THE METAPHOR OF TOOL AND FOUNDATION OF MATHEMATICS

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1 Introduction

The main interest of this paper is to show that in attempting to understand processes of scientific community forming and community interaction in contemporary mathematics, it is interesting to look at the understanding the respective community's members assign to the main metaphors in the scientific discourse under consideration. The case in point is the discussion between category theorists and set theorists about whether or not category-theoretic notions can be of foundational significance in mathematics. This discussion proceeds since the first days when category theorists proposed foundational schemes based on category-theoretic notions. Having been done for a long time in the usual way via publications in scientific journals and talks in meetings, the discussion is continued now more informally on an internet mailing list. This discussion actually seems to be far from an end; one might rather wait for a dying away of the discussion altogether (because of lack of relevance of the problems discussed) than for unanimous decisions on resolutions of the main problems (or even on what is considered to *be* the main problems).

While the mentioned kind of electronic publication is less interesting as far as precise, referee-controlled mathematical results are concerned, the mail utterings offer lots (and even more than traditional publications) of evidence about what will be focussed on here: the understanding assigned to metaphors. Moreover, these mailings are hybrids between personal communication (as done before in letters) and public accounts (as they are publicly accessible — and intended to be published in that way by their writers). So their content is in the same time *not* intended to be eternal, completely checked against error and subjectivity, *and* believed to be of general interest, at least as far as the continuation of the discussion is concerned. This constitutes a difference to most textual sources a historian of mathematics usually works with, and so asks for the development of a suitable methodology.[1]

My paper is, as usual with contributions to a *Novembertagung*, more a preliminary report, a description of a project (or of several projects), intended to make up a discussion of the methods and the interest of that project, than a presentation of results. I wish to thank the participants of Brno *Novembertagung* for many helpful suggestions as to the continuation of my research and the organizers for the opportunity to publish these incomplete remarks. This incompleteness actually includes the fact that many pointers to literature are limited to encyclopedias and the like; I am not yet able to present all the relevant primary sources.

2 Metaphor

When applying the term "metaphor", one should remember that there are whole theories of that notion (even competing ones) both in linguistics and philosophy of language. [6] 284–289, [13] 867–870 and [4] give nice introductions to different sides of this discussion. The main epistemological thesis which is discussed may be the following:

> *metaphor is a medium of fuller, riper knowing than is possible in ordinary language* (Philip Wheelwright; cited after [6] 287)

A similar position is maintained by M. HESSE; see [13] 869. The philosophical question so reads: are there things that can *only* be expressed metaphorically? One might be reminded of WITTGENSTEIN's "zeigen". Brno conference's main text by PHILIP DAVIS contains another citation of LEAVIS in the same vein: "literary sensibility deals with thought that is untranslatable into logic, or into mathematical symbols." ([5] 80).

I will not enter that open debate. My interest is more in how to use the different readings appointed to metaphors as evidence for research in the history of mathematics. For other parts of history, the philosopher of history HANS BLUMENBERG (*1920) was successful in making use of this idea. Let me paraphrase him when he explains his thinkings on how a metaphor works: The metaphor shows a relation to reality, but in the same time its supposed self-evidence serves to defend the background from being turned to foreground (see [12] 126, for instance). I have not yet developed a workable way to apply BLUMENBERG's methods in the history of mathematics, but I intend to do this later.

[1] Which, for lack of space, I will do here only implicitly in trying to *adopt* one.

My view is that metaphors often serve to stand for poorly understood things. But it is not my methodology here to decide whether a given metaphor (or a metaphor in general) is "legitimate", whether it gives a "good" description of the things it is intended to describe. It is precisely the task of the historian to watch this decisionmaking happen and to try to understand what happened — and not to make the decisions. The watching and attempting to understand might be done in analysing how the metaphor is used in a discourse and to what purposes.

3 The metaphor of "tool"

I shall now focus more closely on my case in point: category theory. There is a metaphor with some interesting roles in the history of category theory: the metaphor of "tool". I will not present here a general history of this metaphor in mathematics — which certainly would be indispensable for a complete interpretation of the more particular phenomena I'm interested in.

In a way, to speak of a "mathematical tool" is not really to speak metaphorically. The tool metaphor is what is called in linguistics a dead metaphor: it is used now in everyday talk *as if* it was not a metaphor, but an ordinary expression. The *Oxford Dictionary* [16] 3349 even tells us that this is the case since nearly a thousand years ago. This is not necessarily a problem to my approach, since there are linguists who suppose that to understand how a metaphor works one has to hypothesize a "zero degree of language" ([4] 1314).

