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ENUMERATING 2D AND 3D LATTICE PATHS WITH ARBITRARY STEPS

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Abstract. Let S be a finite set of integer vectors. We consider lattice paths that use only the vectors in S. We focus on paths that use a fixed number of vectors. We generally assume vectors in S have a fixed coordinate sum, which allows us to determine the number of vectors in a path, which we call its length. We count the number of paths with fixed length for various sets of vectors S. We then use our enumeration results to determine the minimal length path given a terminal point. First, we explore this problem in $S \subseteq \mathbb{N}^3$. After solving the problem of enumeration and determining the minimal length for various sets $S \subseteq \mathbb{N}^3$, we solve these problems for a general case $S = \{(1,0),(0,1),(u,v),(v,u)\}$. We conclude with an enumeration problem of paths that stay weakly below the line $y = x$.

Keywords: lattice path; enumeration; shortest path; generating function

MSC 2020: 05A15, 05A19

1. Introduction

Consider paths terminating at $(t_1, t_2, t_3, \ldots, t_d)$ that use only vectors in S with $S \subseteq \mathbb{N}^d$. In this paper, we concentrate on vectors $S \subseteq \mathbb{N}^3$, and in two sections we consider sets $S \subseteq \mathbb{N}^2$. We refer the reader to [3] for a detailed research on the history of lattice paths. An ordered sequence of *steps* which can be shown as $\mathbf{s} = s_1, s_2, \ldots, s_k$ with $s_i \in S$ for all i, is called an S-*walk*. We visualize $\mathbf{s}$ in the plane as a walk that begins at the origin and terminates at the point $s_1 + s_2 + \ldots + s_k$. The number of steps in the walk is called the *length* of that walk.

We enumerate the number of S-walks terminating at a fixed point (a,b), where $a, b \in \mathbb{N}$ is a classical example. For example, when $S = \{(1,0),(0,1)\}$, there are $\binom{a+b}{a}$ such walks, and every S-walk has a length $a+b$. For example, if $S =$

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$\{(1,0),(0,1),(2,1)\}$, the S-walks

$$\mathbf{s} = (1,0),(2,1),(0,1),(2,1)$$

and

$$\mathbf{s}' = (0,1),(1,0),(1,0),(2,1),(0,1),(1,0)$$

are shown in Figure 1, $\mathbf{s}$ is an S-walk with length of 4, while $\mathbf{s}'$ is an S-walk with length of 6, but both walks terminate at $(5,3)$.

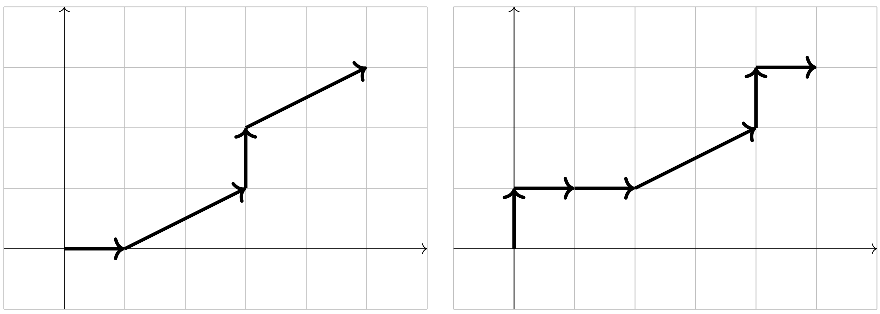

Figure 1. An S-walk with length of 4 (left) and an S-walk with length of 6 (right), both terminating at $(5,3)$.

We aim at enumerating fixed length S-walks for various choices of the set S. To guarantee that every point with nonnegative integer coordinates is reachable, we assume $(1,0),(0,1) \in S$ for $S \subseteq \mathbb{N}^2$, and $(1,0,0),(0,1,0),(0,0,1) \in S$ for $S \subseteq \mathbb{N}^3$.

Evoniuk et al. [1] studied minimal length S-walks for various sets $S \subseteq \mathbb{N}^2$. Iwanojko et al. [4] studied S-walks for $S \subseteq \mathbb{N}^2$ with a new parameter ℓ, which is the total number of *long steps* used in the walk. Long steps are steps other than the standard basis vectors. We let ℓ determine the exact length of the walk, assuming that the long steps have a fixed coordinate sum. Our first goal is to explore this problem for $S \subseteq \mathbb{N}^3$.

