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LIFE AND WORK OF KAREL RYCHLÍK

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1 Life of Karel Rychlík (1885 – 1968)

Karel Rychlík was born on April 16, 1885 in Benešov near Prague as the first of the three children of Barbora Srbová, married Rychlíková (1865 – 1928), and Vilém Evžen Rychlík (1857 – 1923). [1] In October 1904 Karel Rychlík started to study mathematics and physics at the Faculty of Arts of Czech Charles–Ferdinand University in Prague (below only Charles University). He was influenced above all by Professor Karel Petr. In the school year 1907/08 Rychlík was studying at Faculté des Sciences in Paris. He was mainly interested in the lectures of Jacques Hadamard (winter semester) and Émile Picard (summer semester) called *Analyse supérieure*. Besides, Rychlík was attending the lectures of Darboux, Goursat, Raffy, Painlevé and Marie Curie at the same faculty, and the lectures on number theory at Collège de France, read by Georges Humbert. During his stay in Paris Rychlík was also working on his dissertation. On December 16, 1908 he passed the so-called "teacher examination". At the end of the year 1908 he also handed

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[1]Karel Rychlík had a younger brother Vilém (1887 – 1913), who was a brilliant mathematician, too. Karel used to say his brother had been much cleverer than him. It is a riddle because Vilém died very young, at the age of 26. He had just finished the study of mathematics and physics at the Faculty of Arts of Charles University, received the degree Doctor of Philosophy, become an assistant at Czech Technical University in Prague and he had written several treatises. He is told being very lively, loving women and smoking 40 cigarettes a day, which became fateful for him. One day he caught a cold somewhere and within three days he died (that was called a fast consumption).

Their younger sister Jana studied, as an adjunct student, mathematics and biology at the Faculty of Arts and became a biology teacher. But soon she married Václav Špála, later the famous Czech painter, and gave precedence to her husband and children.

in his dissertation, whose most interesting part had already been published [R3]. The review written by Karel Petr and signed also by Jan Sobotka assessed Rychlík's work as excellent. In March of the following year Rychlík passed the two hour lasting rigorosum examination of mathematics and mathematical physics and one hour lasting rigorosum examination of philosophy. On March 30, 1909 Rychlík was awarded the degree Doctor of Philosophy.

From 1909 till 1913 Rychlík worked as an assistant of the mathematical seminar at the Faculty of Arts of Charles University. On January 18, 1912 the board of professors of the university decided unanimously on appointing Rychlík associate professor (Docent). He was officially appointed by the Ministry of Culture and Education on March 15, 1912. The inceptive procedure started on November 17, 1910 and consisted of the assessment of the inceptive treatises [R4] and [R7] devoted to the form theory and of other works published earlier, namely [R1], [R2] and [R3] (the favorable report was written by K. Petr and confirmed by F. Koláček and J. Sobotka), the inception colloquium and the "lecture on trial" named *The Evolution of the Concept of Divisibility*. As a "private associate professor" [2] Rychlík had lectured at the university till 1938. In 1919 the board of professors decided on his appointment adjunct professor, in addition to the present chairs, but their suggestion remained in the ministry and was not put into practice (the financial situation of the school system was not very good). In the end, on November 27, 1920, Rychlík became a professor at the Czech Technical University,[3] where he had been working as an assistant since 1913. In October 1914 he undertook the duties of the professor F. Velísek, who enlisted and died in the war. Rychlík began to read base lectures alternately for students of the first and second year of study, the lecture on probability theory and the lecture on vector analysis.

From today's view, it was a pity that Rychlík remained only private associate professor at Charles University. The main subject of his research was algebra and number theory. It was possible, even necessary, to read such topics at Charles University. In fact, Rychlík was the first who introduced methods and concepts of "modern" abstract algebra in our country – by means of the published treatises as well as his university lectures. Besides, as a professor there he would have had a stronger influence on the young generation of Czech mathematicians. But Rychlík

[2] This position was not paid in general.

[3] First he became an adjunct professor, later, on March 12, 1924, he was appointed full professor (with the validity since December 31, 1923).

spent most of his time (and energy) at the Technical University where he had to adapt his lectures for future engineers. Nevertheless, he approached his work seriously there. In addition to the usual teaching activities, he was a member of many committees, such as organization committee, inceptive committees, etc.

On October 22, 1918 Karel Rychlík married Marie Benešová, whose father was a head supervisor of state railways (let us remark that the word "rychlík" means a "fast train"). Their only daughter, Marie Rychlíková, later an academic sculptor, was born on November 3, 1923.

In 1904 Rychlík became a member of the *Union of Czech Mathematicians and Physicists* (below only the Union) and until World War II he was also a member of its committee. Almost the whole of his life Rychlík lectured in the Union and his lectures were very close related to his scientific research. He was also a member of the *Royal Bohemian Society of Sciences* (elected on January 11, 1922), the *Czech Academy of Sciences and Arts* (May 23, 1924) and the *Czechoslovak National Research Council* under the Academy (May 19, 1925).

Rychlík took part in several international congresses: *5th Congress of Czech Naturalists and Physicians* in Prague (1914; contribution [R11]), *International Congress of Mathematicians* in Strassbourg (1920), *6th Congress of Czechoslovak Naturalists, Physicians and Engineers* in Prague (1928), *International Congress of Mathematicians* in Bologna (1928; contrib. [R28]), *Congress of Mathematicians of Slavonic Countries* in Warszawa (1929; contrib. [R33]) and *Second Congress of Mathematicians of Slavonic Countries* in Prague (1934; the member of the organize committee, the chairman of the first section named *Foundations and Philosophy of Mathematics*, together with V. Hlavatý, and of the second section named *Arithmetics and Algebra,* together with V. Kořínek; contrib. [R42]).

In 1939 all Czech universities were closed, after the war Rychlík was retired. In the last period of his life Rychlík invested his energy to the history of mathematics, above all to the inheritance of Bernard Bolzano, which he had been interested in since his youth, but after the retirement he was engaged in this topic fully.

Karel Rychlík died on May 28, 1968 at the age of 83; he fell prey to the cancer of the urinary bladder.

2 Work of Karel Rychlík

Rychlík's works can be divided into five groups:

1. Algebra and Number Theory (22 works),
2. Mathematical Analysis (7),
3. Works Devoted to Bernard Bolzano (13),
4. Other Works on History of Mathematics (29),
5. Textbooks, Popularization Papers and Translations (16).

In the Czech mathematical community, Rychlík's name is mostly related to his textbooks on elementary number theory ([R36], [R45]) and on the theory of polynomials with real coefficients ([R63]), which are certainly very interesting and useful, but which are not "real" scientific contributions. Worth mentioning is the less known textbook [R43] (1938) on probability theory, written for students of technical university, yet in a very topical way. Here Rychlík builds the probability theory using the axiomatic method that is similar to the one of Kolmogorov [17] and that is thoroughly elaborated – as for the axioms as well as the intimately described proofs. In this context, let us also mention the popularization papers [R5] and [R6] on the special cases ($n = 3, 4, 5$) of the Fermat last theorem, which are cited in the book [38] of P. Ribenboim.

