Linear Differential Transformations of the Second Order

13 Properties of central dispersions

In: Otakar Borůvka (author); Felix M. Arscott (translator): Linear Differential Transformations of the Second Order. (English). London: The English Universities Press, Ltd., 1971. pp. [118]–134.

Persistent URL: http://dml.cz/dmlcz/401683

13 Properties of central dispersions

In this section we shall investigate some elementary properties of central dispersions, particularly their behaviour, their continuity and other properties associated with the existence of derivatives of central dispersions.

13.1 Monotonicity and continuity

1. *The range of each central dispersion of any kind is the interval j.*

For, if Δ is a central dispersion of any kind and $t \in j$ an arbitrary number, then the function Δ takes the value t at the point $\Delta^{-1}(t)$.

2. *Every central dispersion of any kind is an increasing function.*

Proof. As every central dispersion of any kind can be constructed by composition of the fundamental dispersion and its inverse (§ 12.4) it is sufficient to show the truth of this statement for the fundamental dispersion. Let, therefore, δ be a fundamental dispersion of arbitrary kind. In this paragraph we shall denote the fundamental dispersions briefly by ϕ, ψ, χ, ω.

Let $t < x$ be an arbitrary number in the interval j. We have to show that $\delta(t) < \delta(x)$. From (12.1) we have $t < \delta(t)$, $x < \delta(x)$. If $\delta(t) < x$, then we already have $\delta(t) < \delta(x)$. It will therefore be sufficient to show that the inequality $t < x < \delta(t)$ implies the inequality $\delta(t) < \delta(x)$. This we achieve by showing, on the basis of the ordering theorems, the impossibility of the inequality $t < x < \delta(x) < \delta(t)$. The equality relation $\delta(t) = \delta(x)$ is obviously impossible, because the inverse function δ^{-1} exists.

We therefore assume that $t < x < \delta(x) < \delta(t)$, and consider the four kinds of dispersion separately.

(a) $\delta = \phi$. From the definition of the function ϕ, the numbers t, $\delta(t)$ and x, $\delta(x)$ are respectively neighbouring zeros of integrals u, v of the differential equation (q). Obviously these integrals u, v, are independent. From the first ordering theorem (§ 2.3) precisely one zero of the integral v lies between t and $\delta(t)$, which is obviously inconsistent with the above inequalities.

(b) $\delta = \psi$. The proof proceeds on the same lines as (a), using the second ordering theorem.

(c) $\delta = \chi$. From the definition of the function χ, the numbers t, $\delta(t)$ are two neighbouring zeros of an integral v of the differential equation (q) and its derivative v'. The function v' is consequently non-zero between t and $\delta(t)$. Similarly, x, $\delta(x)$ are two neighbouring zeros of an integral u of the differential equation (q) and its derivative u'. Obviously the integrals u, v are independent. From the fourth ordering theorem the function v' has a zero between t and $\delta(x)$, and so (by the above inequalities) a zero between t and $\delta(t)$, which yields a contradiction.

(d) $\delta = \omega$. The proof is similar to that of case (c), using the third ordering theorem, and the proof is complete.

3. *Every central dispersion of arbitrary kind is everywhere continuous.*

This is an immediate consequence of the above theorems 1 and 2.

13.2 The functional equation of the central dispersions

In this paragraph we denote by ϕ, ψ, χ, ω an arbitrary central dispersion of the first, second, third and fourth kinds respectively. Let u, v be arbitrary independent integrals of the differential equation (q).

From the theorem in § 3.12 we conclude that: *In the interval j there hold the following identical relationships*

$$\left.\begin{aligned} u(t)v[\phi(t)] - u[\phi(t)]v(t) &= 0, \\ u'(t)v'[\psi(t)] - u'[\psi(t)]v'(t) &= 0, \\ u(t)v'[\chi(t)] - u'[\chi(t)]v(t) &= 0, \\ u'(t)v[\omega(t)] - u[\omega(t)]v'(t) &= 0. \end{aligned}\right\} \tag{13.1}$$

These relationships are called the *functional equations of the central dispersions.*

We see that the ratios u/v, u'/v' of the integrals u, v and their derivatives u', v' are invariant in the interval j with respect to composition of central dispersions, in the sense of the following formulae:

$$\left.\begin{aligned} \frac{u(t)}{v(t)} &= \frac{u[\phi(t)]}{v[\phi(t)]}, & \frac{u'(t)}{v'(t)} &= \frac{u'[\psi(t)]}{v'[\psi(t)]}, \\ \frac{u(t)}{v(t)} &= \frac{u'[\chi(t)]}{v'[\chi(t)]}, & \frac{u'(t)}{v'(t)} &= \frac{u[\omega(t)]}{v[\omega(t)]}. \end{aligned}\right\} \tag{13.2}$$

13.3 Derivatives of central dispersions

Again let ϕ, ψ, χ, ω denote arbitrary central dispersions of the first, second, third and fourth kinds.

We now prove the following theorem:

(1) *All central dispersions of any kind have at each point $t \in j$ continuous (first) derivatives. These can be represented in terms of arbitrary independent integrals*

u, v of the differential equation (q) *and their derivatives u', v' by the following formulae:*

$$\left.\begin{aligned}
\phi'(t) &= -\frac{u'(t)\,v[\phi(t)] - u[\phi(t)]\,v'(t)}{u(t)\,v'[\phi(t)] - u'[\phi(t)]\,v(t)};\\
\psi'(t) &= -\frac{q(t)}{q[\psi(t)]}\cdot\frac{u(t)\,v'[\psi(t)] - u'[\psi(t)]\,v(t)}{u'(t)\,v[\psi(t)] - u[\psi(t)]\,v'(t)};\\
\chi'(t) &= -\frac{1}{q[\chi(t)]}\cdot\frac{u'(t)\,v'[\chi(t)] - u'[\chi(t)]\,v'(t)}{u(t)\,v[\chi(t)] - u[\chi(t)]\,v(t)};\\
\omega'(t) &= -q(t)\,.\,\frac{u(t)\,v[\omega(t)] - u[\omega(t)]\,v(t)}{u'(t)\,v'[\omega(t)] - u'[\omega(t)]\,v'(t)}.
\end{aligned}\right\} \tag{13.3}$$

Proof. We shall confine ourselves to the case of the central dispersion ϕ. Let u, v be two linearly independent integrals of the differential equation (q). We consider the basis function

$$F(t, x) = u(t)v(x) - u(x)v(t).$$

(see § 2.6; $Q = q$).

