Linear Differential Transformations of the Second Order

1 Introduction

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General properties of ordinary linear homogeneous differential equations of the second order

1 Introduction

1.1 Preliminaries

In this book we shall be concerned with ordinary linear differential equations of the second order of the form

$$y'' = q(t)y \tag{q}$$

We shall suppose that the function q, which for brevity we call the *carrier* of the differential equation (q), is defined in an open bounded or unbounded interval $j = (a, b)$ and belongs to the class C_0. When necessary, the function q will naturally be required to have further properties. The symbol C_0 means as usual the class of functions which are continuous in the interval considered while C_k denotes the class of functions with continuous derivatives up to and including the k-th order (k=1, 2,. . .).

Differential equations of the form (q) are called *Sturm-Liouville* or *Jacobian* differential equations. We shall use the latter term.

A linear differential equation of the second order of the form

$$Y'' + a(x)Y' + b(x)y = 0, \tag{a}$$

whose coefficients a, b are defined in an interval J and belong to the class C_0, can be brought into the Jacobian form (q) by means of a transformation of the independent variable of the form

$$t = t_0 + t_0' \int_{x_0}^{x} \exp\left(-\int_{x_0}^{\sigma} a(\tau)\,\mathrm{d}\tau\right) \mathrm{d}\sigma \tag{1.1}$$

with arbitrary numbers $x_0 \in J$, t_0, $t_0' \neq 0$. The coefficient q is defined and continuous in the interval j, given by the range of the function $t(x)$, $x \in J$ and we have

$$q(t) = -\frac{1}{t_0'^2}\, b(x) \cdot \exp 2 \int_{x_0}^{x} a(\tau)\,\mathrm{d}\tau. \tag{1.2}$$

The connection between solutions of the differential equations (a), (q) is given by the formula

$$Y(x) = y(t) \tag{1.3}$$

in which $x \in J$ and $t \in j$ are homologous values, that is to say connected by the relation (1.1). Hence, if $Y(x)$ is a solution of the differential equation (a) then the function $y(t)$ defined by (3) represents a solution of the differential equation (q), and conversely.

There is another possible way of putting the differential equation (a) into Jacobian form, when the coefficient $a \in C_1$. In this case the transformation

$$Y = \exp\left(-\frac{1}{2}\int_{x_0}^{x} a(\tau)\, d\tau\right) \cdot y$$

of the dependent variable leads to the Jacobian differential equation (q) with the carrier

$$q(x) = \frac{1}{4} a^2(x) + \frac{1}{2} a'(x) - b(x)$$

in the interval J. By a solution of the differential equation (q) we mean a function $y \in C_2$, defined in an interval $i \subset j$ and satisfying (q). In the case when $i = j$ we shall generally use the term *integral* instead of the term *solution*.

It is known that there is precisely one integral y of the differential equation (q) passing through an arbitrary point (t_0, y_0), $t_0 \in j$, with arbitrary gradient y'_0; that is to say, such that $y(t_0) = y_0$, $y'(t_0) = y'_0$. The differential equation (q) is always satisfied by the identically zero function, but usually we shall leave this solution out of our consideration. Sometimes, for convenience, we associate ideas relating to the differential equation (q) with the carrier q itself, for instance we shall speak of solutions or integrals of the carrier q.

1.2 The Wronskian determinant

Given an ordered pair of solutions u, v of the differential equation (q), with the same interval of definition $i \subset j$, there is associated with them the Wronskian determinant or (simply) the Wronskian $w = uv' - u'v$ whose value is constant. The solutions u, v are linearly dependent on or independent of each other according as the Wronskian of the ordered pair u, v or of the pair v, u is equal to zero or different from zero. If the solutions u, v or their first derivatives possess zeros, they are dependent if and only if they or their first derivatives have a common zero. In this case the solutions u, v have all their zeros in common and the same is true for their derivatives u', v'.

