b_0^1 + b_1^1 t + b_2^1 t^2 + \ldots + b_m^1 t^m \in X$$

and $\alpha \in \mathbb{K}$. Now, we see that

$$\begin{aligned}\|\alpha x_1, \ldots, x_n\| &= \max_j |\alpha a_j^1| \times \ldots \times \max_j |a_j^n| = |\alpha| \, \|x_1, \ldots, x_n\|, \\ \|x_1 + y_1, \ldots, x_n\| &= \max_j |a_j^1 + b_j^1| \times \ldots \times \max_j |a_j^n| \\ &\leqslant \Big(\max_j |a_j^1| \times \ldots \times \max_j |a_j^n|\Big) + \Big(\max_j |b_j^1| \times \ldots \times \max_j |a_j^n|\Big) \\ &= \|x_1, \ldots, x_n\| + \|y_1, \ldots, x_n\|.\end{aligned}$$

Therefore, X becomes a linear n-normed space with respect to the n-norm defined by (5.2). Now, we take the product of

$$x_1(t) = a_0^1 + a_1^1 t + a_2^1 t^2 + \ldots + a_m^1 t^m$$
$$y_1(t) = b_0^1 + b_1^1 t + b_2^1 t^2 + \ldots + b_m^1 t^m$$

as

$$(x_1 y_1)(t) = c_0^1 + c_1^1 t + c_2^1 t^2 + \ldots + c_k^1 t^k,$$

where

$$c_k^1 = a_0^1 b_k^1 + a_1^1 b_{k-1}^1 + \ldots + a_k^1 b_0^1.$$

Also, we have

$$\begin{aligned}\|x_1 y_1, \ldots, x_n\| &= \max_j |a_j^1 b_j^1| \times \ldots \times \max_j |a_j^n| \\ &\leqslant \Bigl(\max_j |a_j^1| \times \ldots \times \max_j |a_j^n|\Bigr)\Bigl(\max_j |b_j^1| \times \ldots \times \max_j |a_j^n|\Bigr) \\ &= \|x_1, \ldots, x_n\|\, \|y_1, \ldots, x_n\|.\end{aligned}$$

Thus, X is commutative n-normed algebra with identity $(e = 1)$.

Guediri et al. [10] and Tapdıgoglu [18] studied techniques on Banach algebra and their applications. These two studies focus on the study of Banach algebra structures in multivariable function spaces.

Example 5.4. The space $C[a,b]$ with respect to the n-norm given by (3.1) is a commutative n-Banach algebra with identity $e = 1$, where the product xy is defined as $(xy)(t) = x(t)y(t)$ for all $t \in [a,b]$.

Definition 5.5. Let X be a linear n-normed space and $b_2, \ldots, b_n \in X$. Then an operator $T\colon X \to X$ is called b-bounded if there exists a positive constant M such that

$$\|Tx, b_2, \ldots, b_n\| \leqslant M\|x, b_2, \ldots, b_n\| \text{ for all } x \in X.$$

The norm of T is denoted by $\|T\|$ and is defined as

$$\|T\| = \inf\{M\colon \|Tx, b_2, \ldots, b_n\| \leqslant M\|x, b_2, \ldots, b_n\| \text{ for all } x \in X\}.$$

Remark 5.6. If T is b-bounded on X, the norm of T can be expressed by any one of the following equivalent formulas:

(I) $\|T\| = \sup\{\|Tx, b_2, \ldots, b_n\|\colon \|x, b_2, \ldots, b_n\| \leqslant 1\}$,
(II) $\|T\| = \sup\{\|Tx, b_2, \ldots, b_n\|\colon \|x, b_2, \ldots, b_n\| = 1\}$,
(III) $\|T\| = \sup\{\|Tx, b_2, \ldots, b_n\|/\|x, b_2, \ldots, b_n\|\colon \|x, b_2, \ldots, b_n\| \neq 0\}$.

Also, we have

$$\|Tx, b_2, \ldots, b_n\| \leqslant \|T\| \, \|x, b_2, \ldots, b_n\| \quad \text{for all } x \in X.$$

Example 5.7. Let X_b be the set of all b-bounded linear operators on H, where H is an n-Hilbert space. We define the operations

$$T_1 + T_2 \text{ by } (T_1 + T_2)(x) = T_1(x) + T_2(x), \quad \alpha T \text{ by } (\alpha T)(x) = \alpha T(x),$$
$$T_1 T_2 \text{ by } (T_1 T_2)(x) = T_1(T_2(x)),$$

for every $x \in H$. Define

$$\|T\| = \sup\{\|Tx, b_2, \ldots, b_n\| \colon \|x, b_2, \ldots, b_n\| \leqslant 1\}.$$

Now, it can be easily verified that $(X_b, \|\cdot\|)$ is a Banach space. Also, we get that $\|T_1 T_2\| \leqslant \|T_1\| \, \|T_2\|$. Thus, X_b is a Banach algebra. The identity operator is the identity element of this Banach algebra. Clearly, X_b is not commutative.