Let $t_0 = j$ be arbitrary and $x_0 = \phi(t_0)$ the corresponding value of ϕ. Then we have $F(t_0, x_0) = 0$. (§ 3.12). From § 2.6 there is precisely one continuous function $x(t)$ defined in a neighbourhood i ($\subset j$) of t_0, which takes the value x_0 at the point t_0 and satisfies the equation $F[t, x(t)] = 0$ in the interval i. This function x possesses, in the interval i, the continuous derivative

$$x'(t) = -\frac{u'(t)\,v[x(t)] - u[x(t)]\,v'(t)}{u(t)\,v'[x(t)] - u'[x(t)]\,v(t)}. \tag{13.4}$$

But the function ϕ is defined and continuous in the interval j, and consequently also in the interval i, and satisfies the above equation $F[t, \phi(t)] = 0$ in i; consequently $x(t) = \phi(t)$ for $t \in i$. The existence of $\phi'(t_0)$ follows, and hence, taking account of (4), the first formula (3).

2. *The derivatives of central dispersions may be represented as follows in terms of an integral u of the differential equation* (q) *and its derivative u'*:—

$$\phi'(t) = \begin{cases} \dfrac{u^2[\phi(t)]}{u^2(t)} & \text{for} \quad u(t) \neq 0,\\[2ex] \dfrac{u'^2(t)}{u'^2[\phi(t)]} & \text{for} \quad u(t) = 0; \end{cases} \tag{13.5}$$

$$\psi'(t) = \begin{cases} \dfrac{q(t)}{q[\psi(t)]}\cdot\dfrac{u'^2[\psi(t)]}{u'^2(t)} & \text{for} \quad u'(t) \neq 0,\\[2ex] \dfrac{q(t)}{q[\psi(t)]}\cdot\dfrac{u^2(t)}{u^2[\psi(t)]} & \text{for} \quad u'(t) = 0; \end{cases} \tag{13.6}$$

$$\chi'(t) = \begin{cases} -\dfrac{1}{q[\chi(t)]} \cdot \dfrac{u'^2[\chi(t)]}{u^2(t)} & \text{for} \quad u(t) \neq 0, \\ -\dfrac{1}{q[\chi(t)]} \cdot \dfrac{u'^2(t)}{u^2[\chi(t)]} & \text{for} \quad u(t) = 0; \end{cases} \tag{13.7}$$

$$\omega'(t) = \begin{cases} -q(t) \cdot \dfrac{u^2[\omega(t)]}{u'^2(t)} & \text{for} \quad u'(t) \neq 0, \\ -q(t) \cdot \dfrac{u^2(t)}{u'^2[\omega(t)]} & \text{for} \quad u'(t) = 0. \end{cases} \tag{13.8}$$

Proof. We shall restrict ourselves to the proof of (5); essentially it is obtained by a transformation of the first formula (3).

In the case $u(t) \neq 0$ we multiply the numerator and denominator of the first formula of (3) by $u(t)$ and in the numerator make use of the relationship $u(t)v[\phi(t)] = u[\phi(t)]v(t)$; then we proceed analogously in the denominator, multiplying by $u[\phi(t)]$ and making use of the same relationship. After division by the Wronskian $w(t) = w[\phi(t)]$ of u, v we obtain the first formula (5).

In the case $u(t) = 0$ we have $u'(t)\, u'[\phi(t)] \neq 0$. We multiply, as above, the numerator and denominator by $u'(t)$, then by $u'[\phi(t)]$ and so obtain the second formula (5).

3. *The derivatives of the central dispersions can be represented as follows by means of the first or second amplitudes r, s of an arbitrary basis* (u, v) *of the differential equation* (q):—

$$\left.\begin{aligned} \phi'(t) &= \frac{r^2[\phi(t)]}{r^2(t)}, & \psi'(t) &= \frac{q(t)}{q[\psi(t)]} \cdot \frac{s^2[\psi(t)]}{s^2(t)}, \\ \chi'(t) &= -\frac{1}{q[\chi(t)]} \cdot \frac{s^2[\chi(t)]}{r^2(t)}, & \omega'(t) &= -q(t) \cdot \frac{r^2[\omega(t)]}{s^2(t)}. \end{aligned}\right\} \tag{13.9}$$

Proof. We give the proof of the first formula as an example.

Let (u, v) be a basis of the differential equation (q) and r, s the corresponding first and second amplitudes. Let $t \in j$ be arbitrary. At least one of the two numbers $u(t)$, $v(t)$ is non-zero; let us assume, for definiteness, that it is $u(t)$. Considering the function $\lambda = v/u$ we have (from (2)) $\lambda(t) = \lambda[\phi(t)]$, and consequently

$$\phi'(t) = \frac{u^2[\phi(t)]}{u^2(t)} = \frac{1 + \lambda^2(t)}{1 + \lambda^2[\phi(t)]} \cdot \frac{r^2[\phi(t)]}{r^2(t)} = \frac{r^2[\phi(t)]}{r^2(t)},$$

which is the first formula of (9).

13.4 Higher derivatives

We see from the above results that

If the carrier q of the differential equation (q) *is continuous in the interval j, all central dispersions of the first kind have continuous derivatives of the third order in that interval and all other central dispersions have continuous derivatives of the first order.*

More precisely we have

If the carrier q of the differential equation (q) *belongs to the class* C_k $(k = 0, 1, \ldots)$ *then all central dispersions of the first kind* $\in C_{k+3}$ *and all other central dispersions* $\in C_{k+1}$.

13.5 The connection between central dispersions and the transformation problem

The formulae (5)–(8) have an important bearing on the transformation problem (§ 11).

Let $\phi_\nu, \psi_\nu, \chi_\rho, \omega_\rho$ $(\nu = 0, \pm 1, \ldots; \rho = \pm 1, \ldots)$ be arbitrary central dispersions of the first, second, third and fourth kinds, and let u be an arbitrary integral of (q). Then the above formulae (5)–(8) hold for these central dispersions.

