Mathematica Bohemica

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Mathematica Bohemica, Vol. 151 (2026), No. 3, 495–504

Persistent URL: `http://dml.cz/dmlcz/153719`

ON SOME q-GENERALIZATION OF SUPER CATALAN NUMBERS

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Received December 17, 2024. Published online August 14, 2025.
Communicated by Carlos Alexis Gómez Ruíz

Abstract. The aim of this paper is to generalize the Super Catalan numbers $S(m,n)$ and some properties of them by virtue of q-Calculus. For this purpose, we focus on formulating several weighted sums and different representations of the Super Catalan number sequence. Moreover, we derive some q-identities involving the q-Super Catalan numbers as a generalization of the results obtained for the numbers $S(m,n)$.

Keywords: binomial coefficient; Catalan numbers, Super Catalan numbers, q-identity

MSC 2020: 11B65, 05A10, 11B83

1. Introduction

The sequence of Catalan numbers occurs in many counting problems in number theory and graph theory. It is known that Dyck words, ballot sequences, well formed sequence of parentheses, 2-lines standard-tableaux, ordered and binary trees are counted by Catalan numbers due to their structures directly related to the central binomial coefficients. Some of the important properties and uses of them can be seen in the references [2], [7], [8].

For $C_0 = 1$, nth Catalan number is defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{1.1}$$

where

$$\binom{2n}{n} = \frac{(2n)!}{n!n!} \tag{1.2}$$

is the central binomial coefficient. Catalan numbers are listed as an integer sequence

$$\{C_n\}_{n\geqslant 0} = \{1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots\}.$$

DOI: 10.21136/MB.2025.0166-24

The recursive definition of the Catalan numbers is given by

$$C_n = \frac{4n-2}{n+1} C_{n-1}, \tag{1.3}$$

for $n \geqslant 1$ (see [12]). Several generalizations of the Catalan numbers and their studies can be found in the literature such as [10], [16].

In the present paper, we investigate the number $S(m,n)$ that is known as a generalization of C_n and defined by

$$S(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}. \tag{1.4}$$

These numbers are referred to as Super Catalan numbers, where $m, n \geqslant 0$ and $S(0,0) = 1$ in [6]. It is clearly seen from definition (1.4) that $S(n,1) = S(1,n) = 2C_n$. The numbers $S(m,n)$ are given in the following table.

m/n	0	1	2	3	4	5	6	7	$\dots$
0	1	2	6	20	70	252	924	3432	$\dots$
1	2	2	4	10	28	84	264	858	$\dots$
2	6	4	6	12	28	72	198	572	$\dots$
3	20	10	12	20	40	90	220	572	$\dots$
4	70	28	28	40	70	140	308	728	$\dots$
5	252	84	72	90	140	252	504	1092	$\dots$
6	924	264	198	220	308	504	924	1848	$\dots$
7	3432	858	572	572	728	1092	1848	3432	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1. $S(m,n)$ Super Catalan numbers.

The following facts can be observed from the table:

$$S(n+1,n) = 2\binom{2n}{n}, \tag{1.5}$$

$$S(n+2,n) = 2\binom{2n}{n}\left(2 + \frac{1}{n+1}\right), \tag{1.6}$$

$$C_n = \frac{S(n+2,n) - S(n+1,n)}{2(n+2)}. \tag{1.7}$$

From definition (1.4), the following recursive definition can be clearly given:

$$S(m,n) = \frac{4n-2}{m+n} S(m,n-1). \tag{1.8}$$

A different recurrence relation for $S(m,n)$ was given by Dan Rubenstein as

$$4S(m,n) = S(m+1,n) + S(m,n+1). \tag{1.9}$$

The studies [3], [5], [8], [13], [14], [15] help to understand the content of the Super Catalan numbers more specifically. For example, in [3], Chen and Wang gave an interpretation for the Super Catalan numbers $S(m, m+s)$, for $s \geqslant 3$.