If an n-Banach algebra X does not have an identity, we can extend X, which we show now, so that the extended n-Banach algebra has an identity. Let $\overline{X}$ be the set of all ordered pairs (x, a), where $x \in X$ and $a \in \mathbb{C}$. Now, we show that $\overline{X}$ can be equipped conveniently with algebraic operations and an n-norm such that $\overline{X}$ becomes an n-Banach algebra. We define the operations in $\overline{X}$ as follows:

$$\begin{aligned}(x, a) + (y, b) &= (x + y, a + b),\\ (x, a)(y, b) &= (xy + ay + bx, ab),\\ \alpha(x, a) &= (\alpha x, a\alpha), \quad \alpha \in \mathbb{K}.\end{aligned}$$

We can easily verify that with the above operations, $\overline{X}$ becomes an algebra. The n-norm on $\overline{X}$ is defined as:

$$\|(x_1, c_1), (x_2, c_2), \ldots, (x_n, c_n)\|$$
$$= \begin{cases} \|x_1, x_2, \ldots, x_n\| + |c_1 c_2 \ldots c_n| & \\ & \text{if } (x_1, c_1), \ldots, (x_n, c_n) \text{ are linearly independent,} \\ 0 & \text{if } (x_1, c_1), \ldots, (x_n, c_n) \text{ are linearly dependent,} \end{cases}$$

where $x_1, x_2, \ldots, x_n \in X$ and $c_1, c_2, \ldots, c_n \in \mathbb{C}$. Here, the n-norm axioms are fulfilled. Also, for $x, y, a_2, \ldots, a_n \in X$ and $a, b, b_2, \ldots, b_n \in \mathbb{C}$, we have

$$\begin{aligned}
&\|(x,a)(y,b),(a_2,b_2),\ldots,(a_n,b_n)\| \\
&\quad = \|(xy+ay+bx, ab),(a_2,b_2),\ldots,(a_n,b_n)\| \\
&\quad = \|xy+ay+bx, a_2,\ldots,a_n\| + |abb_2\ldots b_n| \\
&\quad \leqslant \|x,a_2,\ldots,a_n\|\,\|y,a_2,\ldots,a_n\| + |a|\,\|y,a_2,\ldots,a_n\| + \\
&\qquad + |b|\,\|x,a_2,\ldots,a_n\| + |abb_2\ldots b_n| \\
&\quad \leqslant (\|x,a_2,\ldots,a_n\| + |ab_2\ldots b_n|)(\|y,a_2,\ldots,a_n\| + |bb_2\ldots b_n|) \\
&\quad = \|(x,a),(a_2,b_2),\ldots,(a_n,b_n)\|\,\|(y,b),(a_2,b_2),\ldots,(a_n,b_n)\|.
\end{aligned}$$

We now show that $\overline{X}$ is an n-Banach space, i.e., $\overline{X}$ is complete. Clearly, a sequence $\{(x_k, a_k)\}$ is a Cauchy sequence in $\overline{X}$ if and only if $\{x_k\}$ is a Cauchy sequence in X and $\{a_k\}$ is a Cauchy sequence in $\mathbb{C}$. Since X and $\mathbb{C}$ are complete, there exist $x \in X$ and $a \in \mathbb{C}$ such that $\lim\limits_{k\to\infty} x_k = x$ and $\lim\limits_{k\to\infty} a_k = a$. Therefore, $\lim\limits_{k\to\infty} (x_k, a_k) = (x, a)$ is in $\overline{X}$. Also, it is easy to verify that $(x,a)(\theta,1) = (x,a) = (\theta,1)(x,a)$, where θ is the zero element in X. Thus, $\overline{X}$ is an n-Banach algebra with the indentity $(\theta, 1)$.

For 2-normed algebra, augmentation of unity is discussed by Srivastava et al. [17].

5.1. Invertible and noninvertible elements. In this section, we deal with the set of invertible and noninvertible elements in an n-Banach algebra. We remember that G denotes the set of all invertible elements of X. It is easy to verify that G is a group. Let S denote the set of all noninvertible elements of X. Clearly, $S \cup G = X$.