Not only among mathematicians and not only in Bohemia, Rychlík is widely known as the historian of mathematics, above all in the connection with Bernard Bolzano. Rychlík's activities related to Bolzano's manuscripts are discussed in a separate paper in these proceedings, named *Remarks on Bolzano's Inheritance Research in Bohemia.* But also a range of other papers on the history of mathematics more or less relates to Bolzano, namely the works devoted to N. H. Abel ([R87]), A.-L. Cauchy ([R57], [R58], [R59], [R60], [R68], [R85]) and the prize of the Royal Bohemian Society of Sciences for the problem of the solution of any algebraic equation of the degree higher then four in radicals ([R80], [R81]). Some of the remaining papers are only short reports ([R44], [R53], [R54], [R61]) or loose processings of literature ([R75], [R77]), the others contain a good deal of an original work based on primary sources, namely the papers devoted to É. Galois ([R62]), F. Korálek ([R79]), M. Lerch ([R27], [R73]), E. Noether ([R70]), F. Rádl ([R55], [R56]), B. Tichánek ([R25], [R74], [R78]), E. W. Tschirnhaus ([R76]) and F. Velísek ([R20]). Moreover, Rychlík adds his own views and valuable observations, which shows his wide insight and deep interest in the history of mathematics and in mathematics itself. On October 21, 1968 the Czechoslovak Academy of Sciences awarded Rychlík in memoriam a prize for the series of 13 papers on the history of mathematics published after 1957, namely [R57], [R58], [R59], [R62], [R64], [R66], [R67],

[R68], [R70], [R76], [R77], [R81] and [R87].

Rychlík's algebraic works are known only to a relatively narrow circle of mathematicians. In this contribution we will discuss just this first group which includes the most important mathematical papers concerning algebra and number theory. We will omit the works on mathematical analysis, which were rather occasional – although they also contain a lot of interesting ideas. The reader, who would like to know more about Rychlík's life or work, can find further information on Rychlík's internet pages.[4]

The most important mathematical papers of Karel Rychlík can be divided as follows.

Works on Algebra and Number Theory

Principal Papers

- **g**–adic Numbers .... [R11], [R12], [R17], [R21]
- Valuation Theory .... [R14], [R22]
- Algebraic Numbers, Abstract Algebra ... [R15], [R16], [R23], [R24], [R26]
  [R31], [R32], [R33], [R38], [R39]
- Determinant Theory .... [R37], [R42]

Other Works

- Theory of equations .... [R2]
- Theory of Algebraic Forms .... [R4], [R7]
- Group Theory .... [R3]

Figure 2 illustrates the influences in the development of the algebraic number theory. The aim of the scheme is to show Rychlík's place there; hence there are not all existing influences – the predetermination of the figure would be covered up. It just tries to show the two main streams, the *ideal theory* represented by R. J. W. Dedekind and his continuators, and the *divisor theory* represented by L. Kronecker, his student K. Hensel, his student H. Hasse and other mathematicians, including Karel Rychlík. The two approaches are put well in the preface to the book [9] by H. Hasse:

There are two quiet distinct approaches, the divisor–theoretic and the ideal–theoretic, to the theory of algebraic numbers. The first, based on the arithmetic researches of Kummer and Kronecker and on the function–theoretic methods of Weierstrass, was developed by Hensel at the turn of the century; it was expanded by the general field theory

[4] http://euler.fd.cvut.cz/publikace/HTM/Index.html

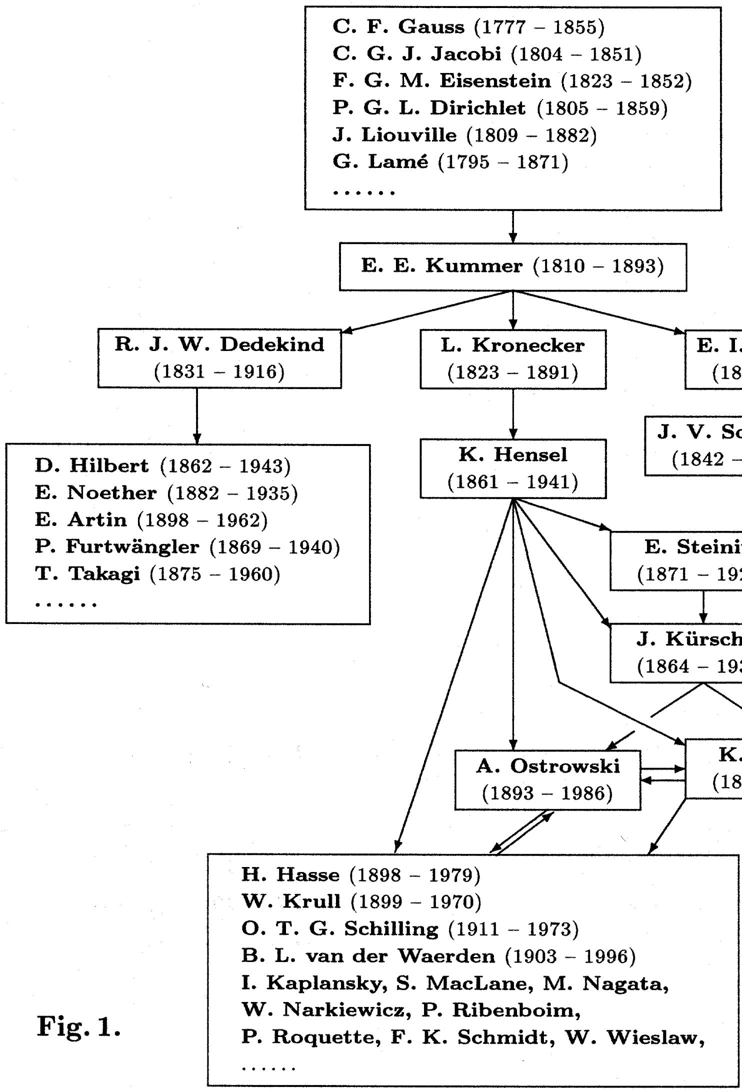

C. F. Gauss (1777 – 1855)
C. G. J. Jacobi (1804 – 1851)
F. G. M. Eisenstein (1823 – 1852)
P. G. L. Dirichlet (1805 – 1859)
J. Liouville (1809 – 1882)
G. Lamé (1795 – 1871)
......
E. E. Kummer (1810 – 1893)
R. J. W. Dedekind (1831 – 1916)
L. Kronecker (1823 – 1891)
E. I. Zolotarev (1847 – 1878)
J. V. Sochocki (1842 – 1927)
D. Hilbert (1862 – 1943)
E. Noether (1882 – 1935)
E. Artin (1898 – 1962)
P. Furtwängler (1869 – 1940)
T. Takagi (1875 – 1960)
......
K. Hensel (1861 – 1941)
E. Steinitz (1871 – 1928)
J. Kürschák (1864 – 1933)
A. Ostrowski (1893 – 1986)
K. Rychlík (1885 – 1968)
H. Hasse (1898 – 1979)
W. Krull (1899 – 1970)
O. T. G. Schilling (1911 – 1973)
B. L. van der Waerden (1903 – 1996)
I. Kaplansky, S. MacLane, M. Nagata,
W. Narkiewicz, P. Ribenboim,
P. Roquette, F. K. Schmidt, W. Wieslaw,
......

Fig. 1.

of Steinitz and the general valuation theory of Kürschák, Ostrowski, and others. The second approach was conceived somewhat earlier by Dedekind, further developed by Hilbert, and was then expanded by the general ideal theory of Emmy Noether, Artin, and others.