The interesting point about the use of "tool" in mathematical discourse is that tools are opposed to objects of study in their own right. MORITZ EPPLE made me sensible to this point; see also his *Habilitationsvortrag* [9]. Before, I thought of the relation between tool and object simply as a dialectical one: a mathematical concept can be both, but not in same time, so there is a certain tension between these two aspects of the concept. So far, so good, but this does not explain very much.[2] Moritz pointed at the relation between research communities (and how they behave, interact and sometimes conflict) and what is regarded as an object by one community, as a tool by another. The example chosen here — the discussion between set theorists and topos theorists — serves as a good example of such a "conflict by differing assignments of

[2]To speak of dialectics here stems from work by Brieskorn [2] and Alexandrov [1] done in the early seventies — and recurrence to the idea of dialectics certainly reflects the spirit of that time.

'toolness' or 'objectness' to a concept", but the example does more: it renders evident that the *very notion* of *being* a tool might be understood differently and interpreted differently by different communities.

4 Some basics and history of category theory

The basics of category theory (CT) should nowadays be known to most mathematicians.[3] The definition of category might seem similar to that of other algebraic structures like groups or monoids, but a look at examples will quickly prove the much larger scope of this notion.

The first obvious and wide class of examples are categories whose objects are all sets with a certain structure and whose arrows are all the corresponding structure preserving maps: sets with functions (the category **Set**), groups with homomorphisms (the category **Grp**), topological spaces with continuous functions (the category **Top**) and so on. Functors intuitively serve to transport structure from one category to another; indeed the historical origin[4] of the very notion lays in algebraic topology, where the purpose was to find algebraic invariants of topological spaces, which amounts to have functors from **Top** to **Grp**, say. The important difference between former invariants and functors is the emphasis not only on the topological spaces, but also on the maps between them. The "naturality" of the transformations is intuitively the

[3] For the convenience of the reader, I give some (informal) definitions here, which differ only slightly from the first definitions given in [7].

A *category* $\mathcal{C}$ consists of a collection of objects $A, B, C, \ldots$ and a collection of arrows $f, g, h, \ldots$; between the arrows there is a law of composition such that:

1. to each arrow f there are assigned two objects $dom(f)$ (the *domain*) and $cod(f)$ (the *codomain*); if $dom(f) = A, cod(f) = B$, write $f : A \longrightarrow B$.

2. the composition of f and g, $g \circ f$, is defined whenever $cod(f) = dom(g)$.

3. composition is associative.

4. for each A there is a unique *identity arrow* 1_A such that $1_A \circ f = f \ \forall f$ with $dom(f) = A$ and $g \circ 1_A = g \ \forall g$ with $cod(g) = A$.

A *functor* $F : \mathcal{C} \longrightarrow \mathcal{C}'$ between two categories $\mathcal{C}, \mathcal{C}'$ is a pair of maps which map objects resp. arrows of $\mathcal{C}$ to objects resp. arrows of $\mathcal{C}'$ and preserve category structure (identities and composition). A *natural transformation* τ of two functors $F, F' : \mathcal{C} \longrightarrow \mathcal{C}'$ is a collection of arrows in $\mathcal{C}'$, where to each object A in $\mathcal{C}$ there is assigned an arrow $\tau_A : F(A) \longrightarrow F'(A)$ in $\mathcal{C}'$ such that for each arrow $f : A \longrightarrow B$ in $\mathcal{C}$ the equation $\tau_B \circ F(f) = F'(f) \circ \tau_A$ holds. Significance of these definitions will become clearer with the examples (all these examples are easily verified to meet the definitions).

[4] For more details on CT history, see [3].

idea to be independent of artificial choices in one's constructions; the most elucidating example of [7] for such a construction is the bidual of a finite–dimensional vector space: This bidual (which is a functor in a category of vector spaces) is isomorphic to the original space (which can be seen as the identity functor on that category that maps each object to itself), and the construction of the isomorphism does not depend on the choice of a basis. For this reason EILENBERG and MACLANE called the isomorphism a *natural* equivalence.

But the arrows need not be maps: there is a category **Htop**, having topological spaces as its objects and equivalence classes of homotopy equivalent maps as its arrows. There are more "nonstandard" examples: Every set is a category with the elements the objects and the only arrows the identities (identified with the objects)—this yields surely a trivial law of composition. Similarly, each group is a category with the neutral element as the only object and all elements as arrows, group composition being category composition. Or take a partially ordered set with objects all elements and arrows all pairs of elements (x, y) such that $x \leq y$. More interesting are the category of categories **Cat**, taking the functors as arrows, or the category $\mathbf{Func}(\mathcal{C}, \mathcal{C}')$ of all functors between two given categories, taking the natural transformations as arrows. The most prominent examples of such functor categories are categories of sheaves.