Theorem 5.8. *Let X be a complex n-Banach algebra with unity e. If $x \in X$ satisfies $\|x, a_2, \ldots, a_n\| < 1$ for every $a_2, \ldots, a_n \in X$, then $e - x$ is invertible and $(e-x)^{-1} = e + \sum\limits_{j=1}^{\infty} x^j$.*

P r o o f. From (5.1) we get

$$\|x^j, a_2, \ldots, a_n\| \leqslant \|x, a_2, \ldots, a_n\|^j \quad \text{for every } a_2, \ldots, a_n \in X,$$

for any positive integer j, so that the infinite series $\sum\limits_{j=1}^{\infty} x^j$ is convergent because $\|x, a_2, \ldots, a_n\| < 1$. Since X is complete, it is easy to verify that the infinite series $\sum\limits_{j=1}^{\infty} x^j$ is convergent to an element in X. Let therefore $s = e + \sum\limits_{j=1}^{\infty} x^j$. If we prove that $s = (e-x)^{-1}$, the proof is finished. Now, by a simple calculation, we obtain

$$(e-x)(e+x+\ldots+x^k) = (e+x+\ldots+x^k)(e-x) = e - x^{k+1}. \tag{5.3}$$

Since $\|x, a_2, \dots, a_n\| < 1$ and so $x^{k+1} \to \theta$ as $k \to \infty$, letting $k \to \infty$ in (5.3) and using the continuity of multiplication in X, we obtain $(e - x)s = s(e - x) = e$. Thus, $s = (e - x)^{-1}$. This proves the theorem. $\square$

In Theorem 5.8, if x is replaced by $(e - x)$, we obtain the following corollary.

Corollary 5.9. *If $x \in X$ satisfies $\|e - x, a_2, \dots, a_n\| < 1$ for every $a_2, \dots, a_n \in X$, then x^{-1} exists and $x^{-1} = e + \sum_{j=1}^{\infty} (e - x)^j$.*

Next, we prove another corollary.

Corollary 5.10. *Let $x \in X$ and λ be a scalar such that $\|x, a_2, \dots, a_n\| < |\lambda|$ for every $a_2, \dots, a_n \in X$. Then $(\lambda e - x)^{-1}$ exists and*

$$(\lambda e - x)^{-1} = \sum_{j=1}^{\infty} \lambda^{-j} x^{j-1}, \ (x^0 = e).$$

Proof. If $y \in X$ is such that $y^{-1} \in X$ and $\alpha \neq 0$ be a scalar, then it is clear that $(\alpha y)^{-1}$ exists and $(\alpha y)^{-1} = \alpha^{-1} y^{-1}$. Having noted this, we can write $\lambda e - x = \lambda(e - x/\lambda)$ and we show that $(e - x/\lambda)^{-1}$ exists. Now, by hypothesis, we have

$$\left\| e - \left(e - \frac{x}{\lambda} \right), a_2, \dots, a_n \right\| = \frac{\|x, a_2, \dots, a_n\|}{|\lambda|} < 1.$$

So, by Corollary 5.9, $(e - x/\lambda)^{-1}$ exists and so $(\lambda e - x)^{-1}$ exists.

For the infinite series representation, we make use of the second part of Corollary 5.9. Now, we have $(\lambda e - x) = \lambda(e - \lambda^{-1} x)$ and it is easy to verify that $(\lambda e - x)^{-1} = \sum_{j=1}^{\infty} \lambda^{-j} x^{j-1}$. This proves the corollary. $\square$

In the following theorem, we obtain a set-theoretic property of invertible and noninvertible elements of X.

Theorem 5.11. *The set G of all invertible elements of X is an open subset.*

Proof. Let $x_0 \in G$. We should show that there exists a sphere around x_0, each element of which belongs to G. Consider the open sphere

$$B_{\{a_2, \dots, a_n\}} \left(x_0, \frac{1}{\|x_0^{-1}, a_2, \dots, a_n\|} \right)$$

with center at x_0 and radius $1/\|x_0^{-1}, a_2, \dots, a_n\|$. Every point x of this sphere satisfies the inequality

$$\|x - x_0, a_2, \dots, a_n\| < \frac{1}{\|x_0^{-1}, a_2, \dots, a_n\|}. \tag{5.4}$$