It seemed at first that the ideal–theoretic approach was superior to the divisor–theoretic, not only because it led to its goal more rapidly and with less effort, but also because of its usefulness in more advanced number theoretic research. For Hilbert and, after him, Furtwängler and Takagi succeeded in constructing on this foundation the imposing structure of class field theory, including the general reciprocity law for algebraic numbers, whereas on Hensel's side no such progress was recorded. More recently however, it turned out, first in the theory of quadratic forms and then especially in the theory of hypercomplex numbers (algebras), not only that the divisor–theoretic or valuation–theoretic approach is capable of expressing the arithmetic structural laws more simply and naturally, by making it possible to carry over the well–known connection between local and global relations from function theory to arithmetic, but also that the true significance of class field theory and the general reciprocity law of algebraic numbers are revealed only through this approach. Thus, the scales now tip in favor of the divisor–theoretic approach.[5]

On figure 2 the survey of quotations in Rychlík's principal algebraic papers can be seen (except the two papers on determinant theory which stay somewhat aside). It is evident that Rychlík was influenced above all by K. Hensel.[6] Notice that the works were published between 1914 and 1932, that is in the period of the birth and formation of the "modern" abstract algebra. Regrettably only a few Rychlík's papers were published in a generally renowned magazine – *Crelle's Journal*; the most of them were published in de facto local Bohemian journals. It was certainly meritorious for the enlightenment in the Czech mathematical public, but although some of the works were written in German, they were not noticed by the mathematical community abroad, even though they were referred in *Jahrbuch* or *Zentralblatt*. On the other hand, Rychlík's papers published in Crelle's Journal became known and they have been cited in the literature. Nevertheless, it was not only Rychlík who published mostly for the Czech audience. In fact, this situation was common that time in the young autonomous republic.

[5] [9] – the quotation from the English translation published in 1972, p. VI.

[6] Besides the cited published papers Rychlík lectured on these subjects in his talks in the Union; already in the "administrative year" (between the two December general meetings) 1908–1909 he had a lecture named *On Algebraic Numbers according to Kurt Hensel.*

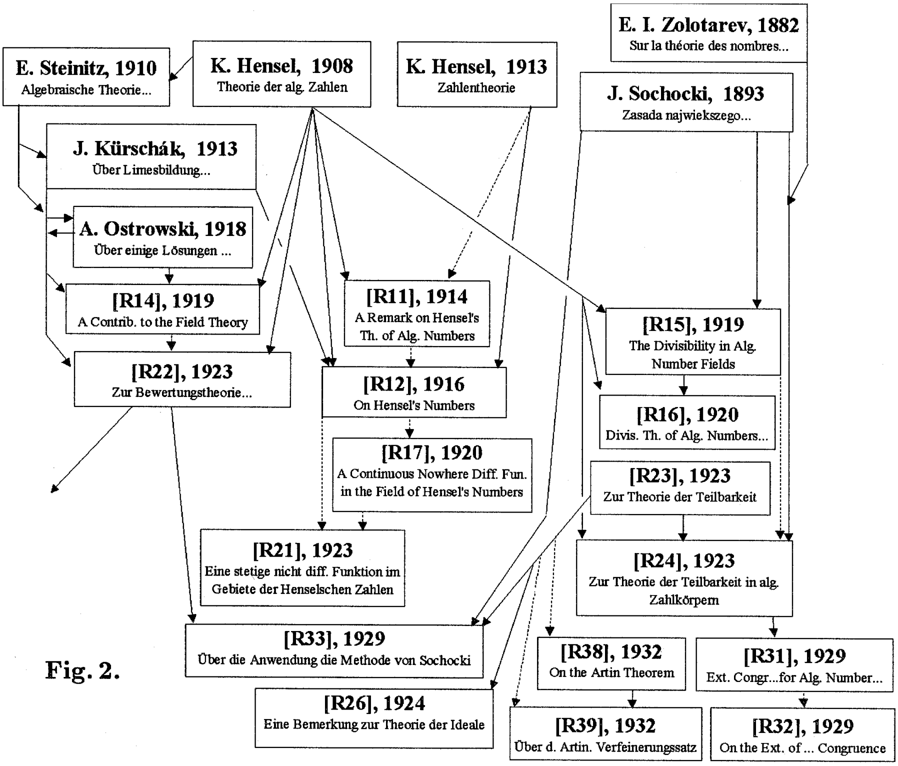


Fig. 2.

In his papers Rychlík mostly came out of a certain work (see fig. 2) and gave some improvement – mainly he based definitions of the main concepts or proofs of the main theorems on another base, in the spirit of abstract algebra, which meant the generalization or simplification. The papers mark out by the brevity, conciseness, topicality as well as the "modern" way of writing (from the point of view of that time).

2.1 g–adic Numbers

Two of Rychlík's papers are devoted directly to the concept of g–adic numbers itself, introduced by Kurt Hensel in his paper [12] (although it was preceded by other works where these ideas had been forming). From all Hensel's works on this topic let us mention only the books [13] and [14] which are often cited in Rychlík's papers.

A Remark on Hensel's Theory of Algebraic Numbers [R11] is the extract of the lecture at the *5th Congress of Czech Naturalists and Physicians*, that took place in Prague in 1914. First Rychlík reminds the additive normal form of a g–adic number, which can be easily transfered to the multiplicative normal form. Then he extends these ideas to algebraic number fields. He cites Hensel's work [15] published in the same year, where the mentioned generalization is made for quadratic number fields. As for the concepts concerning algebraic numbers, Rychlík refers to Hensel's book [13].

On Hensel's Numbers **[R12], 1916**

The second paper is devoted to the introduction and properties of the ring of g–adic numbers. Rychlík cites Hensel's books [13] and [14], and the treatise [43] of E. Steinitz. While Hensel took the way analogical to the construction of the field of real numbers by means of decadic expansions, Rychlík came out – like Cantor – from the concept of fundamental sequence and limit (as he notes, one of the merits of this approach is that directly from the definition it is immediately seen that the ring of g–adic numbers depends only on primes contained in g, not on their powers). Of course, the idea of constructing the field of p–adic numbers (for a prime p) came from Kürschák [22], who introduced the concept of *valuation* (see 2.2). Rychlík generalized the notion of a limit in a little bit different way, closer to Hensel. Moreover, he studied comprehensively rings of g–adic numbers for a composite number g. Kürschák's paper [22] is cited only in the postscript which seems to be written subsequently. It is plausible he came to the idea of the generalization of Cantor's approach independently of Kürschák.[7]

In the mentioned postscript Rychlík generalized Kürschák's technique for the case of the composite number g and defined what was later called a *pseudo-valuation* of a ring R[8] as a mapping $\|\cdot\|$ of R into the set of non-negative real numbers, which satisfies the following conditions:

$$\|a\| > 0 \text{ if } a \in R,\ a \neq 0; \quad \|0\| = 0, \tag{PV1}$$
$$\|a+b\| \leq \|a\| + \|b\| \quad \text{for all } a, b \in R, \tag{PV2}$$
$$\|ab\| \leq \|a\| \,.\, \|b\| \quad \text{for all } a, b \in R. \tag{PV3}$$

[7] We have already mentioned that Rychlík had been involved in this topics at least since 1908/09 (see footnote 6) and trying to improve Hensel's ideas – here the solid foundation of a basic concept was at the first place.

[8] It is almost unknown but interesting that Rychlík defined this concept 20 years before the publication of Mahler's paper [23], which is usually considered as a work where the general pseudo-valuation (Pseudobewertung) was introduced (p. 81, see also e.g. [26], p. 12). At the end of the paper [24] K. Mahler himself remarked that pseudo-valuations had already appeared in the work [2] of M. Deuring (chap. VI, §10, 11) published in 1935, namely for hypercomplex systems, but he had found it out after the printing of the previous paper [23].