In [9], q-analogue of an integer n is defined by the formula

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k, \tag{1.10}$$

where $0 < q < 1$. Note that q-Calculus is based on the fact that

$$\lim_{q \to 1^-} [n]_q = n \tag{1.11}$$

and this property helps us to reduce q-identities to classical Calculus. Meanwhile q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

where

$$(a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$$

is the q-Pochhammer symbol [11]. In [1], [17], q-Super Catalan numbers are introduced as

$$S_q(m,n) = \frac{[2m]_q![2n]_q!}{[m]_q![n]_q![m+n]_q!}. \tag{1.12}$$

The following recurrence relation follows from definition (1.12):

$$S_q(m,n) = (1+q^n)\frac{[2n-1]_q}{[m+n]_q} S_q(m,n-1). \tag{1.13}$$

In [17], for $s, m \geqslant 0$, Warnaar and Zudilin proved that $S_q(m,n)$ is a polynomial with non-negative integer coefficients. In [1], Allen deduced that the q-analogue of Dan Rubenstein's recurrence is

$$(1+q^{n-m}) S_q(m,n) = \frac{q^{n-m}}{1+q^{m+1}} S_q(m+1,n) + \frac{1}{1+q^{n+1}} S_q(m,n+1). \tag{1.14}$$

From definition (1.12), one can obtain that

$$S_q(1,n) = (1+q)C_n(q), \tag{1.15}$$

where

$$C_n(q) = \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q, \quad C_0(q) = 1 \tag{1.16}$$

is the q-Catalan number introduced by Fürlinger and Hofbauer [4].

In this current study, we formulate some weighted sums involving the terms of $S(m,n)$ and C_n. In Section 3, we derive the q-forms of these identities as a generalization of the results of the Super Catalan numbers given in Section 2.

2. Some formulas related to the number $S(m,n)$

In this section, we establish some formulas that give the relations between central binomial coefficients and Super Catalan numbers.

In the following theorem, we deduce a new representation for the Super Catalan numbers.

Theorem 2.1. *For integers $m \geqslant 0$ and $n \geqslant 1$ we have*

$$S(m,n) = 2C_m \frac{(2n-1)!}{(m+2)^{(n-1)}(n-1)!}, \tag{2.1}$$

where $x^{(n)} = x(x+1)(x+2)\dots(x+n-1)$ is the rising factorial.

Proof. By applying the recursive relation (1.8) to $S(m,n)$, n times, we have

$$S(m,n) = \frac{(4n-2)}{m+n}\frac{(4(n-1)-2)}{m+n-1}\frac{(4(n-2)-2)}{m+n-2}\dots\frac{(4-2)}{m+1}S(m,0).$$

Since $S(m,0) = \binom{2m}{m}$, we obtain

$$\begin{aligned} S(m,n) &= \binom{2m}{m}\frac{(4n-2)(4n-6)\dots 2}{(m+1)(m+2)\dots(m+n)} \\ &= 2^n \frac{\binom{2m}{m}}{m+1}\frac{(2n-1)(2n-3)\dots 1}{(m+2)\dots(m+n)} \\ &= \frac{2^n C_m}{(m+2)^{(n-1)}}[(2n-1)(2n-3)\dots 1]. \end{aligned}$$

Then, by some elementary operations, we conclude that

$$S(m,n) = \frac{2^n C_m}{(m+2)^{(n-1)}}\frac{(2n-1)!}{2^{n-1}(n-1)!}.$$

Hence, the proof is done. □

In the next theorem, we derive a recurrence-like formula for the Super Catalan numbers.

