20. Cosets of subgroups

20.1. Definition

Consider a group $\mathfrak{G}$ and let $\mathfrak{A}$ be a subgroup of $\mathfrak{G}$. Let, moreover, $p \in \mathfrak{G}$ be an arbitrary element of $\mathfrak{G}$.

The subset $p\mathfrak{A}$ of $\mathfrak{G}$, i.e., the set of the products of the elements p and each element of $\mathfrak{A}$, is called the *left coset* or *left class of p with regard to* $\mathfrak{A}$ or (if we know it is a question of $\mathfrak{A}$), briefly, the *left coset* or *left class of p*.

Similarly, the subset $\mathfrak{A}p$, i.e., the set of the products of each element of $\mathfrak{A}$ and the element p is called the *right coset* or *right class of p with regard to $\mathfrak{A}$*, briefly the *right coset* or *right class of p*.

Note that the field A of the subgroup $\mathfrak{A}$ is simultaneously both the left and the right coset of the element $\underline{1}$ with regard to $\mathfrak{A}$.

We shall first describe, in a few simple theorems, the properties of the left cosets; as the properties of the right cosets are analogous, we shall not deal with them here and leave it to the reader to consider them himself.

20.2. Properties of the left (right) cosets

Let $\mathfrak{A} \subset \mathfrak{G}$ be a subgroup and p, q arbitrary elements of $\mathfrak{G}$. Then the following theorems are true:

1. *The left coset $p\mathfrak{A}$ contains the element p.*

Indeed, since $\mathfrak{A}$ is a subgroup, there holds $\underline{1} \in \mathfrak{A}$ and we have $p = p\underline{1} \in p\mathfrak{A}$.

2. *If and only if $p \in \mathfrak{A}$, then $p\mathfrak{A} = A$.*

To prove that the above statement applies, let us first assume $p \in \mathfrak{A}$. Since $\mathfrak{A}$ is a subgroup, the product of p and any element of $\mathfrak{A}$ is again included in $\mathfrak{A}$, hence $p\mathfrak{A} \subset A$. Moreover, $p^{-1} \in \mathfrak{A}$ and, for any element $a \in \mathfrak{A}$, there is $p^{-1}a \in \mathfrak{A}$ so that $p(p^{-1}a) \in p\mathfrak{A}$; but $p(p^{-1}a) = (pp^{-1})a = \underline{1}a = a$; hence $a \in p\mathfrak{A}$ and we have $A \subset p\mathfrak{A}$. Consequently, $p\mathfrak{A} = A$. Let us now suppose that for some element $p \in \mathfrak{G}$ there holds $p\mathfrak{A} = A$. Then the product pa, for every $a \in \mathfrak{A}$, is contained in $\mathfrak{A}$ and therefore, in particular (for $a = \underline{1}$), p is an element of $\mathfrak{A}$.

Theorem 2 may be generalized in terms of:

3. *If and only if $p^{-1}q \in \mathfrak{A}$, then $p\mathfrak{A} = q\mathfrak{A}$.*

In fact, if $p^{-1}q \in \mathfrak{A}$, then in accordance with theorem 2, $p^{-1}q\mathfrak{A} = A$ and, consequently, $q\mathfrak{A} = (pp^{-1})q\mathfrak{A} = p(p^{-1}q)\mathfrak{A} = p(p^{-1}q\mathfrak{A}) = p\mathfrak{A}$. Conversely, from $q\mathfrak{A} = p\mathfrak{A}$ there follows $(p^{-1}q)\mathfrak{A} = A$ and so $p^{-1}q \in \mathfrak{A}$, which we were to prove.

4. *The left cosets $p\mathfrak{A}$, $q\mathfrak{A}$ are either disjoint or identical.*

This remarkable property of the left cosets may be verified in the following way: If both left cosets $p\mathfrak{A}$ and $q\mathfrak{A}$ have a common element x and so $x \in p\mathfrak{A}$, $x \in q\mathfrak{A}$, then $p^{-1}x \in \mathfrak{A}$, $q^{-1}x \in \mathfrak{A}$. Hence, in accordance with theorem 3, we have $p\mathfrak{A} = x\mathfrak{A} = q\mathfrak{A}$ and both left cosets $p\mathfrak{A}$ and $q\mathfrak{A}$ are identical.