Then it is possible to set up notions of a limit and a fundamental sequence and to extend a ring to a complete one. The special case is the ring of g–adic numbers, which is the completion of the rational number field $\mathbb{Q}$, provided the value $\|a\| = e^{-\rho}$ for $a = g^\rho \overline{a}$, $\rho \in \mathbb{Z}$, $\overline{a}$ is an integer with respect to g (see footnote 9), $g^{\rho+1}$ does not divide a, is considered.

Back to the article. Rychlík defines the notions of integers, units, divisibility, equivalence (marked $a \sim b$ (g)) and congruence *with respect to g*, according to Hensel's [14].[9] Moreover, Rychlík defines the *order with respect to g*: $a < b$ (g) iff a is divisible by b, but is not equivalent with b. He derives various properties of the above concepts and considers sequences of rational numbers $a_1, a_2, \dots, a_n, \dots$.

DEFINITION 2. *The sequence has a rational number a as a limit with respect to g,* $\lim_{n\to\infty} a_n = a$ (g), *if for every rational number d it is possible to find a positive integer N such that for every* $n \geq N$ *it is* $a_n - a < d$ (g).[10]

Similarly he defines a *fundamental sequence with respect to g* [11] and proves the series of propositions, e.g. that every convergent sequence is also fundamental, but not vice versa. In this very moment Rychlík defines g–adic numbers:

DEFINITION 3. *We will assign to every sequence of rational numbers, which is fundamental with respect to g and which has not a rational number as a limit, a new number which we will call its limit.*

Limits of sequences convergent with respect to g will be called g–adic numbers. So there are included also rational numbers in the domain of g–adic numbers …

We will not assign to every sequence convergent with respect to g a separate limit, but to two sequences $a_1, a_2, \dots, a_n, \dots,$ $b_1, b_2, \dots, b_n \dots,$ *convergent with respect to g, we assign the same g–adic number* $A = B$ (g), $A = \lim_{n\to\infty} a_n$ (g), $B = \lim_{n\to\infty} b_n$ (g), *if*

$$\lim_{n\to\infty} (a_n - b_n) = 0 \ (g).$$

[9] A rational number $A = m/n$, m, n relatively prime, is called an *integer with respect to g* (*in Bezug auf g*), if n and g are relatively prime. *A unit with respect to g* is a rational number E such that both E and $1/E$ are integers with respect to g. A rational number A is *divisible with respect to g* by a rational number B, if A/B is an integer with respect to g. A, B are *equivalent* or *associated*, if A is divisible by B with respect to g and vice versa. A, B are *congruent modulo* g^ρ, $\rho \in \mathbb{Z}$, if their difference $A - B$ is divisible by g^ρ. See [R12], pp. 2–3.

[10] [R12], p. 3; for example, $\lim_{n\to\infty} g^n = 0$ (g).

[11] $\forall d \in \mathbb{Q}\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq N,\ \forall k \in \mathbb{N}:\ a_{n+k} - a_n < d\ (g)$.

In this way the equality of g–adic numbers is defined; and it is acceptable, since the explicit relation is reflexive, symmetric and transitive.[12]

Arithmetic operations are defined quite naturally [13] and it is proved that g–adic numbers form a ring, later denoted by $\mathbb{Q}_g$. If $g = p$ (or, what is essentially the same, $g = p^k$), p is a prime, then it is a field, but if g contains at least two different primes, then there exist non-trivial zero divisors and $\mathbb{Q}_g$ is only a ring.

For $g = pq \ldots r$ Rychlík considers the decomposition of the ring of g–adic numbers into the fields of p–adic, q–adic, ... , r–adic numbers in the sence of a direct sum and proves the theorem on the unique representation of the g–adic number in the additive normal form. The theorem on completeness of the ring of g–adic numbers follows.

Rychlík also considers series in $\mathbb{Q}_g$. The necessary and sufficient condition for convergence is now the zero limit with respect to g of its terms. So the series of the form

$$a_\nu g^\nu + a_{\nu+1} g^{\nu+1} + \ldots, \qquad a_\nu,\ a_{\nu+1},\ \ldots, \qquad \nu \in \mathbb{Z}, \tag{4}$$

which were used by Hensel for the definition of g–adic numbers, converge with respect to g. Rychlík also proves that every g–adic number A can be represented uniquely by the *reduced g–adic expansion*, i.e. in the form (4) with $a_i \in \{0, 1, \ldots, g-1\}$.

A Continuous Non-Different. Function in $\mathbb{Q}_p$ [R17], 1920; [R21], 1922

In 1920 Karel Petr published in the Czech journal *Časopis pro pěstování mathematiky a fysiky* a very simple example of a continuous function without derivative [33]. Only the knowledge of the definition of continuity and derivative and a simple arithmetic theorem is necessary to understand both the construction and the proof of continuity and non–differentiability of the function. Petr's function is defined on the

[12] [R12], pp. 5–6. Notice this corresponds to a definition in terms of equivalence classes.

[13] For $A = \lim_{n\to\infty} a_n\ (g)$, $B = \lim_{n\to\infty} b_n\ (g)$ it is defined: $A + B = \lim_{n\to\infty} (a_n + b_n)\ (g)$, $AB = \lim_{n\to\infty} (a_n b_n)\ (g)$. The correctness of such definition is also proved.

interval $[0, 1]$ as follows:

$$\text{if} \quad x \quad = \quad \frac{a_1}{10^1} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} + \cdots ; \; a_k \in \{0, 1, \ldots, 9\}; \tag{5}$$

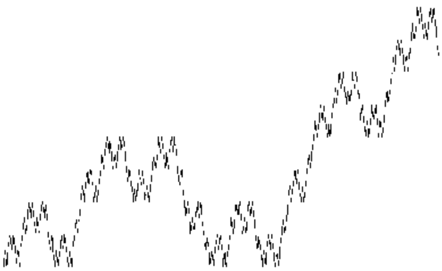
then

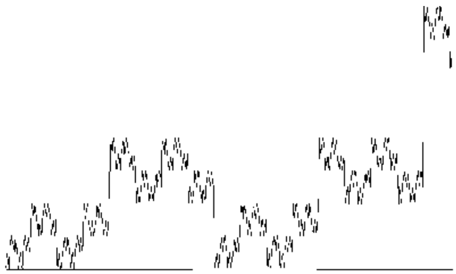
$$f(x) \quad = \quad \frac{b_1}{2^1} \pm \frac{b_2}{2^2} \pm \frac{b_3}{2^3} \pm \frac{b_4}{2^4} \pm \cdots ; \; b_k = \{ \begin{array}{ll} 0 & \text{for even } a_k \\ 1 & \text{for odd } a_k \end{array} \tag{6}$$

the sign before

$$b_{k+1}\{ \begin{array}{l} \text{opposite than the one before } b_k \text{ if } a_k \in \{1,\ 3,\ 5,\ 7\} \\ \text{the same otherwise} \end{array}$$

The graph of an approximation of Petr's function can be seen on the left picture bellow. To show it more graphically, a four–adic number system was used. Comparing with the graph on the right, the necessity of the exception to the rule of sign assignment awarded to the digit 9 can be understood; the result would not be a continuous function:





In the same year and in the same journal Karel Rychlík generalized Petr's function in his paper [R17]; the German version [R21] with the same content was published two years later in Crelle's journal. Rychlík carried the function from the real number field $\mathbb{R}$ to the field of p–adic numbers $\mathbb{Q}_p$ (compare (4)):

if

$$x = a_r p^r + a_{r+1} p^{r+1} + \cdots , \quad r \in \mathbb{Z}, \quad a_i \in \{0, 1, \ldots, p-1\}; \tag{7}$$

then

$$f(x) = a_r p^r + a_{r+2} p^{r+2} + a_{r+4} p^{r+4} + \cdots . \tag{8}$$

The proof that the function described in this way is continuous in $\mathbb{Q}_p$, but has no derivative at any point in this field, is rather elementary.

