BERNARD BOLZANO
(OCTOBER 5, 1781 – DECEMBER 18, 1848)

October 5th, 1961 marked the 180th anniversary of the birth of Prague born Bernard Bolzano. Czechoslovak scientific institutions paid tribute to this day by holding a special meeting. Mathematicians throughout the world know Bolzano primarily as one of the pioneers of that trend of mathematical research in the nineteenth century which aimed at *a critical revision of the basic concepts of mathematical analysis.* In this respect Bolzano's work had much in common with those papers of his great contemporary, A. Cauchy, which dealt with the basic theorems of analysis. However, there is a substantial difference between the work of these two mathematicians. Bolzano was not merely a mathematician but also a philosopher and logician. He said himself that mathematics interested him primarily as a branch of philosophy and a means towards the practice of correct thinking. His work, inasfar as it concerned mathematical analysis, was therefore devoted almost exclusively to obtaining a better foundation of its most basic parts and cannot be compared with the tremendous breadth of Cauchy's work; this certain degree of one-sidedness is probably in connection with the lack of skill in mathematical technique, with which we often meet in Bolzano's work. On the other hand, in studying the basic concepts of mathematical analysis and their interrelations Bolzano goes much further and deeper than any one of his contemporaries. The character of his investigations on the foundations of analysis is perhaps best seen in the two works mentioned below.

The first is "Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege", published in 1817. This treatise contains a proof of the theorem that a function[1]) f, continuous in a closed interval $[a, b]$, assumes all the values between $f(a)$ and $f(b)$ inside this interval (Bolzano's formulation is slightly different). In the introduction Bolzano shows why this "intuitively evident" theorem requires proof and criticizes some incorrect "demonstrations". His concept of continuity (in an interval) is identical with today's concept and is the same as that used by Cauchy in his Cours d'Analyse (1821). Bolzano's proof is based on the theorem on the existence of the least upper bound of a bounded non-empty set of numbers. The theorem on the least upper bound is, in its turn, derived from the so-called *Bolzano-Cauchy condition* for the convergence of a sequence of real numbers (in

[1]) In the present article "function" always means a real function of one real variable.

Cauchy's works this condition appeared four years later). Bolzano also tries to prove the sufficiency of this condition; here, of course, there is a gap in his proof since the theory of real numbers did not exist in his time.

This work shows Bolzano's deep comprehension of the basic questions of analysis. The same sphere of problems is dealt with on a higher level and to a greater extent in a paper found among his manuscripts by M. Jašek around 1920 and published under the title "Functionenlehre" by the Royal Bohemian Learned Society in 1930. This treatise was to have formed a part of a large work entitled "Grössenlehre".

"Functionenlehre", which was probably written in the early thirties, consists of a foreword and two chapters, the first dealing with continuity and the second with derivatives. The depth of the conception as well as the systematic character and clarity of the exposition is apparent particularly in the chapter on the continuity of functions of one variable. In contrast to the earlier formulations by Bolzano and Cauchy, here the former introduces not only continuity in an interval but also *at a point* and even continuity *from the right and left*. That he was fully aware of the importance of this localization of continuity is clear from the fact that he gives an example of a function which is discontinuous at each point as well as an example of a function which is continuous exactly at one point. Bolzano then proves a number of theorems on continuous functions in the following order (we have re-written them in modern notation and terminology):

I. *If* $\limsup_{x\to c} |f(x)| = \infty$, *$f$ is not continuous at the point c.*

II. *A function which is continuous in a closed interval $[a, b]$ is bounded in it.*

III. *If f is continuous in a closed interval $[a, b]$ and if there exists a sequence of numbers $x_n \in [a, b]$ such that* $\lim_{n\to\infty} f(x_n) = c$, *then f assumes the value c in $[a, b]$.*

IV. *Each function, continuous in a closed interval $[a, b]$, assumes there the greatest and smallest value.* (It is of interest how careful Bolzano's formulations are: "greatest in the sense that no other value is greater".)

V. *A function f, continuous in an interval J, has the following property:*
(A) *If $\alpha \in J$, $\beta \in J$, $f(\alpha) \neq f(\beta)$, then f assumes all values between $f(\alpha)$ and $f(\beta)$ in the open interval (α, β).* (This is of course the theorem from "Rein analytischer Beweis ...".)