In Section 2, we consider the case $S = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$. In Section 3, we start with the sets of the form $S = \{(1,0,0),(0,1,0),(0,0,1),(u+1,u,u),(u,u+1,u),(u,u,u+1)\}$, next we generalize this problem to sets of the form $S = \{(1,0,0),(0,1,0),(0,0,1),(u,v,v),(v,u,v),(v,v,u)\}$. In Section 4, we explore this problem for the sets of the form $T_n = \{(1,0,0),(0,1,0),(0,0,1)\} \cup \{(i,j,n-i-j)\colon\ 0 \leqslant i \leqslant n, 0 \leqslant j \leqslant n-i\}$. In Section 5, we give an answer to the open problem left by Evoniuk et al. [1], Problem 10 which considers the sets of the form $S = \{(1,0),(0,1),(u,v),(v,u)\}$. In Section 6, we determine the number of paths that stay weakly below the line $y = x$ for $S = \{(1,0),(0,1),(v+1,v),(v,v+1)\}$, which is the case $u = v+1$ of Section 5.

2. Walks for $S = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$

Lattice paths for $S = \{(1,0),(0,1),(1,1)\}$ have been studied many times. In this section, we explore S-walks for $S = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$. In this case, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ are the *short steps*, and $(1,1,1)$ is the *long step*.

We begin by establishing some notations. We adopt the notations used by Evoniuk et al. [1] and Iwanojko et al. [4] to be consistent with their studies. For a given set of vectors $S \subseteq \mathbb{N}^3$ and a terminating point (a,b,c), $d(a,b,c;S)$ is the minimum length that an S-walk terminating at (a,b,c) can have. Similarly, $d(a,b;S)$ is the minimum length that an S-walk terminating at (a,b) can have for $S \subseteq \mathbb{N}^2$. The set of all S-walks that terminate at (a,b,c) and use exactly ℓ long steps is denoted by $\mathcal{W}(a,b,c;S,\ell)$, and $|\mathcal{W}(a,b,c;S,\ell)|$ denotes the cardinality of that set. We use $|\mathcal{W}(a,b,c;S)|$ to denote the number of S-walks terminating at (a,b,c) with minimal length.

Theorem 2.1. *Let* $S = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$. *For any* $(a,b,c) \in \mathbb{N}^3$,

$$|\mathcal{W}(a,b,c;S,\ell)| = \binom{a+b+c-2\ell}{\ell, a-\ell, b-\ell}. \tag{2.1}$$

Proof. We begin by proving an S-walk with ℓ long steps must use $a-\ell$ of the $(1,0,0)$ steps. Consider the contribution of long steps on the x-axis. We know that the number of long steps is ℓ. Hence, long steps contribute ℓ units on the x-axis. The steps $(0,1,0)$ and $(0,0,1)$ have a contribution of 0 regardless of how many of them are used in the walk. Each $(1,0,0)$ step contributes 1 unit; hence, the walk must have $a-\ell$ of these steps. Considering the contribution of long steps on the y-axis we see that the walk must use $b-\ell$ of the $(0,1,0)$ steps. Likewise, considering the contribution of steps on the z-axis, we see that the walk must use $c-\ell$ of the $(0,0,1)$ steps. Such a walk has a length of

$$\ell + (a-\ell) + (b-\ell) + (c-\ell) = a+b+c-2\ell.$$

From $a+b+c-2\ell$ positions we choose ℓ of them to be long steps, $a-\ell$ of them to be $(1,0,0)$ steps, $b-\ell$ of them to be $(0,1,0)$ steps, and the remaining steps will be $(0,0,1)$ steps. $\square$

As ℓ gets larger, after a certain point, $|\mathcal{W}(a,b,c;S,\ell)|$ will always be 0. From the proof of Theorem 2.1, we get that $|\mathcal{W}(a,b,c;S,\ell)|$ is nonzero if and only if $\ell \leqslant a$, $\ell \leqslant b$, and $\ell \leqslant c$. Thus, we can say that the maximum value of ℓ where $|\mathcal{W}(a,b,c;S,\ell)| > 0$ is $\min(a,b,c)$. When ℓ is this maximum value, we enumerate minimal length S-walks.

Corollary 2.2. *Let* $S=\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$. *For all* $(a,b,c)\in\mathbb{N}^3$, $d(a,b,c;S)=a+b+c-2\min(a,b,c)$, *and*

$$|\mathcal{W}(a,b,c;S)| = \binom{a+b+c-2\min(a,b,c)}{\min(a,b,c),\, a-\min(a,b,c),\, b-\min(a,b,c)}. \tag{2.2}$$

$b \backslash a$	0	1	2	3	4	5	6	7
7	0	7	56	252	840	2310	5544	12012
6	0	6	42	168	504	1260	2772	5544
5	0	5	30	105	280	630	1260	2310
4	0	4	20	60	140	280	504	840
3	0	3	12	30	60	105	168	252
2	0	2	6	12	20	30	42	56
1	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0

Figure 2. Values of $|\mathcal{W}(a,b,1;S,1)|$ for $S=\{(1,0,0),(0,1,0),\ (0,0,1),(1,1,1)\}$ and $0\leqslant a,b\leqslant 7$.