At the end Rychlík mentions that it would be possible to follow the same considerations in any field of p–adic algebraic numbers (introduced by K. Hensel) subsistent to the algebraic number field of a finite degree over $\mathbb{Q}$.

We shall remark that this Rychlík's work was one of the first published papers dealing with p–adic continuous functions. In Hensel's [14] some elementary p–adic analysis can be found, but otherwise it was developed much later; compare e.g. the papers of L. G. Šnirelman [44], J. Dieudonné [4], J. de Groot [5] etc.[14]

2.2 Valuation Theory

We have already mentioned the paper [22] of J. Kürschák, where the concept of *valuation* (*Bewertung*) was introduced as a mapping $\|\cdot\|$ of K into the set of non-negative real numbers, satisfying the following conditions:

$$\begin{aligned} \|a\| &> 0 \quad \text{if } a \in K,\ a \neq 0; \quad \|0\| = 0, && \text{(V1)}\\ \|1+a\| &\leq 1 + \|a\| \quad \text{for all } a \in K, && \text{(V2)}\\ \|ab\| &= \|a\| \,.\, \|b\| \quad \text{for all } a, b \in K, && \text{(V3)}\\ & \exists a \in K:\ \|a\| \neq 0, 1. && \text{(V4)} \end{aligned}$$

As a special case Kürschák considers p–adic valuations defined as follows. Let $K = \mathbb{Q}$ be a rational number field, p a prime. Every $a \in \mathbb{Q}$ can be expressed in the form $a = p^{\alpha} u/v$, $\alpha, u, v \ \in \mathbb{Z}$, where u, v are relative prime to p. We set

$$\|a\| = \mathrm{e}^{-\alpha},\ \|0\| = 0. \tag{13}$$

Instead of (V2) the stronger condition (later called *ultrametric inequality*) can be proved:

$$\|a+b\| \leq \mathrm{Max}(\|a\|, \|b\|) \quad \text{for all } a, b \in K. \tag{V2'}$$

Valuations satisfying (V2') are called *non-archimedean*, otherwise they are called *archimedean*.[15]

The concept of valuation enables Kürschák to generalize Cantor's approach to the construction of the real number field by means of fundamental sequences, and to construct the completion of an arbitrary

[14] For bibliography see e.g. Więsław's paper [48].

[15] This terminology didn't appear in Kürschák's work but it became usual soon (at least since Ostrowski's paper [30]).

valued field.[16] It is not difficult to extend the valuation from a given field to it's completion; the valuation of a limit of a sequence $\{a_n\}$ is defined as a limit of a real sequence $\{\|a_n\|\}$. A special instance is the field $\mathbb{Q}_p$, which is the completion of $\mathbb{Q}$ provided with a valuation (13). The fundamental result of Kürschák's paper is the proof of the following theorem:

THEOREM 2. Every valued field K can be extended to a complete, algebraically closed valued field.

First Kürschák constructs the completion of a given valued field K and extends the valuation from K to the completion as outlined above. Then he extends the valuation from the complete field to its algebraic closure. Finaly he considers the completion of this algebraic closure and proves that the completion is algebraically closed again. Here Kürschák follows the approach used by K. Weierstrass in [47], where a new proof, that the complex number field $\mathbb{C}$ is algebraically closed, was given.

As for the existence of an algebraic closure in the second step, Kürschák refers to the treatise of E. Steinitz [43]. But it is necessary to prolong the valuation to the algebraic extension. Consider a complete field K. It can be easily shown that if α is a root of a monic irreducible polynomial

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n, \quad a_i \in K \quad (a_n = \pm N\alpha),^{17} \tag{15}$$

it is necessary to define its valuation as $\|\alpha\| = \|a_n\|^{\frac{1}{n}}$. To prove that this is a valuation, the main point is to prove the triangle inequality (V2). For this purpose Kürschák generalizes for the case of an arbitrary valued field the results concerning power series in $\mathbb{C}$, given in the thesis [6] of J. Hadamard, and shows that the radius of convergence l of the series

$$\frac{1}{f(x)} = \frac{c_0}{x} + \frac{c_1}{x^2} + \cdots \tag{16}$$

for an irreducible polynomial $f(x)$ of the form (15), is equal to $\|a_n\|^{1/n}$. The inequality (V2) is then equivalent to $l' \leq 1 + l$, where l, l' are the radii of convergence of $1/f(x), 1/f(x-1)$ respectively. This part of the whole proof is relatively lengthy and laborious. But at the beginning of his paper Kürschák remarks that in all cases, where the inequality (V2') holds, i.e. for non-archimedean valuations, it is possible to generalize Hensel's considerations concerning the decomposition of polynomials over $\mathbb{Q}_p$, especially the assertion, later called *Hensel's Lemma*:

[16] Here the sequence $\{a_n\}$ is called *fundamental*, if for each $\varepsilon > 0$ there exists N such that $\|a_n - a_{n+k}\| < \delta$ holds for each $n > N$ and $k > 0$, e. g. in a usual way. The limit is defined similarly. The completion then consists of the elements of the original field K and (ideal) limits of all fundamental sequences in K.

LEMMA 2. If the polynomial (15) is irreducible and $\|a_n\| < 1$, then it is also $\|a_i\| < 1$ for all coefficients a_i, $1 \le i \le n$.

Kürschák shows that then it is easy to derive the triangle inequality for an algebraic extension.[18] He didn't prove Hensel's Lemma for a field K with a non-archimedean valuation – he wrote he had not succeeded in it's generalization for all cases, it means for archimedean valuations too. So he turned to the unified proof based on Hadamard's theorems, valid for all valuations.

A. Ostrowski proved in his paper [30] published in 1918, that every field K with an archimedean valuation is isomorphic to a certain subfield $\overline{K}$ of the complex number field $\mathbb{C}$ in the way that for every $a \in K$ and the corresponding $\overline{a} \in \overline{K}$ it is $\|a\| = |\overline{a}|^\rho$, where $|\cdot|$ is an ordinary absolute value on $\mathbb{C}$, $0 < \rho < 1$, ρ does not depend on a (such valuations are called *equivalent*). In other words, up to isomorphism, the only complete fields for an archimedean valuation are $\mathbb{R}$ and $\mathbb{C}$, where the problem of the extension of valuation is trivial. Hence it is possible to restrict the considerations only to non-archimedean valuations and use the generalization of Hensel's Lemma.

And this is precisely what was done in full details by Karel Rychlík in [R14] (Czech) and [R22] (German).