Let $y = x_0^{-1}x$ and $z = e - y$. Then by (5.1) and (5.4), we have

$$\begin{aligned}\|z, a_2, \ldots, a_n\| &= \|y - e, a_2, \ldots, a_n\| = \|x_0^{-1}x - x_0^{-1}x_0, a_2, \ldots, a_n\| \\ &= \|x_0^{-1}(x - x_0), a_2, \ldots, a_n\| \\ &\leqslant \|x_0^{-1}, a_2, \ldots, a_n\| \, \|x - x_0, a_2, \ldots, a_n\| < 1.\end{aligned}$$

By Theorem 5.8, $e - z$ is invertible, i.e., y is invertible. So, $y \in G$. Now, $x_0 \in G$, $y \in G$ and by our earlier verification, G is a group, so $x_0 y \in G$. But $x_0 y = x_0 x^{-1} x = x$. So, any x satisfying (5.4) belongs to G. This shows that G is an open subset of X. This completes the proof. □

Corollary 5.12. *The set of all noninvertible elements is a closed subset of X.*

Definition 5.13. A linear operator $T\colon X \to X$ is said to be continuous at $x_0 \in X$ if for any open ball $B_{\{e_2,\ldots,e_n\}}(T(x_0), \varepsilon) > 0$ there exists $\delta > 0$ such that

$$T(B_{\{e_2,\ldots,e_n\}}(x_0, \delta)) \subset B_{\{e_2,\ldots,e_n\}}(T(x_0), \varepsilon).$$

Equivalently, for given $\varepsilon > 0$, there exist some $e_2, \ldots, e_n \in X$ and $\delta > 0$ such that for $x \in X$

$$\|x - x_0, e_2, \ldots, e_n\| < \delta \Rightarrow \|T(x) - T(x_0), e_2, \ldots, e_n\| < \varepsilon.$$

Theorem 5.14. *The mapping $T\colon G \to G$ given by $T(x) = x^{-1}$ for all $x \in G$ is continuous.*

P r o o f. Let x_0 be an element of G and x be any other element of G that satisfies the inequality

$$\|x - x_0, a_2, \ldots, a_n\| < \frac{1}{2\|x_0^{-1}, a_2, \ldots, a_n\|}.$$

Now,

$$\begin{aligned}\|x_0^{-1}x - e, a_2, \ldots, a_n\| &= \|x_0^{-1}(x - x_0), a_2, \ldots, a_n\| \\ &\leqslant \|x_0^{-1}, a_2, \ldots, a_n\| \, \|x - x_0, a_2, \ldots, a_n\| < \frac{1}{2}\end{aligned} \tag{5.5}$$

and so by Corollary 5.9, $x_0^{-1}x \in G$ and further

$$x^{-1}x_0 = (x_0^{-1}x)^{-1} = e + \sum_{j=1}^{\infty}(e - x_0^{-1}x)^j. \tag{5.6}$$

Next, we show that the mapping $T\colon G \to G$ given by $T(x) = x^{-1}$ is continuous at x_0. Now, we have

$$
\begin{aligned}
(5.7)\qquad \|Tx - Tx_0, a_2, \ldots, a_n\| &= \|x^{-1} - x_0^{-1}, a_2, \ldots, a_n\| \\
&= \|(x^{-1}x_0 - e)x_0^{-1}, a_2, \ldots, a_n\| \\
&\leqslant \|x_0^{-1}, a_2, \ldots, a_n\| \, \|x^{-1}x_0 - e, a_2, \ldots, a_n\| \\
&= \|x_0^{-1}, a_2, \ldots, a_n\| \left\| \sum_{j=1}^{\infty} (e - x_0^{-1}x)^j, a_2, \ldots, a_n \right\| \text{ [by (5.6)]} \\
&\leqslant \|x_0^{-1}, a_2, \ldots, a_n\| \sum_{j=1}^{\infty} \|e - x_0^{-1}x, a_2, \ldots, a_n\|^j \\
&= \|x_0^{-1}, a_2, \ldots, a_n\| \, \|e - x_0^{-1}x, a_2, \ldots, a_n\| \sum_{j=0}^{\infty} \|e - x_0^{-1}x, a_2, \ldots, a_n\|^j \\
&= \frac{\|x_0^{-1}, a_2, \ldots, a_n\| \, \|e - x_0^{-1}x, a_2, \ldots, a_n\|}{1 - \|e - x_0^{-1}x, a_2, \ldots, a_n\|} \\
&< 2\|x_0^{-1}, a_2, \ldots, a_n\| \, \|e - x_0^{-1}x, a_2, \ldots, a_n\| \text{ [by (5.5)]} \\
&\leqslant 2\|x_0^{-1}, a_2, \ldots, a_n\|^2 \, \|x - x_0, a_2, \ldots, a_n\|,
\end{aligned}
$$

since

$$
\begin{aligned}
\|e - x_0^{-1}x, a_2, \ldots, a_n\| = \|x_0^{-1}x - e, a_2, \ldots, a_n\| &= \|x_0^{-1}(x - x_0)a_2, \ldots, a_n\| \\
&\leqslant \|x_0^{-1}, a_2, \ldots, a_n\| \, \|x - x_0, a_2, \ldots, a_n\|.
\end{aligned}
$$