5. *The left cosets $p\mathfrak{A}$, $q\mathfrak{A}$ are equivalent sets.*

Our object now is to show that there exists a simple mapping of the set $p\mathfrak{A}$ onto $q\mathfrak{A}$. Each element of $p\mathfrak{A}$ ($q\mathfrak{A}$) is the product pa (qa) of the element p (q) and a convenient element $a \in \mathfrak{A}$. Since $pa = pb$ ($qa = qb$) yields $a = b$, the element a is uniquely determined. Conversely, if $a \in \mathfrak{A}$, then $pa \in p\mathfrak{A}$ ($qa \in q\mathfrak{A}$). We observe that $\begin{pmatrix} pa \\ a \end{pmatrix}$ is a simple mapping of the set $p\mathfrak{A}$ onto $\mathfrak{A}$ and, similarly, $\begin{pmatrix} a \\ qa \end{pmatrix}$ is a simple mapping of the subgroup $\mathfrak{A}$ onto $q\mathfrak{A}$. Hence $\begin{pmatrix} a \\ qa \end{pmatrix}\begin{pmatrix} pa \\ a \end{pmatrix}$ is a simple mapping of the set $p\mathfrak{A}$ onto $q\mathfrak{A}$, which was to be proved.

Let us now proceed to the case when $\mathfrak{G}$ contains, besides $\mathfrak{A}$, a further subgroup, $\mathfrak{B}$.

6. *If the left cosets $p\mathfrak{A}$, $q\mathfrak{B}$ are incident, then their intersection $p\mathfrak{A} \cap q\mathfrak{B}$ is the left coset of each of its own elements with regard to the subgroup $\mathfrak{A} \cap \mathfrak{B}$.*

In fact, if the cosets $p\mathfrak{A}$, $q\mathfrak{B}$ have a common element $c \in \mathfrak{G}$, then by theorem 1 and theorem 2, there holds: $p\mathfrak{A} = c\mathfrak{A}$, $q\mathfrak{B} = c\mathfrak{B}$ whence $p\mathfrak{A} \cap q\mathfrak{B} = c\mathfrak{A} \cap c\mathfrak{B}$. Every element $x \in c\mathfrak{A} \cap c\mathfrak{B}$ is the product of c and a convenient element $a \in \mathfrak{A}$ and, at the same time, the product of c and a convenient element $b \in \mathfrak{B}$ and so $x = ca = cb$. Consequently, $a = b \in \mathfrak{A} \cap \mathfrak{B}$ and, therefore, $x \in c(\mathfrak{A} \cap \mathfrak{B})$. Thus we have $p\mathfrak{A} \cap q\mathfrak{B} \subset c(\mathfrak{A} \cap \mathfrak{B})$. Moreover, every element $x \in c(\mathfrak{A} \cap \mathfrak{B})$ is the product of the element c and an element $a \in \mathfrak{A} \cap \mathfrak{B}$, so that $x = ca \in c\mathfrak{A} \cap c\mathfrak{B}$. Consequently, $c(\mathfrak{A} \cap \mathfrak{B}) \subset p\mathfrak{A} \cap q\mathfrak{B}$ and we have the required result.

7. *If $\mathfrak{A} \subset \mathfrak{B}$, then from the incidence of the left cosets $p\mathfrak{A}$, $q\mathfrak{B}$ there follows $p\mathfrak{A} \subset q\mathfrak{B}$.*

Indeed, by 1.10.3, $\mathfrak{A} \subset \mathfrak{B}$ yields $\mathfrak{A} \cap \mathfrak{B} = \mathfrak{A}$; in accordance with theorems 6 and 4, there applies $p\mathfrak{A} \cap q\mathfrak{B} = p\mathfrak{A}$ and, consequently, $p\mathfrak{A} \subset q\mathfrak{B}$ (1.10.3).