A Contribution to the Field Theory [R14], 1919

Rychlík follows the pertinent results of Hensel's book [13] and brings them from $\mathbb{Q}_p$ to a general non-archimedean valued complete field K. He uses the following terminology. *Integers in* K are the elements $a \in K$ for which $\|a\| \le 1$; the elements with $\|a\| = 1$ are called *units*. It is shown that the integers form an integral domain J. An element $a \in K$ is said to be *divisible* by $0 \ne b \in K$, if $a/b \in J$. For a non-unit integer m, i.e. $\|m\| < 1$, the congruence $a \equiv 0 \pmod{m}^*$ denotes $\|a\| < \|m\|$; $a \equiv b \pmod{m}^*$ denotes $a - b \equiv 0 \pmod{m}^*$. The congruence of polynomials is understood coefficientwise, $R(g, h)$ denotes the resultant of polynomials $f(x)$ and $g(x)$. After a detailed preparation including some auxiliary propositions Rychlík derives the assertion later called sometimes *Hensel–Rychlík Lemma* (besides other variants).[19]

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[18] Suppose α is a root of an irreducible polynomial $f(x)$ of the form (15), $\|a_n\| \le 1$. Then $1 + \alpha$ is the root of an irreducible polynomial $f(x-1) = x^n + \cdots + b_n$, $b_n = (-1)^n + a_1(-1)^{n-1} + \cdots + a_n$. If Hensel's Lemma holds in K, then $\|b_n\| \le 1$; thus $\|1+\alpha\| = \|b_n\|^{\frac{1}{n}} \le 1$, i.e. $\|1+\alpha\| \le \mathrm{Max}(1, \|\alpha\|)$.

[19] *Hensel–Rychlík lemma* sometimes denotes the following consequence of lemma 3 (although it is not explicitly stated in the German variant [R22]):
LEMMA. *Let $f(x)$ be a polynomial with integral coefficients in a valued complete field*

LEMMA 3. *If for a polynomial $f(x)$ with integral coefficients in a valued complete field K the congruence 1.) $f(x) \equiv g_0(x)h_0(x) \pmod{r^2}^*$ holds, where $g_0(x)$ and $h_0(x)$ are polynomials of degrees ≥ 1 with integral coefficients in K and $r \neq 0$ is their resultant, then it is also*[20] *2.) $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ are polynomials with integral coefficients in K of the same degrees as $g_0(x)$ and $h_0(x)$ respectively, and it is 3.) $g(x) \equiv g_0(x)$, $h(x) \equiv h_0(x) \pmod{r^2}^*$. Furthermore it is $\|R(g,h)\| = \|R(g_0,h_0)\|$.*[21]

One of the consequences of lemma 3 is the assertion mentioned by Kürschák (in a slightly weakened form) as lemma 2. As it was outlined above, on this base it is not difficult to get over theorem 2, without the help of power series. In this sense Rychlík put the valuation theory on purely algebraic foundations.

***Zur Bewertungstheorie der algebraischen Körper* [R22], 1923**

This is the German variant of the Czech paper [R14] with practically the same content. But only this German work became wide known; it was published in Crelle's journal, while [R14] appeared in a Czech journal *Časopis ...* and was not noticed by the mathematical community abroad.

Rychlík's work [R22] is cited for example by H. Hasse [7], [10], W. Krull [19], [20], M. Nagata [27], W. Narkiewicz [28], A. Ostrowski [31], [32], P. Ribenboim [36], P. Roquette [39], O. F. G. Schilling [40], F. K. Schmidt [10], [41], W. Więsław [49] and others. In the connection with some variant of the above lemma, Rychlík's name is mentioned also without the explicite citation of the work; see e.g. the paper [16] of I. Kaplansky, the recent book [37] of P. Ribenboim etc.

The reader who is interested in the history of the valuation theory will certainly enjoy the erudite work *On the History of Valuation Theory* of P. Roquette [39].

2.3 Theory of Algebraic Numbers, Abstract Algebra

The papers included in this group were published in Czech journals, in Czech or German, and remained almost unknown outside Bohemia.

K. If it is possible to find ξ_0 in the field K, such that 1.) $\|f(\xi_0)\| < \|f'(\xi_0)\|^2$, then the equation $f(x) = 0$ has a root ξ in the field K such that 2.) $\xi \equiv \xi_0 \pmod{1}^*$. [R14], p. 156.

[20] One more assumption should be added to make this assertion true in general: the leading coefficient of $f(x)$ is equal to the product of the leading coefficients of $g_0(x)$ and $h_0(x)$, not only congruent. See also e.g. [32], p. 275.

[21] [R14], p. 154.

Nevertheless they are very interesting and they show Rychlík's approach to scientific work.[22]

The Divisibility in Algebraic Number Fields [R15], 1919; [R16], 1920

In the couple of papers [R15] and [R16] (both in Czech) Rychlík constructs the divisibility theory for algebraic number fields over $\mathbb{Q}$. In [R15] the divisibility with respect to a prime p according to Hensel's [13] is considered, and the existence and uniqueness (up to association) of the greatest common divisor of any two integers with respect to p of a given field is proved. For fields of a finite degree over $\mathbb{Q}$ Rychlík proves that the number of non-associated prime elements with respect to p is finite and that the law of unique factorization into prime elements holds. By using Hensel's concepts he distinctly simplifies the cogitations of J. Sochocki given in [42], where the mentioned results are proved (all for finite extensions).

In the second paper [R16] Rychlík continues the construction of the divisibility theory for algebraic numbers based on the concept of divisors. He quotes Hensels book [13], but his approach is rather different. In comparison with Hensel, Rychlík's concept of divisors, as well as concepts of integrality, divisibility and association, are independent of a considered field. Moreover, Rychlík's definition can be used also for fields of infinite degrees. These are the advantages compared with Hensel's definitions based on the decomposition into prime divisors and therefore fixed to a certain field of a finite degree.

First Rychlík defines *divisors with respect to* a prime p as elements of the factor group A/J where A denotes the commutative multiplicative group of all nonzero algebraic numbers and J is its subgroup made up of units with respect to p. In a concrete algebraic number field K, *divisors of K with respect to p* are divisors that correspond to numbers of K; they form again a commutative group, isomorphic to the factor group A'/J', where A' denotes the group of nonzero algebraic numbers of K and J' the group of units with respect to p of K (compare e.g. [9]). The *group of divisors with respect to a system* of certain primes $p, q, \ldots, r, \ldots$ (their number can be either finite or infinite) is introduced, in todays terminology, as an external direct product of groups of divisors corresponding to primes $p, q, \ldots, r, \ldots$. In the following the divisors with respect to the system of all rational primes, simply divisors, are considered and the divisibility theory in K is built.

[22] Moreover, some of the ideas contained there can be found in later works of other mathematicians, compare e.g. [35].

Let us remark that the paper [R16] is mentioned in the book [28] of W. Narkiewicz.[23]

Zur Theorie der Teilbarkeit [R23], 1923

In the paper [R23] (in German) Rychlík introduces the concept of a *commutative semi-group* and constructs the divisibility theory in its quotient group.[24] A *commutative group* is defined in a usual way as a set G provided with a binary operation (multiplicative notation is used) satisfying the associative and commutative laws and the condition that for all $a, b \in G$ the equation $ax = b$ has a unique solution x. A *commutative semi-group* is a set H with a binary operation which satisfies associative and commutative laws and the couple of conditions: $ab = ab' \Rightarrow b = b'$ for all $a, b, b' \in H$ and $\exists 1 \in H : a1 = a$ for all $a, b \in H$; we would rather say a commutative semi-group with a unit and cancellation.