Two functions $u, v \in C_2$ and defined in the interval j are independent integrals of a differential equation (q) if and only if the Wronskian $w = uv' - u'v$ is constant and not zero. The function q is then given by

$$q = \frac{1}{w}(u'v'' - u''v'). \tag{1.4}$$

If a solution u of the differential equation (q) is everywhere non-zero then the function $v(t)$ given by

$$v(t) = u(t) \cdot \int_{t_0}^{t} \frac{d\sigma}{u^2(\sigma)} \quad (t_0 \in j) \tag{1.5}$$

represents a further solution of (q). The solutions u, v are linearly independent: $w = 1$.

1.3 Bases

The set of all integrals of the differential equation q forms a two-dimensional linear space, the so-called *integral space* r of the differential equation (q). Every ordered pair of linearly independent integrals u, v of the differential equation (q) forms a *basis* (u, v) of r; an arbitrary integral $y \in r$ has uniquely determined constant coordinates c_1, c_2 with respect to the basis (u, v); that is to say $y = c_1u + c_2v$. Conversely, given any ordered pair of constants c_1, c_2 there is precisely one corresponding integral y of the differential equation (q) with the coordinates c_1, c_2. Any basis of r will also be called a basis of the differential equation (q). The bases (u, v), (v, u) will be called *inverse*, and two bases (u, v), (ku, kv), in which k ($\neq 0$) is an arbitrary constant, will be called *proportional*.

1.4 Integral curves

A basis (u, v) of the differential equation (q) determines a plane curve with parametric coordinates $u(t)$, $v(t)$. The curve can therefore be defined by means of the parametric representation $x_1 = u(t)$, $x_2 = v(t)$, with respect to a fixed coordinate system with origin O. We shall call such a curve an *integral curve* of the differential equation (q). If the variable t represents time then the integral curve can be considered as the trajectory of the point $P(t) = P[u(t), v(t)]$. The oriented area traced out by the radius vector $\overrightarrow{OP}$ in the time interval $t_1 < t < t_2$ has the value $-\frac{1}{2}w(t_2 - t_1)$, where w represents of course the Wronskian of u, v. An element of the two-dimensional linear homogeneous transformation group, that is to say a centroaffine plane transformation, transforms any integral curve of the differential equation (q) into another such integral curve. Those properties of an integral curve of the differential equation (q) which are invariant under such transformations, hold for all integral curves of (q) and are determined by appropriate properties of the carrier q of the differential equation (q). Conversely, those properties of the integral curves of the differential equation (q) which arise from special properties of the carrier q, have an invariant character with respect to centroaffine plane transformations.

1.5 Kinematic interpretation of integrals

Occasionally it is useful to regard the value $u(t)$ of an integral u of the differential equation (q) as the directed distance of a point P, moving on an oriented line G, from a fixed point or origin O on the line G. At any instant t ($\in j$) the point P is at a distance $u(t)$ units from the origin O, and lies in the positive or negative direction on the line G according as $u(t) > 0$ or < 0. The instants when P passes through the origin O are given precisely by the zeros of u. We say that the motion of the point P *follows* the integral u of the differential equation (q).

1.6 Types of differential equations (q)

All integrals of the differential equation (q) have the same oscillatory character, that is to say they all have either a finite or an infinite number of zeros in the interval j.

In the first case the differential equation (q) is said to be *of finite type* or *non-oscillatory*. More precisely, it is said to be *of type* (m), m integral, $m \geqslant 1$, if it possesses integrals with m zeros in the interval j but none with $m + 1$ zeros. In the second case the differential equation (q) is said to be *of infinite type*; specifically, it is described as *left* or *right oscillatory* according as the zeros of its integral cluster towards the left-hand end point or the right-hand end point of the interval j, and *oscillatory* if the zeros cluster towards both end points. Alternatively, we describe a differential equation (q) of infinite type as being of the *first*, *second* or *third category*.

Later (§§ 3.6, 3.10, 7.2) we shall have occasion to separate differential equations (q) of finite type (m), $m \geqslant 1$, into general and special differential equations. The term *kind* of a differential equation (q) will have two meanings; if (q) is of finite type, then its "kind" denotes whether it is general or special; if (q) is of infinite type than its "kind" is its category.