Figure 2 shows $|\mathcal{W}(a,b,c;S,\ell)|$ for $c=1$ and $\ell=1$. For $a=b$ (the numbers on the line $y=x$), equation (2.1) gives $|\mathcal{W}(a,a,1;S,1)|=\binom{2a-1}{a-1,a-1,1}=(2a-1)\binom{2a-2}{a-1}$, which is OEIS [6] sequence A002457. The nonzero numbers $1,6,30,140,630,2772,12012,\ldots$, are ath Catalan number multiplied by ath triangular number. For example, if $a=4$, then the fourth Catalan number is 14, and the fourth triangular number is 10, $14\cdot 10=140$. One other thing we noticed is $2|\mathcal{W}(a,a,1;S,1)|=|\mathcal{W}(a+1,a,1;S,1)|$, and by symmetry $2|\mathcal{W}(a,a,1;S,1)|=|\mathcal{W}(a,a+1,1;S,1)|$.

In Figure 3, similarly to Figure 2, we see that $2|\mathcal{W}(a+1, a, 2; S, 2)| = |\mathcal{W}(a+2, a, 2; S, 2)| = |\mathcal{W}(a, a+2, 2; S, 2)|$. The array shown in Figure 3 appears in sequence A129533.

7	0	0	21	168	756	2520	6930	16632
6	0	0	15	105	420	1260	3150	6930
5	0	0	10	60	210	560	1260	2520
4	0	0	6	30	90	210	420	756
3	0	0	3	12	30	60	105	168
2	0	0	1	3	6	10	15	21
1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
b \ a	0	1	2	3	4	5	6	7

Figure 3. Values of $|\mathcal{W}(a, b, 2; S, 2)| = \binom{a+1}{2}\binom{a+b}{a+1}$ for $S = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$ and $0 \leqslant a,\ b \leqslant 7$.

Corollary 2.3. *Let* $S = \{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$. *For* $(a,b,c) \in \mathbb{N}^3$, *if* $b = a + c$, *then*

$$|\mathcal{W}(a, b, c; S, \ell)| = 2|\mathcal{W}(a, b-1, c; S, \ell)|. \tag{2.3}$$

P r o o f. Equation (2.1) gives the number of walks terminating at (a,b,c). If we plug in $b = a + c$, then

$$|\mathcal{W}(a, a+c, c; S, \ell)| = \binom{2a + 2c - 2\ell}{\ell, a-\ell, c-\ell},$$

and

$$|\mathcal{W}(a, a+c-1, c; S, \ell)| = \binom{2a + 2c - 1 - 2\ell}{\ell, a-\ell, c-\ell}.$$

It is easy to see that $2\binom{2a+2c-2\ell}{\ell, a-\ell, c-\ell} = \binom{2a+2c-1-2\ell}{\ell, a-\ell, c-\ell}$. $\square$

3. Walks for $S = \{(1,0,0),(0,1,0),(0,0,1),(u,v,v),(v,u,v),(v,v,u)\}$

In this section, we consider the sets of the form $S = \{(1,0,0),(0,1,0),(0,0,1),(u,v,v),(v,u,v),(v,v,u)\}$, where (u,v,v), (v,u,v) and (v,v,u) are *long steps*, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are *short steps*. We begin by exploring the case $u = v+1$.

Theorem 3.1. *Let $v \geqslant 1$ and $S = \{(1,0,0),(0,1,0),(0,0,1),(v+1,v,v),(v,v+1,v),(v,v,v+1)\}$. For any $(a,b,c) \in \mathbb{N}^3$,*

$$|\mathcal{W}(a,b,c;S,\ell)| = \binom{a+b+c-3v\ell}{\ell}\binom{a+b+c-3v\ell}{a-v\ell,\, b-v\ell}. \tag{3.1}$$

Proof. Let an S-walk use i of the $(v+1,v,v)$ steps and j of the $(v,v+1,v)$ steps. Then this S-walk must use $\ell-i-j$ of the $(v,v,v+1)$ steps. Long steps have a total contribution of

$$i(v+1,v,v)+j(v,v+1,v)+(\ell-i-j)(v,v,v+1) = (v\ell+i, v\ell+j, v\ell+\ell-i-j).$$