Theorem 2.2. *For integers $m \geqslant 0$ and $n \geqslant 1$ we have*

$$\sum_{k=0}^{n-1} \frac{3k-m+1}{k+m+1} S(m,k) = S(m,n) - \binom{2m}{m}. \tag{2.2}$$

P r o o f. From recurrence (1.9) we can write

$$3\sum_{k=0}^{n} S(m,k) = \sum_{k=0}^{n} S(m+1,k) - \sum_{k=0}^{n} (S(m,k) - S(m,k+1)).$$

Since

$$\sum_{k=0}^{n} (S(m,k) - S(m,k+1))$$

is a telescopic sum, we get

$$3\sum_{k=0}^{n} S(m,k) = \sum_{k=0}^{n} S(m+1,k) + S(m,0) - S(m,n+1).$$

Then, we use the formula $S(m+1,k) = 2(2m+1)S(m,k)/(m+k+1)$ to get that

$$\begin{aligned} S(m,n+1) &= \binom{2m}{m} + \sum_{k=0}^{n} \left(2\frac{2m+1}{m+k+1} - 3\right) S(m,k) \\ &= \binom{2m}{m} + \sum_{k=0}^{n} \frac{3k-m+1}{k+m+1} S(m,k). \end{aligned}$$

By substituting $n-1$ for n, we complete the proof. □

As a result of Theorem 2.2, we have the following corollary by taking $m = 1$ and $m = n$ in formula (2.2), respectively.

Corollary 2.3. *For $n \geqslant 1$ we have*

$$C_n = 1 + \sum_{k=0}^{n-1} \frac{3k}{k+2} C_k \tag{2.3}$$

and

$$\sum_{k=0}^{n} \frac{3k-n+1}{k+n+1} S(n,k) = \binom{2n}{n}. \tag{2.4}$$

In the following theorem, we give a weighted summation formula for the Super Catalan numbers.

Theorem 2.4. *For $n \geqslant 1$ we have*

$$\sum_{k=1}^{n} \frac{4^{-k}}{n+k+1} S(n,k) = \frac{1}{2n+1}\left(C_n - 4^{-n}\binom{2n}{n}\right). \tag{2.5}$$

P r o o f. By making some necessary arrangements, we observe that

$$\sum_{k=1}^{n} 4^{-n-k} S(n,k) - \sum_{k=1}^{n} 4^{-n-k-1} S(n,k+1) = 4^{-n-1} S(n,1) - 4^{-2n-1} S(n,n+1).$$

Then, by using recurrence (1.8), we get

$$\sum_{k=1}^{n} 4^{-k}\left(1 - \frac{k+1/2}{n+k+1}\right) S(n,k) = 4^{-1} S(n,1) - 4^{-n-1} S(n,n+1).$$

Hence, we can write

$$\frac{2n+1}{2}\sum_{k=1}^{n} \frac{4^{-k}}{n+k+1} S(n,k) = 4^{-1} S(n,1) - 4^{-n-1} S(n,n+1).$$

Using identity (1.5) in the last equation, we complete the proof. □

3. Some q-identities involving the number $S_q(m,n)$

In this section, we derive some identities for $S_q(m,n)$ via q-Calculus as a generalization of the results given in Section 2.

We give a recursive summation formula for $S_q(m,n)$ in the following theorem.

Theorem 3.1. *For $m,n \geqslant 1$ we have*

$$S_q(m,n) = q^{m+1} \sum_{k=0}^{n-1} \left(\frac{[3k-m+1]_q + [k]_q (q^{k-m} - 1)}{[k+m+1]_q} \right) S_q(m,k) + \begin{bmatrix} 2m \\ m \end{bmatrix}_q. \tag{3.1}$$

P r o o f. By observing

$$\sum_{k=1}^{n} S_q(m,k) + S_q(m,n+1) = \sum_{k=1}^{n+1} S_q(m,k) = \sum_{k=0}^{n} S_q(m,k+1)$$

and using the formula

$$S_q(m,k+1) = (1+q^{k+1})\frac{[2k+1]_q}{[m+k+1]_q} S_q(m,k),$$

we get

$$\sum_{k=0}^{n} S_q(m,k) - S_q(m,0) + S_q(m,n+1) = \sum_{k=0}^{n} (1+q^{k+1})\frac{[2k+1]_q}{[m+k+1]_q} S_q(m,k).$$