The proofs of the theorems are built up systematically from the theorem on the least upper bound and the so-called Bolzano-Weierstrass theorem: Each bounded sequence of numbers has an accumulation point. Bolzano was fully aware of the importance of the compactness of a closed interval: he shows that theorems III and IV are not valid for open intervals. He also emphasizes that property (A) is not sufficient for a function to be continuous in J — but here his arguments are not satisfactory.

The theorems mentioned so far have a common characteristic feature: they show that in addition to the properties stated in the definition of continuity all continuous functions have certain other properties, which are in agreement with the naive conception of "continuity". Nonetheless, Bolzano's study of the notion of continuity follows also another, almost opposite direction. He constructs examples of functions which are continuous and nevertheless possess some properties which disagree with the naive conception of continuity and therefore appeared paradoxal in his time. The most important and best known of them is the so-called *Bolzano's function.*

This function F is defined in an interval $[a, b]$ as the limit of a sequence of piecewise linear functions $y_1, y_2, \ldots$, the graphs of which are obtained from a segment by iterating a certain simple geometric construction. Bolzano shows that the function F, although continuous in $[a, b]$, is not monotonic in any interval and has not a finite derivative at any point of a certain set dense in $[a, b]$. We know today that F has no (neither finite nor infinite) derivative at any point of the interval (a, b), but Bolzano neither proved nor asserted it. Bolzano's proof contains one serious gap: he deduces the continuity of the function F simply from the fact that F is the limit of certain continuous functions y_n. Thus he did not arrive at the concept of uniformity or recognize its importance; this caused some serious mistakes in his work. This is the more surprising since in other places he emphatically warns against errors of this kind.

Bolzano's deep research into the general properties of continuous functions, and particularly the construction of functions with "paradoxal" properties, show him as the immediate forerunner of the modern theory of real functions. It is a pity that his "Functionenlehre" was not published earlier, at a time when it could have still influenced and accelerated the development of this branch of mathematics.

To complete Bolzano's theory of functions of one real variable, one gap had to be filled, namely the *theory of real numbers* had to be created. It is well known that several such theories were put forward around 1870 (Cantor, Dedekind, Méray, Weierstrass). As K. Rychlík reports (Theorie der reellen Zahlen in Bolzanos handschriftlichem Nachlasse, Czech. Math. Journal 7 (82) (1957), pp. 553–567), Bolzano left behind a manuscript containing an attempt at a systematic theory of real numbers; it culminates in a proof of sufficiency of the Bolzano-Cauchy criterion for the convergence of a sequence, of the theorem on the least upper bound, and of Dedekind's theorem. Bolzano's attempt is remarkable for his time but not quite successful. His fault was that he tried to include in his theory all the "expressions" containing infinitely many arithmetical operations with rational numbers ("unendliche Zahlenausdrücke"); under restriction, for example, to series with rational terms, Bolzano's attempt could probably be brought to a successful end.

Just as Bolzano's interest in philosophy and logics is seen in his mathematical work, so his mathematical turn of mind manifests itself in his large book on logic "Wissenschaftslehre" (1837); Bolzano's importance as the forerunner of *mathematical logic* is universally recognized. In this direction, too, new discoveries can be expected from Bolzano's legacy, as is shown in K. Rychlík's report "Betrachtungen aus der Logik in Bolzanos handschriftlichem Nachlasse" (Czech. Math. Journal 8 (83) (1958), pp. 197–202).

The large number of manuscripts left by Bolzano has not yet been fully explored; many sketches and drafts will perhaps remain undecipherable for ever. However, a careful study of this wealth of material can be expected to throw new light on the work of this deep and original thinker.

Bolzano was not only a mathematician, logician and philosopher but also an outstanding *social thinker*. In the suffocating atmosphere of the Austrian absolute monarchy after the French Revolution, when reaction was growing even stronger, he became one of the first to advance new ideas in Bohemia, and paid for this by being relieved of his post of professor at the university. However, his activities along these lines did not cease, and at the beginning of the thirties of the 19th century he published the first and only comprehensive *socialist Utopia* to appear in our country.