We see that long steps contribute $v\ell+i$ units on the x-axis; thus, the walk must use $a-v\ell-i$ of the $(1,0,0)$ steps. Considering the contribution on y and z axes, we see that the walk must use $b-v\ell-j$ of the $(0,1,0)$ steps and $c-v\ell-\ell+i+j$ of the $(0,0,1)$ steps. The length of such a walk is

$$(a-v\ell-i)+(b-v\ell-j)+(c-v\ell-\ell+i+j)+\ell = a+b+c-3v\ell.$$

We first choose ℓ long steps out of $a+b+c-3v\ell$ positions, then we choose i and j of those ℓ long steps to be $(v+1,v,v)$ and $(v,v+1,v)$ steps, respectively. Lastly, we choose which of the remaining $a+b+c-3v\ell-\ell$ steps will be $(1,0,0)$ and $(0,1,0)$ steps. Therefore,

$$\begin{aligned}
&|\mathcal{W}(a,b,c;S,\ell)| \\
&= \sum_{i=0}^{\ell}\sum_{j=0}^{\ell-i}\binom{a+b+c-3v\ell}{\ell}\binom{a+b+c-3v\ell-\ell}{a-v\ell-i,\, b-v\ell-j}\binom{\ell}{i,j} \\
&= \binom{a+b+c-3v\ell}{\ell}\sum_{i=0}^{\ell}\sum_{j=0}^{\ell-i}\binom{a+b+c-3v\ell-\ell}{a-v\ell-i,\, b-v\ell-j}\binom{\ell}{i,j} \\
&= \binom{a+b+c-3v\ell}{\ell}\sum_{i=0}^{\ell}\sum_{j=0}^{\ell-i}\binom{\ell}{i}\binom{a+b+c-3v\ell-\ell}{a-v\ell-i}\binom{\ell-i}{j}\binom{b+c-2v\ell-\ell+i}{b-v\ell-j} \\
&= \binom{a+b+c-3v\ell}{\ell}\sum_{i=0}^{\ell}\binom{\ell}{i}\binom{a+b+c-3v\ell-\ell}{a-v\ell-i}\sum_{j=0}^{\ell-i}\binom{\ell-i}{j}\binom{b+c-2v\ell-\ell+i}{b-v\ell-j}.
\end{aligned}$$

Vandermonde's identity tells us that $\sum\limits_{j=0}^{b-v\ell}\binom{\ell-i}{j}\binom{b+c-2v\ell-\ell+i}{b-v\ell-j}=\binom{b+c-2v\ell}{b-v\ell}$. This equation is the same as $\sum\limits_{j=0}^{\ell-i}\binom{\ell-i}{j}\binom{b+c-2v\ell-\ell+i}{b-v\ell-j}=\binom{b+c-2v\ell}{b-v\ell}$ since $\binom{\ell-i}{j}=0$ for $j>\ell-i$. So,

$$|\mathcal{W}(a,b,c;S,\ell)|=\binom{a+b+c-3v\ell}{\ell}\binom{b+c-2v\ell}{b-v\ell}\sum_{i=0}^{\ell}\binom{\ell}{i}\binom{a+b+c-3v\ell-\ell}{a-v\ell-i}.$$

By a similar argument, $\sum\limits_{i=0}^{\ell}\binom{\ell}{i}\binom{a+b+c-3v\ell-\ell}{a-v\ell-i}=\binom{a+b+c-3v\ell}{a-v\ell}$. Finally,

$$\begin{aligned}|\mathcal{W}(a,b,c;S,\ell)|&=\binom{a+b+c-3v\ell}{\ell}\binom{a+b+c-3v\ell}{a-v\ell}\binom{b+c-2v\ell}{b-v\ell}\\&=\binom{a+b+c-3v\ell}{\ell}\binom{a+b+c-3v\ell}{a-v\ell,b-v\ell}.\end{aligned}$$

□

Corollary 3.2. *Let $v\geqslant 1$ and $S=\{(1,0,0),(0,1,0),(0,0,1)(v+1,v,v),(v,v+1,v),(v,v,v+1)\}$. For $(a,b,c)\in\mathbb{N}^3$ with $a\leqslant b\leqslant c$ and $q=\lfloor\frac{a+b+c}{3v+1}\rfloor$,*

$$d(a,b,c;S)=\begin{cases}a+b+c-3v\left\lfloor\dfrac{a}{v}\right\rfloor & \text{if } a<vq,\\ a+b+c-3vq & \text{if } vq\leqslant a.\end{cases}$$

Proof. It is sufficient to find the greatest value of ℓ where $|\mathcal{W}(a,b,c;S,\ell)|>0$. The proof of Theorem 3.1 shows that $|\mathcal{W}(a,b,c;S,\ell)|>0$ if and only if $\binom{a+b+c-3v\ell}{\ell}>0$ and $\binom{a+b+c-3v\ell}{a-v\ell,b-v\ell}>0$. Furthermore, $d(a,b,c;S)=a+b+c-3v\ell$ for this maximal value of ℓ.