Hence, we obtain

$$S_q(m,n+1) = \sum_{k=0}^{n} \Big((1+q^{k+1})\frac{[2k+1]_q}{[m+k+1]_q} - 1\Big) S_q(m,k) + \begin{bmatrix} 2m \\ m \end{bmatrix}_q.$$

By some elementary operations, one can obtain that

$$(1+q^{k+1})\frac{[2k+1]_q}{[m+k+1]_q} - 1 = q^{m+1}\Big(\frac{[3k-m+1]_q + [k]_q(q^{k-m}-1)}{[m+k+1]_q}\Big).$$

Substituting this into the last equation, we complete the proof. □

As a result of identity (3.1), we have the following corollary.

Corollary 3.2. *For $m, n \geqslant 1$ we have*

$$C_n(q) = \sum_{k=0}^{n-1} q^{k+1}\Big(\frac{[k](q^{k+1}+q+1)}{[k+2]}\Big) C_k(q) + 1, \tag{3.2}$$

$$\begin{bmatrix} 2n \\ n \end{bmatrix}_q = \sum_{k=0}^{n} \Big(\frac{[3k-n+1] + [k](q^{k-n}-1)}{[k+n+1]}\Big) S_q(n,k). \tag{3.3}$$

P r o o f. By taking $m = 1$ and $m = n$ in formula (3.1), we get the desired results, respectively. □

In the next theorem, we give a representation for $S_q(m,n)$ by using q-Catalan numbers.

Theorem 3.3. *For $s \geqslant 4$ we have*

(3.4)
$$S_q(m,m+s) = C_m(q)(-q^{m+s};q^{-1})_{\lceil s/2\rceil}\biggl(\prod_{k=\lceil s/2\rceil}^{s-1}[2m+2k+1]_q\biggr)\biggl(\prod_{k=2}^{\lfloor s/2\rfloor}\frac{1}{[m+k]_q}\biggr),$$

where $\lfloor n\rfloor$ and $\lceil n\rceil$ are the floor and ceiling function.

Proof. From the definition of $S_q(m,n)$ we can write

$$S_q(m,m+s) = \frac{[2m]_q!}{[m]_q![m]_q![m+1]_q}\frac{[2m+2s]_q[2m+2s-1]_q\dots[2m+s+1]_q}{[m+2]_q[m+3]_q\dots[m+s]_q}.$$

By using the formula $[2m]_q/[m]_q = 1+q^m$, we have

$$\begin{aligned}S_q(m,m+s) = C_m(q)(1+q^{m+s})(1+q^{m+s-1})\dots(1+q^{m+s/2+1})\\ \times\frac{[2m+2s-1]_q[2m+2s-2]_q\dots[2m+s+1]_q}{[m+2]_q[m+3]_q\dots[m+s/2]_q}\end{aligned}$$

and

$$\begin{aligned}S_q(m,m+s) = C_m(q)(1+q^{m+s})(1+q^{m+s-1})\dots(1+q^{m+(s+1)/2})\\ \times\frac{[2m+2s-1]_q[2m+2s-2]_q\dots[2m+s+2]_q}{[m+2]_q[m+3]_q\dots[m+(s-1)/2]_q}\end{aligned}$$

for s even and odd, respectively. Combining the last two equations by using the floor and ceiling functions, we complete the proof. □

Corollary 3.4. *For $s \geqslant 4$ we have*

$$S(m,m+s) = C_m 2^{\lceil s/2\rceil}\biggl(\prod_{k=\lceil s/2\rceil}^{s-1}(2m+2k+1)\biggr)\biggl(\prod_{k=2}^{\lfloor s/2\rfloor}\frac{1}{m+k}\biggr). \tag{3.5}$$

Proof. From property (1.11) we clearly have

$$\begin{aligned}&\lim_{q\to1^-} S_q(m,m+s)\\ &= \lim_{q\to1^-}\biggl(C_m(q)(-q^{m+s};q^{-1})_{\lceil s/2\rceil}\biggl(\prod_{k=\lceil s/2\rceil}^{s-1}[2m+2k+1]_q\biggr)\biggl(\prod_{k=2}^{\lfloor s/2\rfloor}\frac{1}{[m+k]_q}\biggr)\biggr).\end{aligned}$$