Bolzano started from a criticism of the contemporary state, and regarded the inequality of classes and property as its greatest evil. He created the image of the *ideal state* which was to be a Republic where the function of parliament was to be replaced by a plebiscite. The state was to be the governing body also in the field of the national economy, the owner of the land, of the means of production and of part of the consumer's goods.The development of production and particularly of machine production would then have unsuspected possibilities. Trade and finance was to be in the hands of the state. Children were to be educated and fed at the state's expense, and the state also was to run medical services for its citizens and to provide for those unable to work. The weakness of the progressive components of Czech society is reflected in Bolzano's writings by his disbelief that his Utopia will come true, by his fundamental rejection of revolutionary methods, and by his belief that class conflicts could be solved by persuasion. Because of the backwardness of economic conditions Bolzano did not believe that the needs of the whole population could be fully satisfied, and hence he preached the necessity of ascetic restriction and of the equalisation of consumption. On the whole, Bolzano's Utopia in content and form belongs to the sphere of socialist Utopias of the 18th century, being probably closest to that of Mably.

Much of the teaching of the great humanist Bolzano certainly belongs to the past. However, in the present tense international conditions whole parts of his work speak to us with great urgency. At the time when an irresponsible decision could bring destruction to large sections of humanity and seriously threaten the civilisation, Bolzano's conviction of the power of human reason, the possibility of agreement between peoples and the sensible and universally beneficient arrangement of the world become very topical. In Bolzano this opinion was not only the result of rational considerations but also of painful personal experience. The Napoleonic Wars which had devastated Europe for decades broke out when Bolzano was a child, and certainly influenced his teaching to a great extent. For this reasons his Addresses contain such vivid descriptions of the horrors of war, in which on the one hand thousands of people die by the sword and on the other hand thousands succumb to hunger, frost and plague. However, in the darkest periods Bolzano believed in the progressive development of society, in the victory of human reason and healthy optimism, and professed prophetically (1811):

"*Es wird – ich sage es mit aller Zuversicht – es wird eine Zeit erscheinen, wo man den Krieg, dieses widersinnige Bestreben, sein Recht durchs Schwert zu beweisen, eben so allgemein verabscheuen wird, wie man den Zweikampf jetzt schon verabscheuet!*"[2])

It is symbolic that today, exactly 150 years later, this question has come to the fore so much, and that not all members of human society have been yet convinced that war is an impermissible method of solving conflicts between states. In such a situation, responsibility for the development of the world lies mainly with those who preserve healthy reasoning so stressed by Bolzano. The role of the intelligentsia to whose conscience Bolzano primarily appealed is increasing. Only when there is agreement and all people of good will unite, when there is peace can Bolzano's motto "*Fortschreiten soll man*" be realised. Only under such conditions can real happiness spread over the earth as described by Bolzano:

"*Unser Geschlecht wird endlich auch immer weiterschreiten in wahrer Glückligkeit, das ist, das Heer der Leiden, welche uns drücken, wird in der Folge der Zeiten sich immer mehr und mehr verringern, je länger, je wirksamer werden die Mittel sein, die man zu ihrer Abhilfe erfunden haben wird, die Zahl derjenigen aus uns, die sich unglücklich fühlen, wird immer kleiner werden, und immer grösser die Anzahl jener, die eine naturgemässe Befriedigung ihrer menschlichen Bedürfnisse auf Erden finden, die ruhig und vergnügt ihr Dasein zubringen, und alt und lebenssatt dem Tode ohne Murren in die Arme sinken, weil auch sie sagen können: dass sie gelebt, und dieser Erde Glück genossen haben.*" (1811).[3])

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[2]) Time will come – I say it with all the responsibility – when War, this contradictory effort to prove one's right by sword, will be condemned as universally as duel is condemned already now.

[3]) Humanity will eventually reach the true felicity, since the reign of distresses which oppress us will in the course of time gradually abate; the means which mankind will find as its remedy will be more and more effective; the number of those of us who feel unhappy will constantly decrease while there will be more and more of those who will find natural satisfaction of their human needs on Earth, and old and sated with life will sink in the arms of Death without grumbling, as even they will be able to say that they lived and shared the happiness of this Earth.