Linear Differential Transformations of the Second Order

15 Differential equations with the same central dispersions of the first kind

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In § 13.12, we stated that to every function ϕ defined in the interval $(j =)\ (-\infty, \infty)$, with the properties (13.42), there correspond infinitely many oscillatory differential equations (q), (with power $\aleph$), whose fundamental dispersion of the first kind is precisely the function ϕ.

The set of all oscillatory differential equations (q) defined in an interval $j = (a, b)$ is partitioned into classes, each being formed of all differential equations (q) with the same fundamental dispersion of the first kind. Now, for a given oscillatory differential equation (q), every 1-central dispersion ϕ_ν is obtained by iteration of the corresponding fundamental dispersion of the first kind or its inverse, i.e. $\phi_\nu = \phi_1^\nu$ for $\nu = 0, \pm 1, \pm 2, \ldots$ (12.2). It follows that for all differential equations (q) contained in a given class the 1-central dispersions ϕ_ν are the same.

In what follows we shall be concerned with differential equations (q) with the same fundamental dispersion of the first kind. For brevity, instead of speaking of differential equations (q), ($\bar{\text{q}}$), with the same fundamental dispersion of the first kind we shall speak of carriers q, $\bar{q}$ with the same fundamental dispersion; also, instead of ϕ_1 we shall write simply ϕ.

We let (q) be, therefore, an oscillatory differential equation in the interval $j = (a, b)$.

15.1 Integral strips

Let ϕ be a function defined in the interval j with the properties (13.42).

By $Q\phi$, or briefly Q, we denote the set of all carriers with the same fundamental dispersion ϕ, and by $(Q\phi)$, more briefly (Q), the set of all differential equations (q), $q \in Q\phi$. All differential equations (q) $\in$ (Q) have therefore the same 1-central dispersions ϕ_ν.

Further, let $J\phi$, more briefly J, denote the set of all integrals of all differential equations (q) contained in the set (Q). Since these differential equations (q) all have the same 1-central dispersions ϕ_ν, J consists of functions with the same zeros; that is to say, two arbitrary integrals y, $\bar{y} \in J$, having one zero in common, have all their zeros in common.

Let $c \in j$ be an arbitrary number. By the *integral strip* of the set (Q) with the *node* c, or more briefly the integral strip (c), we mean the set of all elements of J which vanish at the point c, and we denote this set by Bc. The integral strip Bc thus consists of all integrals vanishing at the point c, of all differential equations (q) which have the common fundamental dispersion of the first kind ϕ. The elements of Bc have therefore the same zeros; these zeros are called the *nodes* of the integral strip Bc. Clearly, the integral strip Bc is uniquely determined by any one of its nodes c': i.e. $Bc = Bc'$.

15.2 Statement of the problem

We are now in a position to describe more precisely the contents of this paragraph; it consists essentially of a study of the following topics:

1. Properties of the integral strips of the set $(Q\phi)$.
2. Relations between carriers with the same fundamental dispersion ϕ; that is, between functions $q, \bar{q} \in Q\phi$.
3. Explicit formulae for carriers with the same fundamental dispersion ϕ.
4. The power of the set $Q\phi$ in the case $j = (-\infty, \infty)$.

15.3 Properties of the integral strips of the set $(Q\phi)$

Let Bc be an integral strip of the set $(Q\phi)$. First we show that:

For every integral $y \in Bc$ the following formulae hold:

$$\int_x^{\phi(x)} \left[\frac{y'^2(c)}{y^2(\sigma)} - \frac{1}{(\sigma - c)^2}\right] d\sigma = \frac{1}{c - x} + \frac{1}{\phi(x) - c}; \qquad (15.1)$$

$$\int_c^{\phi(c)} \left[\frac{y'^2(c)}{y^2(\sigma)} - \frac{1}{(\sigma - c)^2} - \frac{1}{(\sigma - \phi(c))^2}\right] d\sigma = \frac{1 + \phi'(c)}{\phi(c) - c} - \frac{1}{2}\frac{\phi''(c)}{\phi'(c)}; \qquad (15.2)$$

where x is an arbitrary number satisfying the inequalities $x < c < \phi(x)$.

For, let $(q) \in (Q\phi)$ be an arbitrary differential equation. We have already derived the formula (1) (5.45), and so have only to prove (2).