Thus, from (5.7), the continuity of T at x_0 follows. This completes the proof. □

Theorem 5.15. *Let $x \in G$ and $h \in X$ be such that*

$$\|h, a_2, \ldots, a_n\| < \frac{1}{2}\|x^{-1}, a_2, \ldots, a_n\|^{-1}$$

for every $a_2, \ldots, a_n \in X$. Then $x + h \in G$ and

$$\|(x+h)^{-1} - x^{-1} + x^{-1}hx^{-1}, a_2, \ldots, a_n\| \leqslant 2\|x^{-1}, a_2, \ldots, a_n\|^3 \, \|h, a_2, \ldots, a_n\|^2.$$

P r o o f. Since for every $a_2, \ldots, a_n \in X$ we have

$$\|h, a_2, \ldots, a_n\| \, \|x^{-1}, a_2, \ldots, a_n\| < \frac{1}{2},$$

we have

$$\|x^{-1}h, a_2, \dots, a_n\| \leqslant \|x^{-1}, a_2, \dots, a_n\| \, \|h, a_2, \dots, a_n\| < \frac{1}{2}.$$

Thus, by Theorem 5.8, we get that $e + x^{-1}h \in G$. Since G is a group, we have that $x(e + x^{-1}h) \in G$. But $x(e + x^{-1}h) = x + h$ and so $x + h \in G$.

Before proving the second part, we note by Theorem 5.8 that

$$(e + x^{-1}h)^{-1} = e + \sum_{j=1}^{\infty} (-1)^j (x^{-1}h)^j$$

and so

$$\begin{aligned} (5.8) \qquad & \|(e + x^{-1}h)^{-1} - e + x^{-1}h, a_2, \dots, a_n\| \\ & \leqslant \sum_{j=2}^{\infty} \|x^{-1}h, a_2, \dots, a_n\|^j \\ & = \|x^{-1}h, a_2, \dots, a_n\|^2 \sum_{j=0}^{\infty} \|x^{-1}h, a_2, \dots, a_n\|^j \\ & < \|x^{-1}h, a_2, \dots, a_n\|^2 \sum_{j=0}^{\infty} \frac{1}{2^j} = 2\|x^{-1}h, a_2, \dots, a_n\|^2. \end{aligned}$$

Now,

$$\begin{aligned} (x + h)^{-1} - x^{-1} + x^{-1}hx^{-1} &= [x(e + x^{-1}h)]^{-1} - x^{-1} + x^{-1}hx^{-1} \\ &= (e + x^{-1}h)^{-1}x^{-1} - x^{-1} + x^{-1}hx^{-1} \\ &= [(e + x^{-1}h)^{-1} - e + x^{-1}h]x^{-1} \end{aligned}$$

and so taking the n-norm and using (5.8), for every $a_2, \dots, a_n \in X$ we get

$$\begin{aligned} & \|(x + h)^{-1} - x^{-1} + x^{-1}hx^{-1}, a_2, \dots, a_n\| \\ & \qquad \leqslant \|(e + x^{-1}h)^{-1} - e + x^{-1}h, a_2, \dots, a_n\| \, \|x^{-1}, a_2, \dots, a_n\| \\ & \qquad \leqslant 2\|x^{-1}h, a_2, \dots, a_n\|^2 \, \|x^{-1}, a_2, \dots, a_n\| \\ & \qquad \leqslant 2\|x^{-1}, a_2, \dots, a_n\|^3 \, \|h, a_2, \dots, a_n\|. \end{aligned}$$

This proves the theorem. $\square$

5.2. Topological divisor of zero. In this section, we give the idea of a topological divisor of zero in an n-Banach algebra and establish its relationship with noninvertible elements.

Definition 5.16. Let X be an n-Banach algebra. An element $z \in X$ is said to be a topological divisor of zero if there exists a sequence $\{z_k\}$ in X with $\|z_k, a_2, \ldots, a_n\| = 1$ for every $a_2, \ldots, a_n \in X$ and $k = 1, 2, \ldots$ and such that either $zz_k \to \theta$ or $z_k z \to \theta$ as $k \to \infty$.