As we have already mentioned, the properties of the right cosets with regard to $\mathfrak{A}$ are analogous. Between the left and the right cosets with regard to $\mathfrak{A}$ there holds the following relation:

8. *The left coset $p\mathfrak{A}$ is mapped, under the inversion $\boldsymbol{n}$ of the group $\mathfrak{G}$, onto the right coset $\mathfrak{A}p^{-1}$: $\boldsymbol{n}(p\mathfrak{A}) = \mathfrak{A}p^{-1}$. Simultaneously there holds the analogous formula $\boldsymbol{n}(\mathfrak{A}p) = p^{-1}\mathfrak{A}$.*

From $x \in p\mathfrak{A}$ there follows $x = pa$ ($a \in \mathfrak{A}$) and $x^{-1} = a^{-1}p^{-1}$ ($a^{-1} \in \mathfrak{A}$) yields $x^{-1} \in \mathfrak{A}p^{-1}$. Hence $\boldsymbol{n}(p\mathfrak{A}) \subset \mathfrak{A}p^{-1}$. Moreover, every point $y = a'p^{-1} \subset \mathfrak{A}p^{-1}$ ($a' \in \mathfrak{A}$)

is the $\boldsymbol{n}$-image of the element $pa'^{-1} \in p\mathfrak{A}$ ($a'^{-1} \in \mathfrak{A}$). Thus we have $\mathfrak{A}p^{-1} \subset \boldsymbol{n}(p\mathfrak{A})$ and, consequently, $\boldsymbol{n}(p\mathfrak{A}) = \mathfrak{A}p^{-1}$, which completes the proof.

Remark. Both $p\mathfrak{A}$ and $\mathfrak{A}p^{-1}$ are referred to as *mutually inverse cosets*. If one of them is denoted e.g. by $\bar{a}$, then the other is $\bar{a}^{-1}$.

9. *The left coset $p\mathfrak{A}$ and the right coset $\mathfrak{A}q$ are equivalent sets.*

We are to prove that there exists a simple mapping of the set $p\mathfrak{A}$ onto $\mathfrak{A}q$. In accordance with theorem 8 and 7.3.4, the sets $p\mathfrak{A}$ and $\mathfrak{A}p^{-1}$ are equivalent; by the theorem analogous to theorem 5 and valid for the right cosets, $\mathfrak{A}p^{-1}$ and $\mathfrak{A}q$ have the same property. Consequently, by 6.10.7, the assertion is correct.

20.3. Exercises

1. If $\mathfrak{G}$ is Abelian, then the left coset of an element $p \in \mathfrak{G}$ with regard to a subgroup $\mathfrak{A} \subset \mathfrak{G}$ is, at the same time, the right coset and so $p\mathfrak{A} = \mathfrak{A}p$.
2. Let $\mathfrak{A}, \mathfrak{B}$ denote arbitrary subgroups and C a complex in $\mathfrak{G}$. Prove that there holds: a) the sum of all left (right) cosets with regard to $\mathfrak{A}$ which are incident with C coincides with the complex $C\mathfrak{A}$ ($\mathfrak{A}C$); b) the sum $\mathfrak{B}p\mathfrak{A}$ of all left cosets with regard to $\mathfrak{A}$ which are incident with some right coset $\mathfrak{B}p$ ($p \in \mathfrak{G}$) coincides with the sum of all right cosets with regard to $\mathfrak{B}$ which are incident with the left coset $p\mathfrak{A}$.
3. Let $p \in \mathfrak{G}$ be an arbitrary element and $\mathring{\mathfrak{G}}$ the (p)-group associated with $\mathfrak{G}$ (19.7.11). Next, let $\mathfrak{A}$ be an arbitrary subgroup of $\mathfrak{G}$. Prove that: a) the left (right) coset $p\mathfrak{A}$ ($\mathfrak{A}p$) of p with regard to $\mathfrak{A}$ is the field of a subgroup $\mathring{\mathfrak{A}}_l \subset \mathring{\mathfrak{G}}$ ($\mathring{\mathfrak{A}}_r \subset \mathring{\mathfrak{G}}$) of $\mathring{\mathfrak{G}}$; b) the left (right) coset $x \circ \mathring{\mathfrak{A}}_l$ ($\mathring{\mathfrak{A}}_r \circ x$) coincides, for each element x of $\mathring{\mathfrak{G}}$, with the left (right) coset $x\mathfrak{A}$ ($\mathfrak{A}x$).

21. Decompositions generated by subgroups

A most remarkable property of groups is that every subgroup of an arbitrary group determines certain decompositions on the latter.

21.1. Left and right decompositions

Consider the system of all the subsets of the group $\mathfrak{G}$ given by the left cosets with regard to $\mathfrak{A}$. By 20.2.1, every element $p \in \mathfrak{G}$ is included in the left coset $p\mathfrak{A}$ which is, of course, an element of the considered system. By 20.2.4, every two