Let $y \in Bc$ be an integral of the differential equation (q). We choose an arbitrary number t, such that $c < t < \phi(c)$, and apply the formulae (5.43), (5.42) on the intervals $[c, t]$ and $[t, \phi(c)]$

$$\int_c^{t} \left[\frac{y'^2(c)}{y^2(\sigma)} - \frac{1}{(\sigma - c)^2}\right] d\sigma = -\cot \alpha_0(t) + \frac{1}{t - c},$$

$$\int_t^{\phi(c)} \left[\frac{y'^2\phi(c)}{y^2(\sigma)} - \frac{1}{(\sigma - \phi(c))^2}\right] d\sigma = \cot \alpha_1(t) - \frac{1}{t - \phi(c)}.$$

Here α_0 and α_1 are the first phases of the differential equation (q) determined by the initial values:

$$\alpha_0(c) = 0, \quad \alpha_0'(c) = 1, \quad \alpha_0''(c) = 0; \quad \alpha_1\phi(c) = 0, \quad \alpha_1'\phi(c) = 1, \quad \alpha_1''\phi(c) = 0.$$

From the Abel functional equation $\alpha_1\phi = \alpha_1 + \pi$ and from the derivative of this relation, we see that at the point c the functions $\alpha_1, \alpha_1', \alpha_1''$ take the following values: $\alpha_1(c) = -\pi$, $\alpha_1'(c) = \phi'(c)$, $\alpha_1''(c) = \phi''(c)$.

Then, from (5.39), we see that the values taken by the phases $\alpha_0(t)$, $\alpha_1(t)$ at the point t are related by:

$$-\cot \alpha_0(t) + \phi'(c) \cdot \cot \alpha_1(t) = -\frac{1}{2}\frac{\phi''(c)}{\phi'(c)}. \qquad (15.3)$$

The above formulae, which can also be written in the form,

$$\int_c^t \left[\frac{y'^2(c)}{y^2(\sigma)} - \frac{1}{(\sigma - c)^2} - \phi'(c)\frac{1}{(\sigma - \phi(c))^2}\right] d\sigma$$
$$= -\cot \alpha_0(t) + \frac{1}{t - c} + \phi'(c)\left[\frac{1}{t - \phi(c)} - \frac{1}{c - \phi(c)}\right],$$

$$\int_t^{\phi(c)} \left[\frac{y'^2(c)}{y^2(\sigma)} - \phi'(c)\frac{1}{(\sigma - \phi(c))^2} - \frac{1}{(\sigma - c)^2}\right] d\sigma$$
$$= \phi'(c)\left[\cot \alpha_1(t) - \frac{1}{t - \phi(c)}\right] + \frac{1}{\phi(c) - c} - \frac{1}{t - c}$$

give, by addition,

$$\int_c^{\phi(c)} \left[\frac{y'^2(c)}{y^2(\sigma)} - \frac{1}{(\sigma - c)^2} - \phi'(c)\frac{1}{(\sigma - \phi(c))^2}\right] d\sigma$$
$$= -\cot \alpha_0(t) + \phi'(c)\cdot\cot \alpha_1(t) + \frac{1}{\phi(c) - c}[1 + \phi'(c)],$$

and when we apply (3) to this, it gives (2).

As a supplement to this result we remark that by using the relations (5.46), or (5.49) we can derive formulae which generalize (1), (2) above to the case of 1-central dispersions ϕ_ν with arbitrary index v $(= 0, \pm 1, \pm 2, \ldots)$ or derive similar formulae for the 2-central dispersions ψ_ν. We shall not concern ourselves further with this, since the relations (1), (2) are sufficient for our purposes.

We now have some further results:

The (Riemann) integrals which occur on the left of the formulae (1), (2) *are independent of the elements y of the integral strip Bc.*

In other words: these integrals are invariant with respect to the elements y of the integral strip Bc.

A further property of the integral strip Bc is that *the ratio of the derivatives y', $\bar{y}'$ of two arbitrary elements y, $\bar{y}$ in the integral strip Bc takes the same value* ($= k$) *at all nodes of this strip.*

For, let c' be an arbitrary node of Bc. Then we have $c' = \phi_\nu(c)$, ν being some integer, and (13.5) gives

$$(-1)^\nu \frac{y'(c)}{y'(c')} = \sqrt{\phi_\nu'(c)} = (-1)^\nu \frac{\bar{y}'(c)}{\bar{y}'(c')},$$

which proves our statement.