Clearly, every divisor of zero is also a topological divisor of zero. Let Z denote the set of all topological divisors of zero in X. In the next theorem, we give certain connection between Z and the set S of all noninvertible elements.

Theorem 5.17. *The set Z is a subset of S.*

Proof. Let $z \in Z$. Then there exists a sequence $\{z_k\}$ with $\|z_k, a_2, \ldots, a_n\| = 1$ for every $a_2, \ldots, a_n \in X$ and $k = 1, 2, \ldots$ and such that either $zz_k \to \theta$ or $z_k z \to \theta$ as $k \to \infty$. Suppose that $zz_k \to \theta$ as $k \to \infty$. If possible, let $z \in G$. Then z^{-1} exists. Now, as multiplication is a continuous operation, we should have $z_k = z^{-1}(zz_k) \to z^{-1}\theta = \theta$ as $k \to \infty$. This contradicts the fact that $\|z_k, a_2, \ldots, a_n\| = 1$ for every $a_2, \ldots, a_n \in X$ and $k = 1, 2, \ldots$. So $z \in S$ and this proves the theorem. $\square$

Theorem 5.18. *The boundary of S is a subset of Z.*

Proof. By Corollary 5.12, S is a closed subset of X. So, any boundary point of S is in S. Further, for every boundary point of S, there exists a sequence of elements from G that converges to the boundary point. Let x be any boundary point of S, then $x \in S$ and there exists a sequence $\{s_k\}$ of elements from G such that $s_k \to x$ as $k \to \infty$. Now, we have $s_k^{-1}x - e = s_k^{-1}(x - s_k)$. Therefore, if $\|s_k^{-1}, a_2, \ldots, a_n\|$ is a bounded sequence, then because $s_k \to x$ as $k \to \infty$, we get from above that for all large values of k, $\|s_k^{-1}x - e, a_2, \ldots, a_n\| < 1$ and so by Corollary 5.9, $s_k^{-1}x \in G$ and therefore $x = s_k(s_k^{-1}x)$ belongs to G, which contradicts the fact that $x \in S$. Thus, $\|s_k^{-1}, a_2, \ldots, a_n\|$ is not a bounded sequence. Clearly, we may assume that $\|s_k^{-1}, a_2, \ldots, a_n\| \to \infty$ as $k \to \infty$. Let x_k be defined as

$$x_k = \frac{s_k^{-1}}{\|s_k^{-1}, a_2, \ldots, a_n\|},$$

then $\|x_k, a_2, \ldots, a_n\| = 1$ and

$$\begin{aligned} xx_k &= \frac{xs_k^{-1}}{\|s_k^{-1}, a_2, \ldots, a_n\|} = \frac{e + (x - s_k)s_k^{-1}}{\|s_k^{-1}, a_2, \ldots, a_n\|} \\ &= \frac{e}{\|s_k^{-1}, a_2, \ldots, a_n\|} + (x - s_k)x_k \to \theta \text{ as } k \to \infty, \end{aligned}$$

because $\|s_k^{-1}, a_2, \ldots, a_n\| \to \infty$, $s_k \to x$ as $k \to \infty$, and $\|x_k, a_2, \ldots, a_n\| = 1$. Therefore, $x \in Z$ and this proves the theorem. $\square$

5.3. Complex homeomorphism in n-Banach algebra. In this section, we introduce the idea of a complex homeomorphism in an n-Banach algebra and then deduce some of its properties.

Definition 5.19. Let X be a complex n-Banach algebra and T be a bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \ldots \times \langle b_n \rangle$. Then T is called a complex b-homeomorphism if

$$T(xy, b_2, \ldots, b_n) = T(x, b_2, \ldots, b_n)T(y, b_2, \ldots, b_n)$$

for all $x, y \in X$.

Lemma 5.20. *If T is a complex b-homeomorphism defined on $X \times \langle b_2 \rangle \times \ldots \times \langle b_n \rangle$, then $T(e, b_2, \ldots, b_n) = 1$, and if $x \in X$ is invertible, then $T(x, b_2, \ldots, b_n) \neq 0$.*

Proof. Since T is not identically zero, there exists y such that $T(y, b_2, \ldots, b_n) \neq 0$. Now,

$$T(y, b_2, \ldots, b_n) = T(ye, b_2, \ldots, b_n) = T(y, b_2, \ldots, b_n)T(e, b_2, \ldots, b_n).$$