The requirement that $\binom{a+b+c-3v\ell}{\ell}>0$ is satisfied if and only if $a+b+c-3v\ell\geqslant\ell$. This is equivalent to $a+b+c\geqslant(3v+1)\ell$. Furthermore, $\binom{a+b+c-3v\ell}{a-v\ell,b-v\ell}>0$ is satisfied if and only if $\min(a,b,c)\geqslant v\ell$. Our assumption that $a\leqslant b\leqslant c$ makes this requirement equivalent to $a\geqslant v\ell$. Thus, it is sufficient to find the largest value of ℓ for which $a+b+c\geqslant(3v+1)\ell$ and $a\geqslant v\ell$.

We first show that the maximum value that ℓ can attain in the case where $vq\leqslant a$ is q. If $vq\leqslant a$, the requirements that $a+b+c\geqslant(3v+1)q$ and $a\geqslant vq$ both hold. On the other hand, $a+b+c<(3v+1)(q+1)$. Hence, q is the maximum value of ℓ in this case.

We now find the maximum value that ℓ can attain in the case where $a<vq$. When $a<vq$, the point (a,b,c) will no longer be reachable using q long steps since $qv>a$. Because of this, the maximum value of ℓ is $\lfloor\frac{a}{v}\rfloor$. Moreover, $v\lfloor\frac{a}{v}\rfloor\leqslant a<vq$ implies

that $\ell < q$ for this value of ℓ, and the requirement that $a+b+c \geqslant (3v+1)\ell$ holds because $a+b+c \geqslant (3v+1)q > (3v+1)\ell$. On the other hand, $v(\lfloor \frac{a}{v} \rfloor + 1) > a$. Hence, the maximum value of ℓ is $\lfloor \frac{a}{v} \rfloor$. $\square$

Now we turn our attention to the general case $S = \{(1,0,0), (0,1,0), (0,0,1), (u,v,v), (v,u,v), (v,v,u)\}$.

Theorem 3.3. *Let $u > v$ and $S = \{(1,0,0), (0,1,0), (0,0,1), (u,v,v), (v,u,v), (v,v,u)\}$. For any $(a,b,c) \in \mathbb{N}^3$, the number of S-walks terminating at (a,b,c) that use ℓ long steps is*

$$\binom{a+b+c-2\ell(u+2v-1)}{\ell} \times \sum_{i=0}^{\ell} \sum_{j=0}^{\ell-i} \binom{a+b+c-2\ell(u+2v)}{a-ui-v(\ell-i),\, b-uj-v(\ell-j)} \binom{\ell}{i,j}. \tag{3.2}$$

Proof. Let an S-walk use i of the (u,v,v) steps, j of the (v,u,v) steps and $\ell-i-j$ of the (v,v,u) steps. The long steps contribute a factor of

$$i(u,v,v)+j(v,u,v)+(\ell-i-j)(v,v,u) = (ui+v(\ell-i), uj+v(\ell-j), u(\ell-i-j)+v(i+j)).$$

Considering the contribution on x, y and z axes, we see that the walk must use $a-ui-v(\ell-i)$ of the $(1,0,0)$ steps, $b-uj-v(\ell-j)$ of the $(0,1,0)$ steps and $c-u(\ell-i-j)-v(i+j)$ of the $(0,0,1)$ steps. The total length of such a walk can be found as follows: The long steps have a coordinate sum of $u+2v$. The total contribution of long steps is $\ell(u+2v)$. Then the length of such a walk is

$$a+b+c-\ell(u+2v)+\ell = a+b+c-\ell(u+2v-1).$$

Now we choose ℓ steps from $a+b+c-\ell(u+2v-1)$ positions to be long steps, then we choose which i and j of those ℓ long steps will be (u,v,v) and (v,u,v) steps, respectively. From the remaining $a+b+c-\ell(u+2v)$ positions, we choose $a-ui-v(\ell-i)$ of them to be $(1,0,0)$ steps and $b-uj-v(\ell-j)$ of them to be $(0,1,0)$ steps, and remaining steps will be $(0,0,1)$ steps. $\square$

4. Walks with steps of fixed length

In this section, we consider T_n-walks for

$$T_n = \{(1,0,0),(0,1,0),(0,0,1)\} \cup \{(i,j,n-i-j)\colon\ 0 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant n-i\}.$$

As before, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ are *short steps*, and steps of the form $(i,j,n-i-j)$ are *long steps*.