Now, on taking account of the ordering theorems, (§ 2.3), these formulae give the following relationships for all $t \in j$:

$$u(t) = \frac{(-1)^\nu}{\sqrt{\phi_\nu'(t)}}\, u[\phi_\nu(t)], \tag{13.10}$$

$$\frac{u'(t)}{\sqrt{-q(t)}} = \frac{(-1)^\nu}{\sqrt{\psi_\nu'(t)}}\, \frac{u'[\psi_\nu(t)]}{\sqrt{-q[\psi_\nu(t)]}}, \tag{13.11}$$

$$u(t) = \frac{(-1)^\rho}{\sqrt{\chi_\rho'(t)}}\, \frac{u'[\chi_\rho(t)]}{\sqrt{-q[\chi_\rho(t)]}}, \tag{13.12}$$

$$\frac{u'(t)}{\sqrt{-q(t)}} = \frac{(-1)^\rho}{\sqrt{\omega_\rho'(t)}}\, u[\omega_\rho(t)]. \tag{13.13}$$

By formula (10), therefore, the ordered pair $[(-1)^\nu/\sqrt{\phi_\nu'(t)}, \phi_\nu(t)]$ represents a transformation (§ 11.2) of the differential equation (q) into itself in which every integral u of (q) is transformed into itself.

Similarly the formulae (11)–(13) show that if we assume that the function $q\ (< 0) \in C_2$ then the differential equation (q) admits of the associated differential equation $(\hat{q}_1)$. We know (§ 1.9) that the function $u_1(t) = u'(t)/\sqrt{(-q(t))}$ represents an integral of the differential equation $(\hat{q}_1)$, namely the integral associated with u.

The ordered pair $[(-1)^\nu/\sqrt{\psi_\nu'(t)}, \psi_\nu(t)]$ obviously represents a transformation of the associated differential equation $(\hat{q}_1)$ into itself, in which every integral u_1 of $(\hat{q}_1)$ is transformed into itself.

The ordered pair $[(-1)^\rho/\sqrt{\chi_\rho'(t)}, \chi_\rho(t)]$ represents a transformation of $(\hat{q}_1)$ into (q). In this transformation every integral u_1 of the differential equation $(\hat{q}_1)$ is transformed into the associated integral u of (q).

The ordered pair $[(-1)^\rho/\sqrt{\omega_\rho'(t)}, \omega_\rho(t)]$ represents a transformation of (q) into $(\hat{q}_1)$. In this transformation every integral u of the differential equation (q) is transformed into the associated integral u_1 of $(\hat{q}_1)$.

To sum up; *the central dispersions of oscillatory differential equations* (q) *are particular solutions of the Kummer transformation problem for the differential equation* (q) *and its associated differential equation* $(\hat{q}_1)$.

13.6 Relations between derivatives of the central dispersions and the values of the carrier q

1. Let now ϕ, ψ, χ, ω be the *fundamental* dispersions of the first, second, third and fourth kinds. We have the following result:

Theorem. The first derivatives of the fundamental dispersions at every point $t \in j$ may be expressed as ratios of appropriate values taken by the carrier q, as follows

$$\phi'(t) = \frac{q(t_1)}{q(t_3)}, \qquad \psi'(t) = \frac{q(t)\ q(t_4)}{q[\psi(t)]q(t_2)},$$
$$\chi'(t) = \frac{q(t_1)}{q[\chi(t)]}, \quad \omega'(t) = \frac{q(t)}{q(t_2)};$$

in which t_1, t_2, t_3, t_4 denote appropriate numbers ordered as follows

$$t < t_1 < \chi(t) < t_3 < \phi(t); \quad t < t_2 < \omega(t) < t_4 < \psi(t).$$

Proof. For every integral u of the differential equation (q) and arbitrary numbers t, $x \in j$ we have obviously the formula

$$u'^2(x) - u'^2(t) = \int_t^x q(\sigma)[u^2(\sigma)]'\, d\sigma, \tag{13.14}$$

which provides the basis for our proof.

(a) Let u be any integral of the differential equation (q), which vanishes at the point t, so that $u(t) = u'[\chi(t)] = 0$. In formula (14) we set $x = \chi(t)$, so obtaining

$$-u'^2(t) = \int_t^{\chi(t)} q(\sigma)[u^2(\sigma)]'\, d\sigma.$$

Now, in the interval $(t, \chi(t))$ the function uu' and consequently also the function $(u^2)'$ is positive. By the mean value theorem we have

$$\int_t^{\chi(t)} q(\sigma) \cdot [u^2(\sigma)]'\, d\sigma = q(t_1)u^2[\chi(t)],$$

where $t < t_1 < \chi(t)$. The last two formulae give

$$-u'^2(t) = q(t_1)u^2[\chi(t)],$$

and consequently, using (7),

$$\chi'(t) = \frac{q(t_1)}{q[\chi(t)]} \qquad (t < t_1 < \chi(t)).$$

(b) Now let u be any integral of the differential equation (q), whose derivative u' vanishes at the point t: $u'(t) = u[\omega(t)] = 0$.

In formula (14) we set $x = \omega(t)$ giving

$$u'^2[\omega(t)] = \int_t^{\omega(t)} q(\sigma)[u^2(\sigma)]'\, d\sigma.$$

In the interval $(t, \omega(t))$ the function uu', and consequently also $(u^2)'$ is negative. By the mean value theorem we have

$$\int_t^{\omega(t)} q(\sigma)[u^2(\sigma)]' \, d\sigma = -q(t_2)u^2(t),$$

in which $t < t_2 < \omega(t)$. So

$$u'^2[\omega(t)] = -q(t_2)u^2(t),$$

from which, using (8) we obtain

$$\omega'(t) = \frac{q(t)}{q(t_2)} \qquad (t < t_2 < \omega(t)).$$

(c) From the identical relationship

$$\phi(t) = \omega[\chi(t)]$$

we obtain, on applying the above results,

$$\phi'(t) = \omega'[\chi(t)]\chi'(t) = \frac{q[\chi(t)]}{q(t_3)} \cdot \frac{q(t_1)}{q[\chi(t)]} = \frac{q(t_1)}{q(t_3)},$$

in which $\chi(t) < t_3 < \phi(t)$. Consequently we have

$$\phi'(t) = \frac{q(t_1)}{q(t_3)} \qquad (\chi(t) < t_3 < \phi(t)).$$

(d) Similarly, we obtain from the identity

$$\psi(t) = \chi[\omega(t)]$$

the formula

$$\psi'(t) = \frac{q(t)}{q(t_2)} \cdot \frac{q(t_4)}{q[\psi(t)]} \qquad (\omega(t) < t_4 < \psi(t)).$$

2. The above formulae apply, as we have said, to the fundamental dispersions of appropriate kinds. More generally, for $n = 1, 2, \ldots$ we have the following relations.