We have now a feeling for the importance of CT constructions in past–war mathematics.[5] What lacks for our present discussion is information about the attempts to make category-theoretic notions to a foundation of mathematics. This started with WILLIAM LAWVERE who in [11] proposed to leave the traditional foundations of mathematics — everything in mathematics can be expressed in terms of set theory, Zermelo-Fraenkel (ZF) for instance — behind and to replace them by an analogue in categorial terms: He gave a language (defining an alphabet, rules of expression and formula forming and of logical inference) and axioms for CT. Clearly this language has symbols $\circ$, *dom* and so on. He was then able to provide a definition of $\in$ (not identical with the usual set-theoretical one) in terms of this theory. So, he *started* not with the notion of set and membership, but with that of category. The little attraction this attempt was able to produce nonwithstanding, similar ideas were pursued later on with the notion of *(elementary) topos*,

[5]To be more precise here, the rapid development of CT applications started only after the introduction of additional concepts such as universal property, adjoint functor, abelian category in the fifties, especially by Grothendieck.

following ideas of Grothendieck, Lawvere and Tierney; see [14]. There, **Set** is only one (the most intuitive) example of a topos.

5 The discussion on the `fom`-list

These attempts didn't remain without echo, even complaint. I do not plan to describe the whole history of reaction on the proposals here (this will be done in my thesis); I will rather concentrate on the most recent events in this series: the discussion in the world wide web. The contributions cited in the sequel are found on the following URL:

`http://www.math.psu.edu/simpson`

The exact path after `/simpson` will be given with each citation. Inside the texts, "foundations of mathematics" are sometimes referred to by "fom" or "f.o.m.".

Let us first see how "tool" is used by a scholar who, by closer inspection to the whole discussion, will prove to be opponent against the alleged foundational relevance of category-theoretic notions. The concepts mentioned in his text — cohomology and topos theory — are not just the same as category theory. But these concepts are nowadays (re)formulated by category-theoretic means.

> *cohomology is and always will be a technical tool, by and for pure mathematics, not a fundamental concept.*
> (*Steve Simpson*, `fom/postings/9709/msg00005.html`)

> *Maybe topos theory doesn't really have any f.o.m. motivation or content. Maybe topos theory is to be viewed as simply a tool or technique in pure mathematics.*
> (*Steve Simpson*, `fom/postings/9801/msg00127.html`)

It is clear that SIMPSON in a way *opposites* "tool" and "fundamental concept". There are complaints about this opposition in some replies to SIMPSON. A reply to the first phrase stays rather vague:

> *While you are right that cohomology is a mathematical tool with mathematical applications, one might wonder about the meta-mathematical significance of the pervasive role of cohomology in modern mathematics.*
> (*D. Marker*, `fom/postings/9709/msg00016.html`)

A reply in the same vein, but to the second phrase of SIMPSON's, is by COLIN MCLARTY. This scholar is a proponent of foundational relevance of topos theory.[6]

> *Set theorists tend to think of each given theory (ZF, group theory, Feferman's theory of collections and operations, topos theory ...) as either* being *foundational or* not being. *If there is some non-foundational use of topos theory, then topos theory must be per se not foundational.*
> (*Colin McLarty*, `fom/postings/9801/msg00130.html`)

Besides the (only partially accepted) opposition of tools and fundamental concepts, there are more features of "tool" to explore in the discussion. See for instance the following answer of MCLARTY to JOE SHIPMAN.[7]

> $>$ *My impression is that the results achievable from categori-*
> $>$ *cal foundations can be smoothly developed set-theoretically*
> $>$ *(i.e. as if categories had never been invented)*
>
> *You can do it without categorical* foundations. *Much math today would be humanly impossible without categorical* methods.
> (*Colin McLarty*, `fom/postings/9801/msg00222.html`)

This is (implicitly) "tool" in the full sense of the word (cf. again [16]): something that helps man to do something that he or she cannot do by his own forces. Is this the way conceptual progress works? Since outside of mathematical logic there are rarely demonstrations for a concept of being necessary, the "cannot" seems simply to surround the problem. But let us see how the idea helps to understand where the heart of the objections set theorists may have lies. This is best exemplified in a contribution by the eminent set theorist Harvey Friedman:

> *I have been completely unable to see how one can begin to think that the notion of category helps in understanding the*

[6]McLarty speaks of the opponents as "set theorists"; while it happens that the opponents are in fact recruted from their rows, it is not at all clear that being a set theorist means automatically being against foundational attempts by category-theoretic means. Rather set theorists may tend to distrust category theory because of open questions concerning the consistency of category theory — a question I will not focus on here.