Denote by G the quotient group of a semigroup (in the above sense) H. The elements of H are called *integers of* G, an element $a \in G$ is said to be *divisible by* $b \in G$, if $ab^{-1} \in H$; similarly with the other arithmetical operations. We can see that Rychlík defines the divisibility not only for integers, but for all elements of the quotient group G. Similarly he defines the *greatest common divisor* in both G and H and gives their relation, similarly with the least common multiple. He derives properties of the arithmetic based on the above concepts – in general as well as under some sharper assumptions (e.g. the existence of the greatest common divisor in H for any two elements of H – *axiom g.c.d.*). Besides general theorems Rychlík gives various examples and applications of the introduced concepts.

Zur Theorie der Teilbarkeit in algebraischen Zahlkörpern [R24], 1923

In the paper [R24] (in German) Rychlík comes back to the divisibility theory of algebraic numbers, based again on the concept of a *divisor*. Although he does not explicitly quote his Czech works [R15] and [R16] on

[23] Nevertheless with a note that it was not accessible to him.

[24] Rychlík gives no reference here, but the concept of a *semi-group* was defined in the paper [3] (1905) of L. E. Dickson as a set G provided with a binary (multiplicative) operation, where the associative law holds and for any $a, x, y \in G$ either (separately) of $ax = ay$ and $xa = ya$ implies $x = y$. But the purpose of Dickson's definition was other than the divisibility theory. Nevertheless, Rychlík was aware of this paper, since he had mentioned it in his lecture *On Algebraic Number Fields* held in the Union on June 13, 1918 (and in the abstract of it, that was published in *Časopis* ... in the same year).

the same topic, he develops the ideas contained there. He quotes again Sochocki's [42], new is the quotation of *"fast vergessene Abhandlung"* of Zolotarev [50]. Compared with the Czech works, this one is more detailed and the approach is a little bit different, taking advantage of the results of [R23].

Eine Bemerkung zur Theorie der Ideale [R26], 1924

The aim of this paper is the refinement of the usual definition of an ideal of an algebraic number field K,[25] which depends on a considered field so that it is not possible to compare two ideals of two different fields. For this purpose Rychlík again uses the results of [R23].

Let K be an algebraic number field, which is a subfield of an algebraic number field L. Let $\alpha_1, \dots, \alpha_a \in K$ be fixedly chosen, $a \geq 1$. Denote by $\mathfrak{O}_L$ the ring of all algebraic integers contained in an algebraic number field L.[26] Rychlík defines an *ideal* I_K^L to be the set $\mathfrak{a} = \{\alpha_1, \dots, \alpha_a\}_K^L = \{\mu_1\alpha_1 + \dots + \mu_a\alpha_a, \quad \mu_1, \dots, \mu_a \in \mathfrak{O}_L\}$, and then he constructs the ideal theory based on this definition. One of the results is the proof of the assertion that non-zero ideals I_K^L form a group, which is a quotient group of the semi-group formed by non-zero integral ideals I_K^L.[27] To compare ideals in different fields, it suffices to consider as L the field of all algebraic numbers (over $\mathbb{Q}$).

Über die Anwendung der Methode von Sochocki ... [R33], 1929

We have already cited the treatise [42] of J. Sochocki in the connection with Rychlík's papers [R15], [R16] and [R24]. It was mentioned that the existence of the greatest common divisor of any two elements in a ring of integers with respect to p of a finite algebraic extension of the rational number field $\mathbb{Q}$ was proved there, or, in other words, that every ideal in this ring is principal. Using the results of [R23], Rychlík generalized this assertion for a finite algebraic extension of an arbitrary valued field with a prime element.[28]

[25] Here an ideal I is understood to be a system of numbers of K, closed for the addition and for the multiplication by elements of K, with the property that there exists $g \in \mathbb{Z}$ such that every number of I multiplied by g is an (algebraic) integer (it corresponds to *fractional ideal*). If all numbers in I are integers, then I is called *integral ideal.*

[26] Rychlík didn't use this symbol; we write it for the sake of lucidity.

[27] An ideal I_K^L is called *integral*, if $\alpha_1, \dots, \alpha_a$ are algebraic integers. The product of two ideals I_K^L, $\mathfrak{a} = \{\alpha_1, \dots, \alpha_a\}_K^L$ and $\mathfrak{b} = \{\beta_1, \dots, \beta_b\}_K^L$, is defined as the ideal $\mathfrak{ab} = \{\alpha_1\beta_1, \dots, \alpha_a\beta_1, \alpha_1\beta_2, \dots, \alpha_a\beta_b\}_K^L$; the correctness of such definition is proved. Obviously, $\{1\}_K^L\mathfrak{a} = \mathfrak{a}$.

[28] It means a field K with a non-archimedean valuation, where an element p exists,

***On the Extension of the Notion of Congruence ...* [R31], [R32], 1929**

Let K be an algebraic number field of a finite degree n over $\mathbb{Q}$, $\mathfrak{P}$ a nonempty set of prime ideals, finite or infinite. In [R31] Rychlík considers the divisibility *with respect to* $\mathfrak{P}$ for ideals of K.[29] Let $\mathfrak{m}$ be an ideal of K, integral with respect to $\mathfrak{P}$. If $\alpha, \beta \in K$ are integral with respect to $\mathfrak{P}$, such that $\alpha - \beta$ is divisible by m with respect to $\mathfrak{P}$, then they are called *congruent* mod m *with respect to* $\mathfrak{P}$ and it is denoted by $\alpha \equiv \beta \pmod{\mathfrak{m}; \mathfrak{P}}$.

Residue classes (mod $\mathfrak{m}; \mathfrak{P}$) form a ring $O(\mathfrak{m}; \mathfrak{P})$, which is proved to be isomorphic to the ring $O(\mathfrak{m})$ of residue classes (mod $\mathfrak{m}$). In the paper [R32] the special case $K = \mathbb{Q}$ is considered; otherwise, the results are the same as in [R31].

***On the Artin Theorem* [R38], [R39] 1932**

In this paper Rychlík cites the second volume of van der Waerden's *Moderne Algebra* [45] published in 1931, where the exposition of the ideal theory for integral domains was given. In a footnote on page 107, in the connection with ideals in domains where the axiom of the finiteness of ascending chains of ideals ("Teilerkettensatz") doesn't hold, van der Waerden quoted Artin's "Verfeinerungssatz".[30] Rychlík doesn't explicitly quote his German treatise [R23], but he develops considerations contained there and proves the Artin theorem for a commutative group, where the divisibility based on the concept of a semigroup is established.

Let $\mathfrak{H}$ be a semi-group and $\mathfrak{G}$ its quotient group. Suppose the axiom g.c.d. holds (see p. 275). After two auxiliary propositions Rychlík proves:

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such that $\|p\| < 1$ and every $a \in K,\ a \neq 0$, can be expressed in the form $a = p^r e,\ r \in \mathbb{Z},\ \|e\| = 1$.

[29] Ideals of the form $\mathfrak{p}_1^{k_1}\mathfrak{p}_2^{k_2}\dots\mathfrak{p}_m^{k_m}$, $\mathfrak{p}_i \in \mathfrak{P}$, $k_i \in \mathbb{Z}$, are called *representable in* $\mathfrak{P}$. Any ideal $\mathfrak{a} \neq (0)$ can be expressed as $\mathfrak{a} = |\mathfrak{a}|_{\mathfrak{P}} |\mathfrak{a}|_{\overline{\mathfrak{P}}} (1)$, $|\mathfrak{a}|_{\mathfrak{P}}$, $|\mathfrak{a}|_{\overline{\mathfrak{P}}}$ representable in $\mathfrak{P}$, $\overline{\mathfrak{P}}$ respectively; $\mathfrak{a}$ is called *integral with respect to* $\mathfrak{P}$, if the ideal $|\mathfrak{a}|_{\mathfrak{P}}$ is integral in a usual sence (see footn. 25). For an algebraic number $\alpha \in K$ consider a principal ideal (α); if it is an integral ideal with respect to $\mathfrak{P}$, then α itself is called an *integer with respect to* $\mathfrak{P}$. These integers form an integral domain, whose quotient field is K.