$$\phi_n'(t) = \frac{q(t_1)}{q(t_3)} \cdot \frac{q(t_5)}{q(t_7)} \cdots \frac{q(t_{4n-3})}{q(t_{4n-1})};$$

$$\chi_n'(t) = \frac{q(t_1)}{q(t_3)} \cdot \frac{q(t_5)}{q(t_7)} \cdots \frac{q(t_{4n-7})}{q(t_{4n-5})} \cdot \frac{q(t_{4n-3})}{q[\chi_n(t)]};$$

$$\phi_{\mu-1}(t) < t_{4\mu-3} < \chi_\mu(t) < t_{4\mu-1} < \phi_\mu(t); \quad \mu = 1, 2, \ldots, n.$$

$$\psi_n'(t) = \frac{q(t)}{q(t_2)} \cdot \frac{q(t_4)}{q(t_6)} \cdots \frac{q(t_{4n-4})}{q(t_{4n-2})} \cdot \frac{q(t_{4n})}{q[\psi_n(t)]};$$

$$\omega_n'(t) = \frac{q(t)}{q(t_2)} \cdot \frac{q(t_4)}{q(t_6)} \cdots \frac{q(t_{4n-4})}{q(t_{4n-2})};$$

$$\psi_{\mu-1}(t) < t_{4\mu-2} < \omega_\mu(t) < t_{4\mu} < \psi_\mu(t); \quad \mu = 1, 2, \ldots, n.$$

$$\phi'_{-n}(t) = \frac{q(t_{-1})}{q(t_{-3})} \cdot \frac{q(t_{-5})}{q(t_{-7})} \cdots \frac{q(t_{-(4n-3)})}{q(t_{-(4n-1)})};$$

$$\chi'_{-n}(t) = \frac{q(t_{-1})}{q(t_{-3})} \cdot \frac{q(t_{-5})}{q(t_{-7})} \cdots \frac{q(t_{-(4n-7)})}{q(t_{-(4n-5)})} \cdot \frac{q(t_{-(4n-3)})}{q[\chi_{-n}(t)]};$$

$$\phi_{-\mu}(t) < t_{-(4n-1)} < \chi_{-\mu}(t) < t_{-(4\mu-3)} < \phi_{-\mu+1}(t); \quad \mu = 1, 2, \ldots, n.$$

$$\psi'_{-n}(t) = \frac{q(t)}{q(t_{-2})} \cdot \frac{q(t_{-4})}{q(t_{-6})} \cdots \frac{q(t_{-(4n-4)})}{q(t_{-(4n-2)})} \cdot \frac{q(t_{-4n})}{q[\psi_{-n}(t)]};$$

$$\omega'_{-n}(t) = \frac{q(t)}{q(t_{-2})} \cdot \frac{q(t_{-4})}{q(t_{-6})} \cdots \frac{q(t_{-(4n-4)})}{q(t_{-(4n-2)})};$$

$$\psi_{-\mu}(t) < t_{-4\mu} < \omega_{-\mu}(t) < t_{-(4\mu-2)} < \psi_{-\mu+1}(t); \quad \mu = 1, 2, \ldots, n.$$

These formulae for ϕ'_n, χ'_n, ψ'_n, ω'_n are easily proved by induction, using the relations (12.7) and the above formulae for ϕ', χ', ψ', ω'.

Moreover, from the relations $\phi\phi_{-1}(t) = t$, $\psi\psi_{-1}(t) = t$, and the formulae for ϕ', ψ' there follow the further relations

$$\phi'_{-1}(t) = \frac{q(t_{-1})}{q(t_{-3})}; \qquad \psi'_{-1}(t) = \frac{q(t)}{q(t_{-2})} \cdot \frac{q(t_{-4})}{q[\psi_{-1}(t)]}; \tag{13.15}$$

in which $\phi_{-1}(t) < t_{-3} < \chi_{-1}(t) < t_{-1} < t$; $\psi_{-1}(t) < t_{-4} < \omega_{-1}(t) < t_{-2} < t$. The formulae for ϕ'_{-n}, χ'_{-n}, ψ'_{-n}, ω'_{-n} are easily proved by induction, making use of the relations (12.7), (13.15), and the formulae for χ', ω'.

13.7 Relations between central dispersions and phases

Let α, β be respectively a first and second phase of the basis (u, v) of the differential equation (q). We assume, for definiteness, that these phases are adjacent in the mixed phase system of the basis (u, v) (§ 5.14); that is to say, one of the relations $0 < \beta - \alpha < \pi$, $-\pi < \beta - \alpha < 0$ is satisfied in the interval j. By virtue of our assumption that $q < 0$ we have $\operatorname{sgn} \alpha' = \operatorname{sgn} \beta'$ $(= \varepsilon)$. In what follows ϕ, ψ, χ, ω denote *fundamental* dispersions of the four kinds.

Let $x \in j$ be an arbitrary number.

First we consider an integral y of the differential equation (q), which vanishes at the point x. From (5.27) we have

$$\left.\begin{aligned} y(t) &= k \cdot r(t) \cdot \sin [\alpha(t) - \alpha(x)], \\ y'(t) &= \pm k \cdot s(t) \cdot \sin [\beta(t) - \alpha(x)], \end{aligned}\right\} \tag{13.16}$$

where naturally k $(\neq 0)$ is an appropriate constant.

For simplicity, let us set $A(t) = \alpha(t) - \alpha(x)$. Obviously, we have $A(x) = 0$. In the interval j the function A tends monotonically to $+\infty$ or $-\infty$ according as $\varepsilon = 1$ or $\varepsilon = -1$. Consequently, there is a number x_1 $(> x)$, for which the function A takes the value $\varepsilon\pi$. From (16), however, x_1 is the first zero of the integral y to the right of x; we have therefore $x_1 = \phi(x)$ and moreover

$$\alpha\phi(x) = \alpha(x) + \varepsilon\pi. \tag{13.17}$$

Now we consider the function $B(t) = \beta(t) - \alpha(x)$. This function also tends monotonically in the interval j to $+\infty$ or $-\infty$, according as $\varepsilon = 1$ or $\varepsilon = -1$. Consequently there is a number x_3 ($> x$), for which the function B takes the value $\frac{1}{2}(\varepsilon + 1)\pi$ or $\frac{1}{2}(\varepsilon - 1)\pi$, according as $0 < \beta(x) - \alpha(x) < \pi$ or $-\pi < \beta(x) - \alpha(x) < 0$. From (16), however, x_3 is the first zero of the function y' to the right of x. We have therefore $x_3 = \chi(x)$ and moreover

$$\left.\begin{aligned} \beta\chi(x) &= \alpha(x) + \frac{1}{2}(\varepsilon + 1)\pi \quad \text{if} \quad 0 < \beta(x) - \alpha(x) < \pi; \\ \beta\chi(x) &= \alpha(x) + \frac{1}{2}(\varepsilon - 1)\pi \quad \text{if} \quad -\pi < \beta(x) - \alpha(x) < 0. \end{aligned}\right\} \tag{13.18}$$