[7]Sorry about the problems LaTeX might have with the ¿-signs in the email texts.

> *notion of collection. Obviously, the way that the concept of collection is assimilated in little children is by placing two or three actual physical objects into a bunch or group and talking about the bunch or group instead of just talking about the actual physical objects.*
> (*Harvey Friedman*, `fom/postings/9801/msg00185.html`)

Friedman seems to claim that a test for the ability of a mathematical "tool" to be *helpful in understanding a concept* is whether the way the tool treats the concept and the way little children learn the concept coincide. At the same time, for him (as for the proponent he replies to, actually) it seems to be precisely the task of research in foundations of mathematics to understand concepts. Before *this* background it is clear that a *tool* (as something additional, multiplying man's possibilities) cannot be foundational *per se*. There are objections to some of the arguments supporting this view (for instance, the concepts of "counting" and "number", nowadays regarded as elementary and learned by little children, did only become accessible to mankind in a slow development in history; see [10]), but I will not discuss this here. So far, it should have become clear that the discussion presented is characteristic as a community conflict (in fact, I did not present the harder stuff with lots of *ad hominem* arguments) and that the conflict may be partly due to misunderstandings as to what each "side" means by certain metaphors, as "tool", for instance. I finish with a mailing that tries to reconcile the parties and in the same time explains the problem quite clearly:

> *I'd like to agree [...] that SET and TOP and Cat have different powers, different tool-kits. We are in no position to claim with insightful understanding that one is* the *tool-kit.*
>
> *Rather, each is in need of "local foundations", a rigorous local-f.o.m. clarification of for what the tools in each tool-kit are most appropriate. E.g. modelling real analysis in TOP looks like tool-abuse, as for example using a hammer to put in a screw.*
> (*Robert Tragesser*, `fom/postings/9801/msg00315.html`)

References

[1] Alexandrov, Alexander, *Mathematik und Dialektik*, in: [15] 47–63.

[2] Brieskorn, Egbert, *Über die Dialektik in der Mathematik*, in: [15] 220–286.

[3] Corry, Leo, *Modern algebra and the rise of mathematical structures*, Science Networks. Historical Studies. 17 Basel: Birkhäuser 1996. Zbl.858.01022; MR97i:01023.

[4] Dascal, Marcelo et al. (ed.), *Sprachphilosophie. Ein internationales Handbuch zeigenössischer Forschung*, Berlin: De Gruyter 1996. Article “Metaphor” by Umberto Eco, 1313–1323

[5] Davis, Philip J., *Thirty years after the two culture controversy: A mathematician's view* in: Essays in Humanistic Mathematics, ed. A. White, MAA 1993, 79-90.

[6] Edwards, Paul (ed.) *The Encyclopedia of Philosophy*, vol.5. New York: Macmillan 1967.

[7] Eilenberg, Samuel; Mac Lane, Saunders, *General Theory of Natural equivalences*, in: Trans. AMS **58** (1945), 231–294.

[8] Eilenberg, Samuel et al. (ed.), *Proceedings of the Conference on Categorical Algebra* in La Jolla. New York: Springer 1965.

[9] Epple, Moritz, *Genies, Ideen, Institutionen, mathematische Werkstätten: Formen der Mathematikgeschichte*, unveröffentlichtes Vortragsmanuskript, 1998.

[10] Ifrah, Georges, *Universalgeschichte der Zahlen*, Campus 1986.

[11] Lawvere, F. William, *The Category of Categories as a foundation of mathematics*, in: [8] 1–21; 1965. MR34 #7332.

[12] Lutz, Bernd (ed.), *Metzler Philosophenlexikon*, J.B.Metzler 2.Auflage 1995. Article “Blumenberg, Hans” by Jörg Villwock, 123–127.

[13] Mittelstra, Jürgen (ed.), *Enzyklopädie Philosophie und Wissenschaftstheorie*, vol.II. Mannheim: BI 1980.

[14] Osius, G., *Kategorielle Mengenlehre: Eine Charakterisierung der Kategorie der Klassen und Abbildungen*, in: Math. Annalen **210** (1974), 171–196.

[15] Otte, Michael (ed.), *Mathematiker über die Mathematik*, Springer 1974.

[16] *The Compact Edition of the Oxford English Dictionary*, vol.II (P–Z), Oxford University Press 1971.

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