Theorem 4.1. *For all $n \geqslant 2$ and $a,b,c,\ell \in \mathbb{N}$, the number of T_n-walks that terminate at (a,b,c) and use ℓ long steps is given by the generating function*

$$\sum_{(a,b,c)\in\mathbb{N}^3} |\mathcal{W}(a,b,c;T_n,\ell)| x^a y^b z^c = \binom{a+b+c-(n-1)\ell}{\ell}\Bigl(\sum_{i=0}^{n}\sum_{j=0}^{n-i} x^i y^j z^{ni-j}\Bigr)^{\ell} (x+y+z)^{a+b+c-n\ell}. \tag{4.1}$$

b \ a	0	1	2	3	4	5	6
6	25	186	826	2856	8442	22260	53592
5	16	105	426	1379	3864	9702	22260
4	9	52	195	600	1610	3864	8442
3	4	21	76	230	600	1379	2856
2	1	6	24	76	195	426	826
1	0	1	6	21	52	105	186
0	0	0	1	4	9	16	25

Figure 4. Values of $|\mathcal{W}(a,b,1;T_3,1)|$ for and $0 \leqslant a,\ b \leqslant 6$.

Proof. To determine all possible ways to create an ordered list of ℓ *long steps*, we expand $\Bigl(\sum\limits_{i=0}^{n}\sum\limits_{j=0}^{n-i} x^i y^j z^{n-i-j}\Bigr)^{\ell}$. The total number of *short steps* required to reach

(a, b, c) is $a+b+c-n\ell$. Expanding $(x+y+z)^{a+b+c-n\ell}$ gives all possible ways to form an ordered list of $a+b+c-n\ell$ short steps. By multiplying these, we account for all ordered lists consisting of ℓ long steps followed by $a+b+c-n\ell$ short steps. Finally, we select ℓ positions from the total $a+b+c-(n-1)\ell$ positions to assign as long steps, given by $\binom{a+b+c-(n-1)\ell}{\ell}$. □

Figure 4 shows the values of $|\mathcal{W}(a, b, c; T_3, \ell)|$ for $c = 1$ and $\ell = 1$. We noticed that the numbers in the rows for $b = 0, 1$ appear in OEIS [6]. When $b = 0$, we get $(a-1)^2$ for $a \geqslant 1$, and it appears as OEIS [6] sequence A000290. When $b = 1$, the nonzero numbers $1, 6, 21, 52, 105, 186\ldots$ is OEIS [6] sequence A069778.

5. Walks for $S = \{(1,0), (0,1), (u,v), (v,u)\}$

One of the open problems left by Evoniuk et al. [1], Problem 10 asks for determining $d(a, b; S)$ and $\mathcal{W}(a, b; S)$ for $S = \{(1,0), (0,1), (u,v), (v,u)\}$ with $u > v \geqslant 1$. Iwanojko et al. [4], Theorem 3 solved this problem for the case $u = v+1$. Here, we give an answer to the general case of this problem. We say $(1,0)$ and $(0,1)$ are *short steps*, (u,v) and (v,u) are *long steps*.

Theorem 5.1. *Let $u > v \geqslant 1$ and $S = \{(1,0), (0,1), (u,v), (v,u)\}$. For any $(a, b) \in \mathbb{N}^2$,*

$$|\mathcal{W}(a, b; S, \ell)| = \binom{a+b-\ell(u+v-1)}{\ell} \sum_{i=0}^{\ell} \binom{a+b-\ell(u+v)}{a-iu-(\ell-i)v} \binom{\ell}{i}. \tag{5.1}$$

Proof. Let an S-walk use i of the (u,v) steps and $\ell-i$ of the (v,u) steps. The long steps contribute to the walk a factor of

$$i(u,v) + (\ell-i)(v,u) = (iu+(\ell-i)v, (\ell-i)u+iv).$$

We see that long steps contribute $iu+(\ell-i)v$ units on the x-axis and $(\ell-i)u+iv$ units on the y-axis. Hence, the number of $(1,0)$ steps must be $a-iu-(\ell-i)v$, and the number of $(0,1)$ steps must be $b-(\ell-i)u-iv$. The length of such a walk can be found as follows: The long steps have a coordinate sum of $u+v$. The total contribution of long steps is $\ell(u+v)$. Then the length of such a walk is

$$a+b-\ell(u+v)+\ell = a+b-\ell(u+v-1).$$

Now we choose ℓ steps from $a+b-\ell(u+v-1)$ positions to be long steps, then we choose i of those ℓ long steps to be (u,v) steps. Lastly, we choose $a-iu-(\ell-i)v$ steps to be $(1,0)$ steps from the remaining $a+b-\ell(u+v)$ positions. □