In the second place we consider an integral y of the differential equation (q) whose derivative y' vanishes at the point x. From (5.27) we have

$$\begin{aligned} y(t) &= k \cdot r(t) \cdot \sin[\alpha(t) - \beta(x)], \\ y'(t) &= \pm k \cdot s(t) \cdot \sin[\beta(t) - \beta(x)]. \end{aligned}$$

By an analogous method to that used above, we deduce from formula (17) the relationship

$$\beta\psi(x) = \beta(x) + \varepsilon\pi, \tag{13.19}$$

and then

$$\left.\begin{aligned} \alpha\omega(x) &= \beta(x) + \frac{1}{2}(\varepsilon - 1)\pi \quad \text{if} \quad 0 < \beta(x) - \alpha(x) < \pi, \\ \alpha\omega(x) &= \beta(x) + \frac{1}{2}(\varepsilon + 1)\pi \quad \text{if} \quad -\pi < \beta(x) - \alpha(x) < 0. \end{aligned}\right\} \tag{13.20}$$

The formulae (17), (18), (19), (20) are known as the *Abel functional equations* for the fundamental dispersions.

If one combines these relations with the relations (12.8), one obtains the *general Abel functional equations for the central dispersions*, or more briefly the *Abel functional equations*

$$\left.\begin{aligned} \alpha\phi_\nu(x) &= \alpha(x) + \varepsilon\nu\pi, \\ \beta\psi_\nu(x) &= \beta(x) + \varepsilon\nu\pi. \end{aligned}\right\} \tag{13.21}$$

Further, in the case $0 < \beta(x) - \alpha(x) < \pi$, we have

$$\left.\begin{aligned} \beta\chi_\mu(x) &= \alpha(x) + \frac{1}{2}((2\mu - \operatorname{sgn}\mu)\varepsilon + 1)\pi, \\ \alpha\omega_\mu(x) &= \beta(x) + \frac{1}{2}((2\mu - \operatorname{sgn}\mu)\varepsilon - 1)\pi \end{aligned}\right\} \tag{13.22}$$

and, in the case $-\pi < \beta(x) - \alpha(x) < 0$,

$$\left.\begin{aligned} \beta\chi_\mu(x) &= \alpha(x) + \frac{1}{2}((2\mu - \operatorname{sgn}\mu)\varepsilon - 1)\pi, \\ \alpha\omega_\mu(x) &= \beta(x) + \frac{1}{2}((2\mu - \operatorname{sgn}\mu)\varepsilon + 1)\pi, \\ (x \in j;\quad \varepsilon = \operatorname{sgn}\alpha' &= \operatorname{sgn}\beta';\quad \nu = 0, \pm 1, \pm 2, \ldots;\quad \mu = \pm 1, \pm 2, \ldots). \end{aligned}\right\} \tag{13.23}$$

13.8 Representation of central dispersions and their derivatives by means of phases

The Abel functional equations for the central dispersions obviously give a representation, in the interval j, of the central dispersions in terms of the phases α, β.

The representation of the central dispersions of the first and second kinds is obtained from the formulae (21), and is

$$\phi_\nu(t) = \alpha^{-1}[\alpha(t) + \nu\pi \cdot \operatorname{sgn}\alpha'], \qquad \psi_\nu(t) = \beta^{-1}[\beta(t) + \nu\pi \cdot \operatorname{sgn}\beta'] \quad (\nu = 0, \pm 1, \pm 2, \ldots). \tag{13.24}$$

A similar representation is obviously possible for the central dispersions of the third and fourth kinds, χ_μ, ω_μ ($\mu = \pm 1, \pm 2, \ldots$), and this is obtained by application of the formulae (22) and (23); an explicit statement of the corresponding formulae is, however, not needed here. These representations have, as immediate consequences, the properties 1–3 of central dispersions given in § 13.1 as well as the continuous differentiability of all central dispersions. For, every first or second phase takes all real values, is continuously increasing or decreasing, and belongs to the class C_3 or C_1.

By differentiation of the Abel functional equations we obtain the following representations, in the interval j, of the derivatives of central dispersions

$$\left.\begin{aligned} \phi_\nu'(t) &= \frac{\alpha'(t)}{\alpha'[\phi_\nu(t)]}, \qquad \psi_\nu'(t) = \frac{\beta'(t)}{\beta'[\psi_\nu(t)]}, \\ \chi_\mu'(t) &= \frac{\alpha'(t)}{\beta'[\chi_\mu(t)]}, \qquad \omega_\mu'(t) = \frac{\beta'(t)}{\alpha'[\omega_\mu(t)]} \\ &(\nu = 0, \pm 1, \pm 2, \ldots;\quad \mu = \pm 1, \pm 2, \ldots). \end{aligned}\right\} \tag{13.25}$$

13.9 Structure of the Abel functional equations

Every Abel functional equation for the central dispersions obviously has the form

$$\gamma(X) = \bar{\gamma}(t); \tag{13.26}$$

in which each of the symbols γ, $\bar{\gamma}$ represents a first or second phase of the arbitrary basis (u, v) of the differential equation (q) and X is a central dispersion.

If, conversely, we choose a first or second phase γ of the basis (u, v) and allow $\bar{\gamma}$ to run through all the phases of the first or second phase system of the basis (u, v) then the function

$$X(t) = \gamma^{-1}[\bar{\gamma}(t)] \tag{13.27}$$

runs through all central dispersions of a particular kind κ (= 1, 2, 3, 4). If γ is a first (second) phase and $\bar{\gamma}$ runs through the first or second phase system, then X runs through the central dispersions of the first or fourth (third or second) kinds.

13.10 Representation of the central dispersions by normalized polar functions

For simplicity, we shall restrict ourselves to the representation of the fundamental dispersions, which we again here denote by ϕ, ψ, χ, ω. The extension to central dispersions with arbitrary index presents no difficulty.

1 *Representation of the fundamental dispersions* ϕ, ω *by* 1-*normalized polar functions*

Let $h(\alpha)$ be a 1-normalized polar function of the differential equation (q), and $\alpha(t)$, $\beta(t)$ the corresponding phases. We have therefore $\beta = \alpha + h(\alpha)$ at every point $t \in j$. Because of the oscillatory character of the differential equation (q) the definition interval of h coincides with the interval $(-\infty, \infty)$ and, by (6.30), we have in this interval

1. $h \in C_1$;
2. $n\pi < h < (n + 1)\pi$ (n integral)
3. $h^{\backslash} > -1$.