All zeros of integrals of the differential equation (q) are isolated.

1.7 The Schwarzian derivative

We shall now consider a bi-rational transformation T defined in a three-dimensional real coordinate space S_3, specified by the formulae*

$$\left.\begin{aligned} X' &= \frac{1}{\dot{x}}, & \dot{x} &= \frac{1}{X'}, \\ X'' &= -\frac{\ddot{x}}{\dot{x}^3}, & \ddot{x} &= -\frac{X''}{X'^3}, \\ X''' &= 3\frac{\ddot{x}^2}{\dot{x}^5} - \frac{\dddot{x}}{\dot{x}^4}, & \dddot{x} &= 3\frac{X''^2}{X'^5} - \frac{X'''}{X'^4}. \end{aligned}\right\} \tag{1.6}$$

It thus associates with each other two points X' $(\neq 0)$, X'', X'''; and $\dot{x}$ $(\neq 0)$, $\ddot{x}$, $\dddot{x}$. This transformation T leaves the function

$$K(X) = \frac{X''^2}{X'^3} \tag{1.7}$$

invariant, that is to say

$$K(X) = K(x). \tag{1.8}$$

The so-called Schwarzian function

$$S(X) = \frac{1}{2}\frac{X'''}{X'} - \frac{3}{4}\frac{X''^2}{X'^2} \tag{1.9}$$

is transformed as follows:

$$\frac{S(X)}{X'} + \frac{S(x)}{\dot{x}} = 0. \tag{1.10}$$

* It is useful to note the convention, adopted throughout, that differentiation with respect to t is indicated by a prime, and with respect to T by a dot. In (1.6), however, primes and dots serve only to label the coordinates X', $\dot{x}$, etc. (Trans.)

The transformation T is of particular importance in the study of relationships between the values of two (mutually) inverse functions of one variable and of their derivatives.

Let $X(t)$, $x(T)$ be two inverse functions whose intervals of definition we shall denote by i, I respectively. We naturally suppose that the functions X, x are monotonic in the intervals $i = x(I)$, $I = X(i)$. For convenience of terminology we call the numbers t, T *homologous* if they are related by the formulae $T = X(t)$, $t = x(T)$, sometimes the number $t(T)$ will be described as the number homologous to $T(t)$.

We assume that the functions X, x are three times differentiable in the intervals i, I, and $X' \neq 0$, $\dot{x} \neq 0$. Then the rules of differentiation give the following formulae, holding at two homologous numbers $t \in i, T \in I$:

$$\left.\begin{aligned} X'\dot{x} &= 1, \\ X''\dot{x} + X'^2\ddot{x} = 0; \qquad \ddot{x}X' + \dot{x}^2X'' &= 0, \\ X'''\dot{x}^2 + 3X''\ddot{x} + \dddot{x}X'^2 &= 0. \end{aligned}\right\} \tag{1.11}$$

Thus the bi-rational transformation T has the property that the values of the derivatives of two inverse functions X, x go over into each other at homologous points.

1.8 Projective property of the Schwarzian derivative

Let $X(t)$ be a three times differentiable function in the interval j, whose derivative X' is everywhere non-zero, i.e. $X'(t) \neq 0$ for $t \in j$. By the *Schwarzian derivative* of the function X we mean the Schwarzian function $S(X)$ formed with the derivatives X', X'', X'''. The value of the Schwarzian derivative of X at the point $t \in j$ will be denoted by $\{X, t\}$, that is to say

$$\{X, t\} = \frac{1}{2}\frac{X'''(t)}{X'(t)} - \frac{3}{4}\frac{X''^2(t)}{X'^2(t)}. \tag{1.12}$$

A simple calculation yields the relationship

$$\{X, t\} = -\sqrt{|X'|}\left(\frac{1}{\sqrt{|X'|}}\right)'', \tag{1.13}$$

in which on the right-hand side we take the value of the function at the point t. Schwarzian derivatives are of particular value in the linear transformation of functions. A fundamental theorem is the following:

Theorem. The Schwarzian derivatives $\{X, t\}$, $\{Y, t\}$ of two functions $X, Y \in C_3$ in an interval j are identical if and only if X and Y are related projectively, that is

$$Y(t) = \frac{c_{11}X(t) + c_{10}}{c_{21}X(t) + c_{20}}, \tag{1.14}$$

$t \in j$, $c_{10}, c_{11}, c_{20}, c_{21} = \text{const.}$

Proof. Simple calculation shows that the condition for the identity of the two Schwarzian derivatives is sufficient; we have therefore only to prove its necessity.

Let $\{X, t\} = \{Y, t\}$ in the interval j; for brevity, set $\{X, t\} = \{Y, t\} = q(t)$, then $q(t)$ is a continuous function. The formulae (13) shows that the functions $1/\sqrt{|X'|}$, $1/\sqrt{|Y'|}$ are integrals of the differential equation (−q). Since these functions are everywhere non-zero then (from (5)), $X/\sqrt{|X'|}$, $Y/\sqrt{|Y'|}$ are also integrals of the same differential equation (−q), the integrals $1/\sqrt{|X'|}$, $X/\sqrt{|X'|}$ and also the integrals $1/\sqrt{|Y'|}$, $Y/\sqrt{|Y'|}$ being linearly independent. Consequently there exist constants $c_{10}, c_{11}, c_{20}, c_{21}$ such that

$$\frac{Y}{\sqrt{|Y'|}} = c_{11}\frac{X}{\sqrt{|X'|}} + c_{10}\frac{1}{\sqrt{|X'|}},$$
$$\frac{1}{\sqrt{|Y'|}} = c_{21}\frac{X}{\sqrt{|X'|}} + c_{20}\frac{1}{\sqrt{|X'|}},$$

and from this the relation (14) follows. This completes the proof.

We must now mention two further properties of the Schwarzian derivative which are important for our further studies. In this, $X(t)$, $x(T)$ are three times differentiable functions, inverse to each other in the intervals i, I.

1. By easy calculation we find that the following relationship holds at arbitrary points $t \in i$, $T \in I$;

$$\left.\begin{aligned} \frac{\{X, t\}}{X'} &= \frac{1}{4}\frac{X''^2}{X'^3} - \frac{1}{2}\left(\frac{1}{X'}\right)'', \\ \frac{\{x, T\}}{\dot{x}} &= \frac{1}{4}\frac{\ddot{x}^2}{\dot{x}^3} - \frac{1}{2}\left(\frac{1}{\dot{x}}\right)^{\cdot\cdot}. \end{aligned}\right\} \tag{1.15}$$

In particular, if the numbers t, T are homologous, then formulae such as (6), (8), are valid and we obtain the following result:

At two homologous points $t \in i$, $T \in I$ there holds the symmetric relationship

$$\frac{\{X, t\}}{X'} + \frac{1}{2}\left(\frac{1}{X'}\right)'' = \frac{\{x, T\}}{\dot{x}} + \frac{1}{2}\left(\frac{1}{\dot{x}}\right)^{\cdot\cdot}. \tag{1.16}$$

2. Let Z be a three times differentiable function in an interval h, such that $Z(t) \subset i$ and $Z'(t) \neq 0$ for $t \in h$.

Then the composite function XZ exists in the interval h; its Schwarzian derivative also exists there and we have the relationship

$$\{XZ, t\} = \{X, Z(t)\}\, Z'^2(t) + \{Z, t\}. \tag{1.17}$$

1.9 Associated differential equations

In this section we assume that the carrier q of the differential equation (q) in the interval j is everywhere non-zero and $\in C_2$. We then define in the interval j the following differential equation, which we call the *first associated differential equation* ($\hat{q}_1$) of (q):

$$y'' = \hat{q}_1(t) y_1, \tag{q$_1$}$$

where

$$\hat{q}_1(t) = q(t) + \sqrt{|q(t)|}\left(\frac{1}{\sqrt{|q(t)|}}\right)''. \tag{1.18}$$