Corollary 5.2. *Let* $u > v \geqslant 1$ *and* $S = \{(1,0),(0,1),(u,v),(v,u)\}$. *For* $(a,b) \in \mathbb{N}^2$ *with* $a \leqslant b$, *write* $q = \lfloor \min(\frac{a}{v}, \frac{a+b}{u+v}) \rfloor$. *Then*

$$d(a,b;S) = \begin{cases} a+b-(u+v-1)q & \text{if } qu - \Big\lfloor \dfrac{a-qv}{u-v} \Big\rfloor (u-v) \leqslant b, \\ a+b-(u+v-1)(q-1) & \text{if } qu - \Big\lfloor \dfrac{a-qv}{u-v} \Big\rfloor (u-v) > b. \end{cases} \tag{5.2}$$

Proof. By Theorem 5.1, the length of an S-walk is $a+b-\ell(u+v-1)$, where ℓ is the total number of long steps used. To find the smallest length, we need to find the maximum value of ℓ where $|\mathcal{W}(a,b;S,\ell)|$ is nonzero. It follows from the proof of Theorem 5.1 that if $|\mathcal{W}(a,b;S,\ell)|$ is nonzero, then $a+b-\ell(u+v-1) \geqslant \ell$, which is equivalent to saying $\ell \leqslant \frac{a+b}{u+v}$. Assuming $a \leqslant b$ and $v < u$, we have $a \geqslant \ell v$. In conclusion, $\ell \leqslant \lfloor \min(\frac{a}{v}, \frac{a+b}{u+v}) \rfloor = q$.

The sum of q long steps might still not terminate at point (c,d), where $c \leqslant a$ and $b \leqslant d$. For example, if $(a,b) = (3,5)$, $u = 3$, and $v = 1$, then $q = 2$. The sum of 2 long steps can terminate at $(6,2)$, $(4,4)$ or $(2,6)$. In this case, none of the terminal points work for us. However, as we will show later in the proof, there is always a set of $q-1$ long steps that works.

Now we analyze under which conditions q long steps work. Assume q long steps are used. We can write $i+j = q$, where i denotes the number of (v,u) steps and j denotes the number of (u,v) steps. We start by considering all long steps are (v,u). The contribution of long steps on the x-axis is qv. Changing a (v,u) step to a (u,v) step increases this contribution by $u-v$. We claim that when $j = \lfloor \frac{a-qv}{u-v} \rfloor$, the contribution on the x-axis will be its maximum value while still not being greater than a.

Taking $j = \lfloor \frac{a-qv}{u-v} \rfloor$, the contribution of long steps on the y-axis would be $qu - \lfloor \frac{a-qv}{u-v} \rfloor (u-v)$. When $qu - \lfloor \frac{a-qv}{u-v} \rfloor (u-v) \leqslant b$ is satisfied, there exists a set of q long steps which results in the minimal length path. Otherwise, at least one (v,u) step must be changed to a (u,v) step, which would overshoot on the x-axis.

Now, we show that there always exists a set of $q-1$ long steps that allows the creation of a path terminating at (a,b). Let $j = \lfloor \frac{a-qv}{u-v} \rfloor$ as before. Then the contribution of long steps on the y-axis is $(q-1)u - \lfloor \frac{a-qv}{u-v} \rfloor (u-v)$. Note that $\lfloor \frac{a-qv}{u-v} \rfloor (u-v) \geqslant a-qv-(u-v)$, which implies $(q-1)u - \lfloor \frac{a-qv}{u-v} \rfloor (u-v) \leqslant (q-1)u - (a-qv-u+v)$. Since $q \leqslant \lfloor \frac{a+b}{u+v} \rfloor$,

$$(q-1)u - (a-qv-u+v) = q(u+v) - a - v \leqslant a+b-a-v < b.$$

This implies that summing j of the (u,v) steps and $q-j-1$ of the (v,u) steps terminates at (c,d), where $c \leqslant a$ and $d < b$. The rest of the path to (a,b) can be completed with short steps, resulting in a minimal length path. $\square$

6. Walks for $S = \{(1,0),(0,1),(v+1,v),(v,v+1)\}$ weakly below $y = x$

In [7], lattice paths with certain restrictions were studied. In [2], a study was made to determine the number of paths containing exactly k points that lie above a linear boundary for $S = \{(0,1),(1,0)\}$. Inspired by this, we modified the problem to find the number of S-walks containing exactly ℓ long steps and staying weakly below the line $y = x$ for $S = \{(1,0),(0,1),(v+1,v),(v,v+1)\}$. Recall that in Section 5, we considered sets of the form $S = \{(1,0),(0,1),(u,v),(v,u)\}$. In this section, we work with a special case $u = v+1$.