We now choose a number $t_0 \in j$ and set $\alpha_0 = \alpha(t_0)$, $\alpha' = \alpha'(t_0)$. Then (6.28) shows that at two homologous points $\alpha(t) = \alpha$ and $\alpha^{-1}(\alpha) = t \in j$ we have the formula

$$t = t_0 + \frac{1}{\alpha_0'} \int_{\alpha_0}^{\alpha} \left(\exp 2 \int_{\alpha_0}^{\sigma} \cot h(\rho)\, d\rho\right) d\sigma. \tag{13.28}$$

We apply this formula at the point $\phi(t)$ and make use of the Abel functional equation $\alpha[\phi(t)] = \alpha(t) + \varepsilon\pi$ ($\varepsilon = \operatorname{sgn} \alpha'$). Then, for two homologous numbers t, α, we have

$$\phi(t) = t_0 + \frac{1}{\alpha_0'} \int_{\alpha_0}^{\alpha + \varepsilon\pi} \left(\exp 2 \int_{\alpha_0}^{\sigma} \cot h(\rho)\, d\rho\right) d\sigma, \tag{13.29}$$

which can obviously be written

$$\phi(t) = t + \frac{1}{\alpha_0'} \int_{\alpha}^{\alpha + \varepsilon\pi} \left(\exp 2 \int_{\alpha_0}^{\sigma} \cot h(\rho)\, d\rho\right) d\sigma. \tag{13.30}$$

From this we have, by differentiation

$$\phi'(t) = \exp 2 \int_{\alpha}^{\alpha + \varepsilon\pi} \cot h(\rho)\, d\rho, \tag{13.31}$$

and further

$$\frac{\phi''(t)}{\phi'(t)} = 2\alpha_0'[\cot h(\alpha + \varepsilon\pi) - \cot h(\alpha)] \exp\left(-2\int_{\alpha_0}^{\alpha} \cot h(\rho)\, d\rho\right). \quad (13.32)$$

Similarly, formula (28) gives, for two homologous numbers t, α

$$\omega(t) = t + \frac{1}{\alpha_0'}\int_{\alpha}^{\alpha + h(\alpha) - n\pi - \frac{1}{2}(1-\varepsilon)\pi} \left(\exp 2\int_{\alpha_0}^{\sigma} \cot h(\rho)\, d\rho\right) d\sigma. \quad (13.33)$$

2 *Representation of the fundamental dispersions* ψ, χ *by* 2*-normalized polar functions*

Now let $-k(\beta)$ be a 2-normalized polar function of the differential equation (q), and $\alpha(t)$, $\beta(t)$ the corresponding phases; we have therefore $\alpha = \beta + k(\beta)$ at every point $t \in j$. Because of the oscillatory character of the differential equation (q) the interval of definition of $-k$ coincides with the interval $(-\infty, \infty)$ and we have in this interval (from (6.37))

1. $-k \in C_1$;
2. $n\pi < -k < (n + 1)\pi$ (n integral);
3. $-k^{\backprime} < 1$.

We choose an arbitrary number $t_0 \in j$ and put $\beta_0 = \beta(t_0)$, $\alpha_0' = \alpha'(t_0)$. Then (by (6.35)) at any two homologous points $\beta(t) = \beta$ and $\beta^{-1}(\beta) = t \in j$ we have

$$t = t_0 + \frac{1}{\alpha_0'}\int_{\beta_0}^{\beta} [1 + k^{\backprime}(\sigma)] \cdot \exp\left(-2\int_{\beta_0}^{\sigma} [1 + k^{\backprime}(\rho)] \cot k(\rho)\, d\rho\right) d\sigma. \quad (13.34)$$

We apply this formula at the point $\psi(t)$ and make use of the Abel functional equation $\beta[\psi(t)] = \beta(t) + \varepsilon\pi$ ($\varepsilon = \operatorname{sgn} \alpha' = \operatorname{sgn} \beta'$) and so obtain, for any two homologous points t, β

$$\psi(t) = t_0 + \frac{1}{\alpha_0'}\int_{\beta_0}^{\beta + \varepsilon\pi} [1 + k^{\backprime}(\sigma)] \exp\left(-2\int_{\beta_0}^{\sigma} [1 + k^{\backprime}(\rho)] \cot k(\rho)\, d\rho\right) d\sigma. \quad (13.35)$$

This may obviously also be written

$$\psi(t) = t + \frac{1}{\alpha_0'}\int_{\beta}^{\beta + \varepsilon\pi} [1 + k^{\backprime}(\sigma)] \exp\left(-2\int_{\beta_0}^{\sigma} [1 + k^{\backprime}(\rho)] \cot k(\rho)\, d\rho\right) d\sigma, \quad (13.36)$$

and there follows, by differentiation, the result

$$\psi'(t) = \frac{1 + k^{\backprime}(\beta + \varepsilon\pi)}{1 + k^{\backprime}(\beta)} \exp\left(-2\int_{\beta}^{\beta + \varepsilon\pi} [1 + k^{\backprime}(\rho)] \cot k(\rho)\, d\rho\right). \quad (13.37)$$

Similarly, formula (34) gives, for any two homologous numbers t, β, the following formula

$$\chi(t) = t + \frac{1}{\alpha_0'} \int_{\beta}^{\beta + k(\beta) + n\pi + \frac{1}{2}(1+\varepsilon)\pi} [1 + k^{\backprime}(\sigma)] \exp\left(-2\int_{\beta_0}^{\sigma} [1 + k^{\backprime}(\rho)] \cot k(\rho)\, d\rho\right) d\sigma. \tag{13.38}$$

13.11 Differential equations of the central dispersions

The central dispersions of the first kind of the differential equation (q) satisfy a non-linear differential equation of the third order; the same is true for central dispersions of all higher kinds when the carrier $q \in C_2$. These non-linear differential equations of the third order are, as we shall see, special cases of the Kummer differential equation (11.1), whose significance is fundamental for the theory of transformations. We now wish to derive these third order differential equations from the Abel functional equation.

Let X be a central dispersion of the kind κ (= 1, 2, 3, 4). We choose an arbitrary basis (u, v) of the differential equation (q). Then, as we have seen in (26), we have the Abel functional equation

$$\gamma(X) = \bar{\gamma}(t), \tag{13.39}$$

holding in the interval j for appropriate first or second phases γ, $\bar{\gamma}$ of the basis (u, v).