The function $\hat{q}_1$, the so-called *first associated carrier of* q, can obviously be put into the form

$$\hat{q}_1(t) = q(t) - \frac{1}{2}\frac{q''(t)}{q(t)} + \frac{3}{4}\frac{q'^2(t)}{q^2(t)} \tag{1.19}$$

or alternatively

$$\hat{q}_1(t) = q(t) - \left\{\int_{t_0}^{t} q(\sigma)\,d\sigma, t\right\} \quad (t_0 \in j). \tag{1.20}$$

The significant connection between the differential equations (q), ($\hat{q}_1$) lies in the fact that given any integral y of the differential equation (q) the function

$$y_1(t) = \frac{y'(t)}{\sqrt{|q(t)|}} \tag{1.21}$$

is an integral of the differential equation ($\hat{q}_1$). We have also the converse relationship:

For every integral y_1 of the differential equation ($\hat{q}_1$) *the function $y_1\sqrt{|q(t)|}$ represents the derivative y' of precisely one integral y of* (q).

Proof. Let y_1 be an integral of ($\hat{q}_1$). We choose an arbitrary number $t_0 \in j$.

(a) We suppose that there is an integral y of the differential equation (q) such that in the interval j

$$y_1\sqrt{|q|} = y'. \tag{1.22}$$

At the point t_0 the integral y and its derivative y' obviously take the values

$$y_0 = \frac{1}{q(t_0)}\,[y_1\sqrt{|q|}]'_{t=t_0}; \qquad y'_0 = y_1(t_0)\sqrt{|q(t_0)|}. \tag{1.23}$$

We see that there is at most one integral y of (q) satisfying the relation (22), namely that integral of (q) determined by the initial values (23).

(b) We now define, in the interval j, the function y as follows

$$y(t) = \frac{1}{q(t_0)}\,[y_1\sqrt{|q|}]'_{t=t_0} + \int_{t_0}^{t} y_1(\sigma)\sqrt{|q(\sigma)|}\,d\sigma.$$

The function y and its derivative obviously take the values (23) at the point t_0. Moreover, the condition (22) clearly holds in the interval j. Then it follows easily that y_1 satisfies the differential equation ($\hat{q}_1$), and by application of the formula (19) we see that it also satisfies the equation

$$y''' - y''\frac{q'}{q} - y'q = 0$$

which may be written

$$\left[\frac{y'' - qy}{q}\right]' = 0.$$

Consequently we have

$$y'' - qy = kq \qquad (k = \text{const}),$$

and then equation (22) and the initial values given in (23) show that $k = 0$. The function y is consequently an integral of the differential equation (q), and the proof is complete.

The mapping P of the integral space r of (q) on the integral space r_1 of $(\hat{q}_1)$, by which each integral $y \in r$ is mapped into the integral $y_1 = y'/\sqrt{|q|} \in r_1$, is called the *projection* of the integral space r onto the integral space r_1. We also say that y_1 $(= Py)$ is the projection of y, and call the integrals y, y_1 *associated.* The reader may easily verify that the Wronskians of two bases (u, v), (Pu, Pv), of r and r_1 respectively, have the same value.

The differential equation $(\hat{q}_1)$ represents the first associated differential equation of (q). The n-th *associated differential equation* $(\hat{q}_n)$ of (q) is defined as the first associated differential equation of $(\hat{q}_{n-1})$. For example, if we take the Bessel differential equation

$$y'' = -\left(1 + \frac{1 - 4\nu^2}{4t^2}\right) y \quad (j = (0, \infty),\ \nu = \text{const}) \tag{1.24}$$

then the first associated differential equation belonging to this is

$$y'' = -\left(1 + \frac{1 - 4\nu^2}{4t^2} + \frac{12(1 - 4\nu^2)}{(4t^2 + 1 - 4\nu^2)^2}\right) y. \tag{1.25}$$