We begin by establishing notations and terminology that will be used in this section. We use $\mathcal{W}(a,b;S,\ell \mid x \geqslant y)$ to denote the set of paths from origin to (a,b) that stay weakly below the line $y = x$, and $|\mathcal{W}(a,b;S,\ell \mid x \geqslant y)|$ denotes the cardinality of that set. Let (e_1,e_2) be a step in S. We say $e_1 - e_2$ is the *x-factor* of that step. When enumerating S-walks that stay weakly below the line $y = x$, we consider the x-factors of the steps instead of the steps themselves. For example, if $\mathbf{s} = (1,0),(2,3),(1,0),(0,1)$, we consider the sequence 1, -1, 1, -1. The sum of all elements in the sequence must be $a-b$, and the sum of the first k elements for $1 \leqslant k \leqslant |\mathbf{s}|$, where $|\mathbf{s}|$ is the length of the path, must be non-negative. We call such a sequence *good sequence*. This will work because we want $x - y \geqslant 0$ for all points on $\mathbf{s}$ and x-factor is the increment of this value. Notice that the number of long steps only changes the length of the path, so this problem has a connection with the case when $S = \{(0,1),(1,0)\}$.

For $S = \{(0,1),(1,0)\}$ and $a \geqslant b$, $|\mathcal{W}(a,b;S,0 \mid x \geqslant y)| = \frac{a-b+1}{a+b+1}\binom{a+b+1}{a+1}$ ($\ell = 0$ because S does not have any long steps). These numbers are called *ballot numbers* and appear in sequence A009766. Krattenthaler [5], Corollary 10.3.2 proves this. Moreover, when $a = b$, we get Catalan numbers, which is OEIS [6] sequence A000108.

Theorem 6.1. *Let $S = \{(1,0),(0,1),(v+1,v),(v,v+1)$. For $(a,b) \in \mathbb{N}^2$ with $a \geqslant b$,*

$$|\mathcal{W}(a,b;S,\ell \mid x \geqslant y)| = \frac{a-b+1}{a+b-2\ell v+1}\binom{a+b-2\ell v+1}{a-\ell v+1}\binom{a+b-2\ell v}{\ell}. \tag{6.1}$$

Proof. We mentioned that $\frac{a-b+1}{a+b+1}\binom{a+b+1}{a+1}$ gives the number of S-walks that stay weakly below the line $y = x$ for $S = \{(1,0),(0,1)\}$. The x-factors of these two steps are 1 and (-1). This means $\frac{a-b+1}{a+b+1}\binom{a+b+1}{a+1}$ is the number of good sequences with a amount of 1's and b amount of (-1)'s.

Now we turn back to $S = \{(1,0),(0,1),(v+1,v),(v,v+1)$. Let i be the number of $(v+1,v)$ steps. Then the number of $(v,v+1)$ steps is $\ell - i$, the number of

$(1,0)$ steps is $a-i(v+1)-(\ell-i)v=a-\ell v-i$, and the number of $(0,1)$ steps is $b-iv-(\ell-i)(v+1)=b-\ell v-\ell+i$.

The total number of steps with an x-factor of 1 is $i+a-\ell v-i=a-\ell v$, and the total number of steps with an x-factor of (-1) is $\ell-i+b-\ell v-l+i=b-\ell v$. The number of good sequences with $a-\ell v$ amount of 1's and $b-\ell v$ amount of (-1)'s is

$$\frac{a-b+1}{a+b-2\ell v+1}\binom{a+b-2\ell v+1}{a-\ell v+1}.$$

Next, we choose which i of those $a-\ell v$ amount of 1's will correspond to $(v+1,v)$ steps, and which $\ell-i$ of those $b-\ell v$ amount of (-1)'s will correspond to $(v,v+1)$ steps. Therefore,

$$\begin{aligned}|\mathcal{W}(a,b;S,\ell\mid x\geqslant y)|&=\sum_{i=0}^{\ell}\binom{a-\ell v}{i}\binom{b-\ell v}{\ell-i}\frac{a-b+1}{a+b-2\ell v+1}\binom{a+b-2\ell v+1}{a-\ell v+1}\\&=\frac{a-b+1}{a+b-2\ell v+1}\binom{a+b-2\ell v+1}{a-\ell v+1}\sum_{i=0}^{\ell}\binom{a-\ell v}{i}\binom{b-\ell v}{\ell-i}\\&=\frac{a-b+1}{a+b-2\ell v+1}\binom{a+b-2\ell v+1}{a-\ell v+1}\binom{a+b-2\ell v}{\ell}.\end{aligned}$$

In the second line, we used Vandermonde's identity to collapse the summation. $\square$

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