When $\kappa = 1$ or 2 respectively this equation holds if both γ, $\bar{\gamma}$ are first phases or if both are second phases of the basis (u, v); in the cases $\kappa = 3$ or 4 it holds if γ is a second and $\bar{\gamma}$ a first phase, or γ is a first and $\bar{\gamma}$ a second phase respectively.

From formula (39) it follows that for all $t \in j$, apart from the singular points where the functions $\gamma(X)$, $\bar{\gamma}(t)$ are odd multiples of $\frac{\pi}{2}$,

$$\tan \gamma(X) = \tan \bar{\gamma}(t). \tag{13.40}$$

If the central dispersion X is of class C_3, then at every non-singular point $t \in j$ we can take the Schwarzian derivative of this relation. When we take account of (1.17), this gives

$$-\{X, t\} - \{\tan \gamma, X\} \cdot X'^2(t) = -\{\tan \bar{\gamma}, t\}.$$

Now, from (5.18), (5.24), we have

$$-\{\tan \gamma, t\} = q(t) \qquad \text{or} \qquad -\{\tan \gamma, t\} = \hat{q}_1(t),$$

according as γ is a first or second phase of the basis (u, v) and an analogous formula for $\tan \bar{\gamma}$, the right-hand side of (40). Here, $\hat{q}_1$ denotes of course the carrier of the first associated differential equation $(\hat{q}_1)$ of (q):

$$\hat{q}_1(t) = q(t) + \sqrt{|q(t)|}\left(\frac{1}{\sqrt{|q(t)|}}\right)''. \tag{13.41}$$

We come thus to the following theorem:

Theorem. All the central dispersions ϕ of the first kind, with arbitrary indices, satisfy in the interval j the non-linear third order differential equation

$$-\{\phi, t\} + q(\phi) \cdot \phi'^2(t) = q(t). \tag{qq}$$

Moreover, if the carrier $q \in C_2$ all the central dispersions ψ, χ, ω of the second, third and fourth kinds, with arbitrary indices, satisfy in the interval j the equations:

$$-\{\psi, t\} + \hat{q}_1(\psi) \cdot \psi'^2(t) = \hat{q}_1(t). \tag{$\hat{q}_1\hat{q}_1$}$$

$$-\{\chi, t\} + \hat{q}_1(\chi) \cdot \chi'^2(t) = q(t), \tag{$\hat{q}_1 q$}$$

$$-\{\omega, t\} + q(\omega) \cdot \omega'^2(t) = \hat{q}_1(t). \tag{$q\hat{q}_1$}$$

These are the so-called *non-linear third order differential equations for central dispersions*; more precisely for central dispersions of the first, second, third and fourth kinds.

13.12 Solutions of the Abel functional equations with unknown phase functions α, β

The problem of determining the differential equation (q) from a knowledge of its central dispersions leads us to consider the Abel functional equations for the central dispersions with unknown phase functions α, β.

Let ϕ_ν or ψ_ν be a central dispersion of the first or second kind of the differential equation (q). A phase function α, β (§ 5.7) of class C_3 or C_1 respectively, which satisfies the Abel functional equation (21) in the interval j, represents a first or second phase of a differential equation ($\bar{q}$), whose ν-th central dispersion of the first or second kind coincides with the function ϕ_ν or ψ_ν. The carrier $\bar{q}$ is thus determined, respectively, in these two cases by (5.16) or by means of a certain solution of a non-linear second order differential equation (§ 5.12).

Now let χ_μ or ω_μ be a central dispersion of the third or fourth kind of a differential equation (q). If we have two phase functions α, $\beta \in C_3$, C_1 respectively, which are related in the interval j by a formula such as (5.34), and satisfy the Abel functional equation (22), say, then these represent a first and second phase of the same basis of a differential equation ($\bar{q}$), whose μ-th central dispersion of the third or fourth kind coincides with the function χ_μ or ω_μ. The carrier $\bar{q}$ is determined uniquely from the phase α, by the formula (5.16).

We shall restrict ourselves from now on to the solution of the Abel functional equation by phase functions, of class C_3, of a given fundamental dispersion of the first kind.

First we observe that by (12.1) and §§ 13.1, 13.3, 13.4 the fundamental dispersion of the first kind ϕ of every differential equation (q) has the following properties in the interval $j = (a, b)$.

$$\left.\begin{array}{ll} 1. & \phi(t) > t, \\ 2. & \lim_{t \to a+} \phi(t) = a, \quad \lim_{t \to b-} \phi(t) = b, \\ 3. & \phi \in C_3, \\ 4. & \phi'(t) > 0. \end{array}\right\} \tag{13.42}$$

Now we have the following result due to E. Barvínek [2].

Theorem. Corresponding to every function ϕ defined in the interval j and with properties 1 to 4 above, there exist infinitely many phase functions $\alpha \in C_3$ which are solutions of the Abel functional equation

$$\alpha(\phi) = \alpha(t) + \pi \cdot \operatorname{sgn} \alpha' \tag{13.43}$$

and which may, moreover, be obtained constructively.

Proof. Let ϕ be a function defined in the interval j with the above properties 1–4. We shall, as an illustration, construct an increasing solution $\alpha \in C_3$ of (43).

We select a number $t_0 \in j$ and put $t_\nu = \phi^\nu(t_0)$ for $\nu = 0, \pm 1, \pm 2, \ldots$. Then the interval j separates into sub-intervals $j_\nu = [t_\nu, t_{\nu+1})$.

Now in the interval j_0 we choose any function $f \in C_3$ with a continuous positive derivative f', which has the following behaviour in a left neighbourhood of t_1:

$$\lim_{t \to t_1-} f(t) = f(t_0) + \pi,$$

$$\lim_{t \to t_1-} f'(t) = \frac{f'^{+}(t_0)}{\phi'(t_0)}.$$

$$\lim_{t \to t_1-} f''(t) = \frac{1}{\phi'(t_0)} \left(\frac{f'^{+}(t)}{\phi'(t)}\right)'^{+}_{t_0},$$

$$\lim_{t \to t_1-} f'''(t) = \frac{1}{\phi'(t_0)} \left[\frac{1}{\phi'(t)} \left(\frac{f'^{+}(t)}{\phi'(t)}\right)'^{+}\right]'^{+}_{t_0}.$$

Here, the $+$ symbol indicates a right derivative so that, for instance, $f'^{+}(t_0)$ is the right derivative of f at the point t_0.