This implies that $T(e, b_2, \ldots, b_n) = 1$. If x is invertible, then

$$T(x, b_2, \ldots, b_n)T(x^{-1}, b_2, \ldots, b_n) = T(xx^{-1}, b_2, \ldots, b_n) = T(e, b_2, \ldots, b_n) = 1.$$

So, $T(x, b_2, \ldots, b_n) \neq 0$. This proves the lemma. □

Theorem 5.21. *Let $x \in X$ and $\|x, a_2, \ldots, a_n\| < 1$ for every $a_2, \ldots, a_n \in X$. Then $|T(x, b_2, \ldots, b_n)| < 1$ for every complex b-homeomorphism T defined on $X \times \langle b_2 \rangle \times \ldots \times \langle b_n \rangle$.*

Proof. Let λ be a complex number such that $|\lambda| \geqslant 1$. Then $\|x/\lambda, a_2, \ldots, a_n\| < 1$ and so by Theorem 5.8, $e - \lambda^{-1}x$ is invertible. Therefore, by Lemma 5.20, $T(e - \lambda^{-1}x, b_2, \ldots, b_n) \neq 0$. But

$$\begin{aligned} T(e - \lambda^{-1}x, b_2, \ldots, b_n) &= T(e, b_2, \ldots, b_n) - \lambda^{-1}T(x, b_2, \ldots, b_n) \\ &= 1 - \lambda^{-1}T(x, b_2, \ldots, b_n). \end{aligned}$$

So, $T(x, b_2, \ldots, b_n) \neq \lambda$, which shows that $|T(x, b_2, \ldots, b_n)| < 1$. This proves the theorem. □

In the following theorem, we deal with the converse part of Lemma 5.20, i.e., we derive Gleason, Kahane, Zelazko type theorem with respect to complex b-homeomorphism.

Theorem 5.22. *Let T be a bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \ldots \times \langle b_n \rangle$ such that $T(e, b_2, \ldots, b_n) = 1$ and $T(x, b_2, \ldots, b_n) \neq 0$ for every invertible $x \in X$. Then T is complex b-homeomorphism defined on $X \times \langle b_2 \rangle \times \ldots \times \langle b_n \rangle$.*

Proof. Let N be the null space of T, i.e.,

$$N = \{x \in X\colon\ T(x, b_2, \ldots, b_n) = 0\}.$$

Let $x \in X$ and consider the element $x - \beta e$, where $\beta = T(x, b_2, \ldots, b_n)$. Then

$$\begin{aligned} T(x - \beta e, b_2, \ldots, b_n) &= T(x, b_2, \ldots, b_n) - \beta T(e, b_2, \ldots, b_n) \\ &= T(x, b_2, \ldots, b_n) - \beta = 0 \end{aligned}$$

and so $x - \beta e \in N$. Let $x - \beta e = a$. Then every element $x \in X$ can be expressed as $x = a + T(x, b_2, \ldots, b_n)e$, where $a \in N$. Similarly, if $y \in X$, let $y = b + T(y, b_2, \ldots, b_n)e$, where $b \in N$. So,

$$x = a + T(x, b_2, \ldots, b_n)e, \quad y = b + T(y, b_2, \ldots, b_n)e,$$

where $a, b \in N$ and therefore

$$xy = ab + T(y, b_2, \ldots, b_n)a + T(x, b_2, \ldots, b_n)b + T(x, b_2, \ldots, b_n)T(y, b_2, \ldots, b_n)e.$$

Thus,

$$T(xy, b_2, \ldots, b_n) = T(x, b_2, \ldots, b_n)T(y, b_2, \ldots, b_n) + T(ab, b_2, \ldots, b_n). \tag{5.9}$$

Therefore, the theorem will be proved if $T(ab, b_2, \ldots, b_n) = 0$, i.e., if

$$ab \in N \text{ whenever } a \in N \text{ and } b \in N. \tag{5.10}$$

Suppose that a special case of (5.10) is true, i.e.,

$$a^2 \in N \text{ if } a \in N. \tag{5.11}$$

Assuming (5.11), we can deduce (5.10) as follows. In (5.9), we assume $x = y$ and so $a = b$. Then because of (5.11), we get

$$\begin{aligned} T(x^2, b_2, \ldots, b_n) &= T(a^2, b_2, \ldots, b_n) + [T(x, b_2, \ldots, b_n)]^2 \\ &= [T(x, b_2, \ldots, b_n)]^2, \quad x \in X. \end{aligned} \tag{5.12}$$

In (5.12), we replace x by $x+y$ and get

$$T(xy+yx, b_2, \ldots, b_n) = 2T(x, b_2, \ldots, b_n)T(y, b_2, \ldots, b_n), x, y \in X.$$