[30] Consider an integral domain, which is integrally closed in its quotient field. A relation called *Quasigleichheit* of two ideals $\mathfrak{a}, \mathfrak{b}$ of this domain is defined so that $\mathfrak{a}^{-1} = \mathfrak{b}^{-1}$; it is an equivalence and it is denoted by $\mathfrak{a} \sim \mathfrak{b}$. Artin's "Verfeinerungssatz" asserts, that if two decompositions of an ideal $\mathfrak{a}$ are given: $\mathfrak{a} \sim \mathfrak{b}_1\mathfrak{b}_2\dots\mathfrak{b}_m \sim \mathfrak{c}_1\mathfrak{c}_2\dots\mathfrak{c}_n$, then it is possible to factorize both products further, so that the factors coincide – up to the order and *Quasigleichheit*: $\mathfrak{b}_\lambda \sim \prod_\mu \mathfrak{b}_{\lambda\mu}$, $\mathfrak{c}_\nu \sim \prod_\omega \mathfrak{c}_{\nu\omega}$, $\mathfrak{b}_{\lambda\mu} \sim \mathfrak{c}_{\nu\omega}$ for some assignment; $\lambda = 1, \dots, m$, $\nu = 1, \dots, n$.

THEOREM 3. If $a_1 a_2 \dots a_r \sim b_1 b_2 \dots b_s$, where $a_1, \dots, a_r, b_1, \dots, b_s \in \mathfrak{H}$, then it is possible to factorize each factor into factors in $\mathfrak{H}$:

$$a_i \sim a_1^{(i)} a_2^{(i)} \dots a_{r_i}^{(i)} \quad b_i \sim b_1^{(i)} b_2^{(i)} \dots a_{s_i}^{(i)},$$

so that $\{a_k^{(i)},\ i = 1, \dots, r,\ k = 1, \dots r_i\} = \{b_k^{(i)},\ i = 1, 2, \dots, s,\ k = 1, 2, \dots s_i\}$.

The German variant of this paper was published in the same year as [R39].

2.4 Determinant Theory

The paper *Eine Bemerkung zur Determinantentheorie* [R37] published in Crelle's journal in 1931 concerns the assertion that the determinant of a matrix $A \in K^{n \times n}$, $n > 1$, where two rows or columns are identical, is zero, which can be easily proved for the case that the characteristic of the given field K is not 2.[31] Rychlík cites the book [8] of H. Hasse, where a completely general proof using the Laplace's "Entwicklungssatz" is given. Then he gives a simple proof of the considered assertion just for the field K of characteristic 2. He steps as follows. Consider the determinant of a matrix $X = (x_{ij})$ as polynomial over $\mathbb{Z}$ in indeterminates x_{ij}. If a matrix X^* has two identical rows (columns), then it is $|X^*| = 0$ in a ring which arises from $\mathbb{Z}$ by adjunction of the elements of X^*; it is also $|X^*| \equiv 0 \pmod 2$. This implies $|X^*| = 0$ in a ring which arises from a prime field of K by the adjunction of the elements of X^*, hence also in a ring which arises from K by this adjunction. If A is a matrix with elements of K and with two identical rows (columns), then the determinant $|A|$ is received from $|X^*|$ by substituting the elements of A for the elements of X^*, so it is $|A| = 0$.

This Rychlík's paper didn't remain completely unknown – it was cited for example by O. Haupt in the third edition of his *Einführung in die Algebra I* [11].

The second paper concerning determinants [R42] published in 1934 is written in Czech and it comes out of the paper [34] of K. Petr, where the determinant theory is based on the definition of a determinant as an alternating m–linear form. Rychlík generalizes Petr's considerations for the case of an arbitrary field K of an arbitrary characteristic. For this

[31] It is based on the assertion that a mutual exchange of two rows (columns) leads to the opposite sign of the determinant; if we exchange the two rows (columns) that are identical, then $|A| = -|A|$, i. e. $|A| + |A| = 2|A| = 0$.

purpose it is necessary to give a suitable definition of an alternating m–linear form (equivalent to Petr's one for fields of characteristics different from 2).

More details concerning the mentioned papers as well as some others can be found at the cited internet pages (see footnote 4). Instead of the conclusion, let us only refer to the introductory paragraphs of the second part of this paper, where a characterization of Rychlík's work was briefly outlined.

3 Appendix

The abbreviations of magazines used bellow:

Bull. = *Bulletin internat. Acad. Boheme*; **ČPM(F)** = *Časopis pro pěstování mathematiky (a fysiky)*; **ČMŽ** = *Čechoslovackij matematičeskij žurnal – Czechoslovak Mathematical Journal*; **Crelle** = *Journal für die reine und angewandte Mathematik*; **MŠ** = *Matematika ve škole*; **Pokroky** = *Pokroky matematiky, fyziky a astronomie*; **Rozhledy** = *Rozhledy matematicko–fysikální*; **Rozpravy** = *Rozpravy II. tř. České akademie věd a umění*; **Věstník** = *Věstník Královské české společnosti nauk – Mémoires de la société royale des sciences de Bohême.*

References to the following reference magazines are given in the list 3.1:

J = *Jahrbuch über die Fortschritte der Mathematik*; **MR** = *Mathematical Reviews*; **RM** = *Referativnyj žurnal matěmatika*; **ZBL** = *Zentralblatt für Mathematik und ihre Grenzgebiete.*

3.1 The List of Publications of Karel Rychlík

[R1] *Poznámky k theorii interpolace* [Remarks on Interpolation Theory], ČPMF **36** (1907), 13–44; . **J 38**(1907), 309 Petr

[R2] *O resolventách se dvěma parametry* [On Resolvents with two Parameters], Rozpravy **17**(1908), Nr. 31, 5 pp.; **J 39**(1908), 131 Petr.

[R3] *O grupě řádu 360* [On the Group of the Rank of 360], ČPMF **37**(1908), 360–379; **J 39**(1908), 205 Petr.

[R4] *Příspěvek k theorii forem* [A Contribution to the Theory of Forms], Rozpravy **19** (1910), Nr. 49, 13 pp.; **J 41**(1910), 159 Petr.

[R5] *O poslední větě Fermatově pro $n = 4$ a $n = 3$* [On Fermat Last Theorem for $n = 4$ and $n = 3$], ČPMF **39**(1910), 65–86; **J 41**(1910), 249.

[R6] *O poslední větě Fermatově pro $n = 5$* [On Fermat Last Theorem for $n = 5$], ČPMF **39**(1910), 185–195, 305–317; **J 41**(1910), 249.

[R7] *Příspěvek k theorii forem II* [A Contribution to the Theory of Forms II], Rozpravy **20**(1911), Nr. 1, 5 pp.; **J 42**(1911), 146 Petr.

[R8] *Geometrické znázornění řetězců* [The Geometric Representation of Continued Fractions], ČPMF **40**(1911), 225–236; **J 42**(1911), 247.

[R9] *Sestrojení pravidelného sedmnáctiúhelníku* [The Construction of the Regular 17–gon], ČPMF **41**(1912), 81–93; **J 43**(1912), 586 Petr.

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