Hence, we obtain

$$S(m,m+s) = C_m(-1^{m+s};1)_{\lceil s/2\rceil}\Bigg(\prod_{k=\lceil s/2\rceil}^{s-1} 2m+2k+1\Bigg)\Bigg(\prod_{k=2}^{\lfloor s/2\rfloor}\frac{1}{m+k}\Bigg).$$

Since

$$(-1;1)_{\lceil s/2\rceil} = \prod_{k=0}^{\lceil s/2\rceil-1}(1+1^k) = 2^{\lceil s/2\rceil},$$

we get the desired result. $\square$

In the following theorem, we give two representations for $C_n(q)$ and $S_q(m,n)$ that are the q-analogues of (1.7) and (2.1), respectively.

Theorem 3.5. *For $m \geqslant 0$ and $n \geqslant 1$ we have the following:*

$$C_m(q) = \frac{S_q(m,m+2) - S_q(m,m+1) - (q-1)\begin{bmatrix}2m\\ m\end{bmatrix}_q (q^{m+1}+[2m+4]_q)}{(q^{m+2}+1)[m+2]_q}, \tag{3.6}$$

$$S_q(m,n) = (1+q^n)C_m(q)\frac{[2n-1]_q!}{[m+2]_q^{(n-1)}[n-1]_q!}. \tag{3.7}$$

P r o o f. To prove equation (3.6), we use definition (1.12) and write

$$\begin{aligned} S_q(k,k+2) - S_q(k,k+1) &= (q^{k+2}+1)\Bigg(q^{k+2}\begin{bmatrix}2k\\ k\end{bmatrix}_q + [k+2]_q C_k(q)\Bigg) \\ &\quad - (q^{k+1}+1)\begin{bmatrix}2k\\ k\end{bmatrix}_q \\ &= q^{k+1}\begin{bmatrix}2k\\ k\end{bmatrix}_q (q^{k+3}-1) + \begin{bmatrix}2k\\ k\end{bmatrix}_q (q^{k+2}-1) \\ &\quad + (q^{k+2}+1)[k+2]_q C_k(q). \end{aligned}$$

Then, by making necessary operations, we get

$$\begin{aligned} (q^{k+2}+1)[k+2]_q C_k(q) &= S_q(k,k+2) - S_q(k,k+1) \\ &\quad - (q-1)\begin{bmatrix}2k\\ k\end{bmatrix}_q (q^{k+1}[k+3]_q + [k+2]_q). \end{aligned}$$

By using $q^{k+1}[k+3]_q + [k+2]_q = q^{k+1} + [2k+4]$, we get the desired result. To prove (3.7), we apply recursion (1.13) to $S_q(m,n)$ n times and we obtain

$$S_q(m,n) = C_m(q)\frac{\prod\limits_{k=0}^{n-1}(1+q^{n-k})}{[m+2]_q^{(n-1)}}\,\frac{[2n-1]_q[2n-2]_q[2n-3]_q\ldots[1]_q}{[2n-2]_q[2n-4]_q[2n-6]_q\ldots[2]_q}.$$

By using the property $[2(n-k)]_q = (1+q^{n-k})[n-k]_q$, we obtain from the last equation that

$$S_q(m,n) = C(m)\frac{\prod\limits_{k=0}^{n-1}(1+q^{n-k})}{[m+2]_q^{(n-1)}[n-1]_q!}\,\frac{[2n-1]_q[2n-2]_q[2n-3]_q\dots[1]_q}{\prod\limits_{k=1}^{n-1}(1+q^{n-k})}.$$

Since $\prod\limits_{k=0}^{n-1}(1+q^{n-k}) = (1+q^n)\prod\limits_{k=1}^{n-1}(1+q^{n-k})$, the proof is done. $\square$

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