By means of this function f, we can now define in the interval j the function α as follows

$$\alpha(t) = \begin{cases} f(t) & \text{for} \quad t \in j_0, \\ \alpha[\phi^{-1}(t)] + \pi & \text{for} \quad t \in j_\nu, \quad \nu > 0, \\ \alpha[\phi(t)] - \pi & \text{for} \quad t \in j_\nu, \quad \nu < 0. \end{cases}$$

This function α is obviously an increasing phase function $\in C_3$ and satisfies the Abel functional equation (43), so the proof is complete.

We know that when a first phase α of the differential equation (q) is given, the carrier q is uniquely determined by the formula (5.16). From this fact and the above result we expect that to every function ϕ defined in the interval j with the properties (42) there will correspond in general infinitely many oscillatory differential equations (q) having the function ϕ as fundamental dispersion of the first kind. Later, (§ 15.10), we shall show that the power of the set of all differential equations (q) in the interval $(-\infty, \infty)$ with the same fundamental dispersion of the first kind ϕ is independent of the choice of the latter and is equal to the power $\aleph$ of the continuum.

For the solution of the Abel functional equations, by means of phase functions $\in C_1$, with given fundamental dispersions of the second, third or fourth kinds, we refer to the papers by J. Chrasitna [38] and F. Neuman [54]. In general, the situation is as follows:

Let $\lambda(t)$ be a function defined in the interval $j = (a, b)$ with the following properties

$$\left.\begin{array}{l} 1.\ \lambda(t) > t, \\ 2.\ \lim\limits_{t\to a+} \lambda(t) = a, \quad \lim\limits_{t\to b-} \lambda(t) = b, \\ 3.\ \lambda \in C_1, \\ 4.\ \lambda'(t) > 0. \end{array}\right\} \tag{13.44}$$

(a) There are in the interval j infinitely many oscillatory differential equations (q) with $q < 0$, whose fundamental dispersion of the second kind ψ coincides with λ; $\psi(t) = \lambda(t)$ for $t \in j$.

(b) Let $t_0 \in j$ be an arbitrary number. There are in the interval $[t_0, b)$ infinitely many right oscillatory differential equations (q) with $q < 0$, whose fundamental dispersion χ of the third kind coincides with λ: $\chi(t) = \lambda(t)$ for $t \in [t_0, b)$.

(c) Let $t_0 \in j$ be an arbitrary number. There are in the interval $(a, t_0]$ infinitely many left oscillatory differential equations (q) with $q < 0$, whose fundamental dispersion ω of the fourth kind coincides with λ in the interval $(a, \lambda^{-1}(t_0)]$: $\omega(t) = \lambda(t)$ for $t \in (a, \lambda^{-1}(t_0)]$.

It should be noted that problems of this kind are associated with the solutions of appropriate non-linear differential equations of the second order with delayed argument.

13.13 Consequences of the above results

Now we derive some consequences of the above theory.

2. *Monotonic character of the differences.* $\phi_\nu(t) - t$, $\psi_\nu(t) - t$, $\chi_\nu(t) - t$, $\omega_\nu(t) - t$. Consider the differences

$$\phi_n(t) - t, \quad \psi_n(t) - t, \quad \chi_n(t) - t, \quad \omega_n(t) - t, \tag{13.45}$$

$$\phi_{-n}(t) - t, \quad \psi_{-n}(t) - t, \quad \chi_{-n}(t) - t, \quad \omega_{-n}(t) - t \tag{13.46}$$

$$(n = 1, 2, \ldots)$$

in the interval j.

From the formulae in § 13.6 we deduce that if the carrier q is a non-increasing or a decreasing function in the interval j, then the quantities

$$\phi_n'(t) - 1, \quad \psi_n'(t) - 1, \quad \chi_n'(t) - 1, \quad \omega_n'(t) - 1$$

are respectively $\leqslant 0$ or < 0, and the quantities

$$\phi_{-n}'(t) - 1, \quad \psi_{-n}'(t) - 1, \quad \chi_{-n}'(t) - 1, \quad \omega_{-n}'(t) - 1$$

are $\geqslant 0$ or > 0 respectively.

Hence

If the carrier q is a non-increasing or a decreasing function in the interval j, then the same is true of all the differences (45), *while the differences* (46) *are non-decreasing or increasing functions.*

Similarly we show that

If the carrier q is a non-decreasing or an increasing function in the interval j, then the same is true of the differences (45), *while the differences* (46) *are non-increasing or decreasing functions.*

Corollary. Let

$$\cdots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \cdots$$

be the sequence of zeros of any integral of the differential equation (q) and let

$$\cdots < t_{-2m} < t_{-m} < t_0 < t_m < t_{2m} < \cdots \quad (m \geqq 1) \tag{13.47}$$

be a subsequence of it.

If the carrier q in the interval j is non-increasing or if it is decreasing, then (47) denotes respectively a concave or a strictly concave sequence. If the carrier q in the interval j is non-decreasing or increasing then (47) is a convex or a strictly convex sequence. For the case $m = 1$ this gives a result of Sturm-Szegö.

An analogous result holds for the sequence of zeros of the derivative of an arbitrary integral of the differential equation (q).

2. *Derivatives of composite functions.* According to a classical result, the composition of two or more functions of a class C_k ($k = 0, 1, \ldots$) is a function of the same class. It can happen, however, that the function arising from such composition belongs to a higher class than the original functions. Our results on derivatives of central dispersions lead to many situations of this kind. We shall content ourselves with a few remarks on this topic, since a deeper investigation would be beyond the scope of this book.

We shall show that:

Let q be a negative continuous function in the interval $j = (a, b)$, such that the differential equation (q) *is oscillatory. Then there are in the interval j two functions X, Y satisfying the inequality $t < X(t) < Y(t)$, such that the function $q[X(t)]/q[Y(t)]$ belongs to the class C_2. If the function q is strictly monotonic, then there are continuous functions X, Y with the property quoted.*

For we obtain functions X, Y of the kind described if we evaluate $X(t)$, $Y(t)$ at every point $t \in j$ according to the theorem of § 13.6, taking $X(t) = t_1$, $Y(t) = t_3$; $t < t_1 < t_3$. The function $q[X(t)]/q[Y(t)]$ $(= \phi'(t))$ has at the point t a continuous derivative of the second order, as we know from § 13.4. If the function q is strictly monotonic, then it follows from the result in § 13.6 and the relationship $\omega\chi = \phi$ that

$$X(t) = q^{-1}[q[\chi(t)] \cdot \chi'(t)]; \qquad Y(t) = q^{-1}[q[\chi(t)] : \omega'[\chi(t)]],$$

where naturally q^{-1} denotes the inverse function of q. It follows from these formulae that the functions X, Y are continuous in the interval j.