Therefore,

$$xy + yx \in N \quad \text{if } x \in N, \quad y \in X. \tag{5.13}$$

By a simple calculation we obtain

$$(xy-yx)^2 + (xy+yx)^2 = 2[x(yxy) + (yxy)x]. \tag{5.14}$$

If $x \in N$, then the right side of (5.14) belongs to N because of (5.13). Also, for the same reason $xy+yx \in N$ and so by application of (5.12), $(xy+yx)^2 \in N$. Therefore, $(xy-yx)^2 \in N$ and another application of (5.12) gives that

$$xy - yx \in N \quad \text{if } x \in N, \quad y \in X. \tag{5.15}$$

Adding (5.13) and (5.15), we get (5.10) and so the theorem is proved provided we establish (5.11), which we do now.

Since by hypothesis $T(x, b_2, \ldots, b_n) \neq 0$, for every invertible $x \in X$, N contains no invertible element. So, if $\alpha \in N$, then by Corollary 5.9,

$$\|e - \alpha, b_2, \ldots, b_n\| \geqslant 1.$$

Clearly, $\alpha/\lambda \in N$ for any complex number λ and so $\|e - \alpha/\lambda, b_2, \ldots, b_n\| \geqslant 1$. Therefore,

$$\|\lambda e - \alpha, b_2, \ldots, b_n\| = |\lambda| \left\| e - \frac{\alpha}{\lambda}, b_2, \ldots, b_n \right\| \geqslant |\lambda| = |T(\lambda e - \alpha, b_2, \ldots, b_n)|. \tag{5.16}$$

As in the first part of the proof, we can show that every element $x \in X$ can be expressed as $x = \lambda e - \alpha$, where $\alpha \in N$ and λ is a complex number. So, by (5.16), we get that

$$|T(x, b_2, \ldots, b_n)| \leqslant \|x, b_2, \ldots, b_n\|$$

for all $x \in X$. Thus, T is a bounded b-linear functional defined on $X \times \langle b_2 \rangle \times \ldots \times \langle b_n \rangle$ with norm 1.

Let $a \in N$ and without loss of generality, we may assume that $\|a, b_2, \ldots, b_n\| = 1$. Now, we define

$$f(\lambda) = \sum_{j=0}^{\infty} \frac{T(a^j, b_2, \ldots, b_n)}{j!} \lambda^j,$$

where λ is a complex number. Now,

$$|T(a^j, b_2, \ldots, b_n)| \leqslant \|a^j, b_2, \ldots, b_n\| \leqslant \|a, b_2, \ldots, b_n\|^j = 1$$

for all j and so f is an entire function and further

$$|f(\lambda)| \leqslant \sum_{j=0}^{\infty} \frac{|T(a^j, b_2, \ldots, b_n)|}{j!} |\lambda|^j \leqslant \sum_{j=0}^{\infty} \frac{|\lambda|^j}{j!} = e^{|\lambda|}.$$

Also, by hypothesis $f(0) = T(e, b_2, \ldots, b_n) = 1$ and since $a \in N$, we get $f'(0) = T(a, b_2, \ldots, b_n) = 0$.

Thus, if we can prove that $|f(\lambda)| > 0$ for every complex number λ, then Lemma 5.20 gives that $f(\lambda) = 1$ for all λ, i.e., $f''(0) = 0$. But $f''(0) = T(a^2, b_2, \ldots, b_n)$ and so $T(a^2, b_2, \ldots, b_n) = 0$, i.e., $a^2 \in N$ and this is (5.11).

Consider the series

$$L(\lambda) = \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a^j,$$

where λ is a complex number and this converges in the n-norm of X. Since T is b-bounded, we obtain

$$\begin{aligned} T(L(\lambda), b_2, \ldots, b_n) &= T\left(\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} a^j, b_2, \ldots, b_n\right) \\ &= \sum_{j=0}^{\infty} \frac{T(a^j, b_2, \ldots, b_n)}{j!} \lambda^j = f(\lambda). \end{aligned} \tag{5.17}$$

From the expression for $L(\lambda)$ we can verify, as in the classical case, the following functional equation $L(\lambda + \mu) = L(\lambda)L(\mu)$. We have in particular $L(\lambda)L(-\lambda) = L(0) = e$, which shows that $L(\lambda)$ is an invertible element in X. By hypothesis, $T(L(\lambda), b_2, \ldots, b_n) \neq 0$, i.e., from (5.17) we get $f(\lambda) \neq 0$. This completes the proof. $\square$

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