We show further that:

Every two elements y, $\bar{y}$ of the integral strip Bc are related in the interval j by:

$$\int_t^{\phi(t)} \left[\frac{y'^2(c)}{y^2(\sigma)} - \frac{\bar{y}'^2(c)}{\bar{y}^2(\sigma)}\right] d\sigma = 0. \tag{15.4}$$

To establish this, we first observe that formulae (1) and (2) show that (4) is valid if the number t $(= x)$ satisfies the inequalities $t \leqslant c < \phi(t)$. Now let t be an arbitrary number $\in j$; obviously, there is one node c' of Bc which satisfies the inequalities $t \leqslant c' < \phi(t)$, so that formula (4) holds with c' in place of c. Now, from the above results we have $y'^2(c') = \lambda y'^2(c)$, $\bar{y}'^2(c') = \lambda \bar{y}'^2(c)$, in which λ (> 0) is some constant, and this fact completes the proof.

15.4 A sufficient condition for two differential equations to have the same fundamental dispersion

We now wish to examine how far the above properties characterize the integral strips of the set $(Q\phi)$; to that end, let us consider two oscillatory differential equations (q), ($\bar{\text{q}}$) in the interval $j = (a, b)$ with the fundamental dispersions ϕ, $\bar{\phi}$ of the first kind.

We assume that the differential equations (q), ($\bar{\text{q}}$) possess integrals y, $\bar{y}$ which have the same zeros and also the property that the ratio of their derivatives $y'/\bar{y}'$ ($= k$) is the same at each of these zeros x. Finally, let at least one (always the same one) of the two relations

$$\int_t^{\phi(t)} \left[\frac{k}{y^2(\sigma)} - \frac{1}{k}\,\frac{1}{\bar{y}^2(\sigma)}\right] d\sigma = 0, \qquad \int_t^{(\bar{\phi}t)} \left[\frac{k}{y^2(\sigma)} - \frac{1}{k}\,\frac{1}{\bar{y}^2(\sigma)}\right] d\sigma = 0. \quad (15.5)$$

be satisfied at every $t \in j$ except for the zeros x.

Then the fundamental dispersions ϕ, $\bar{\phi}$ coincide, $\phi = \bar{\phi}$, and consequently y, $\bar{y}$ are elements of the integral strip Bx of the set $(Q\phi)$.

To show this, let us assume that the first relation (5) holds and let $t \in j$ be arbitrary. If t is a (common) zero of the integrals y, $\bar{y}$, then from the definition of ϕ, $\bar{\phi}$ it follows immediately that $\phi(t) = \bar{\phi}(t)$. We therefore assume that $y(t) \neq 0$, $\bar{y}(t) \neq 0$.

Let c_{-1}, c, c_1 be successive zeros of y and $\bar{y}$ determined by the inequalities $c_{-1} < t < c < c_1$; then $\phi(t)$, $\bar{\phi}(t)$ lie between c and c_1.

From (5) we have

$$\int_t^{\phi(t)} \left[\frac{y'^2(c)}{y^2(\sigma)} - \frac{1}{(\sigma - c)^2}\right] d\sigma = \int_t^{\phi(t)} \left[\frac{\bar{y}'^2(c)}{\bar{y}^2(\sigma)} - \frac{1}{(\sigma - c)^2}\right] d\sigma.$$

From this and (5.44) it follows that $\cot \bar{\alpha}\phi(t) = \cot \bar{\alpha}(t)$; $\bar{\alpha}$ being the phase of the differential equation ($\bar{\text{q}}$) determined by the initial values $\bar{\alpha}(c) = 0$, $\bar{\alpha}'(c) = 1$, $\bar{\alpha}''(c) = 0$. We have therefore $\bar{\alpha}\phi(t) = \bar{\alpha}(t) + m\pi$, m being an integer. But since $\phi(t)$ lies between c and c_1, $m = 1$, and consequently $\bar{\alpha}\phi(t) = \bar{\alpha}(t) + \pi$. When we compare this relationship with the Abel functional equation $\bar{\alpha}\bar{\phi}(t) = \bar{\alpha}(t) + \pi$, then we obtain $\bar{\phi}(t) = \phi(t)$, and the proof is complete.

15.5 Ratios of elements of an integral strip

We consider again an integral strip Bc of the set $(Q\phi)$. Let $y, \bar{y} \in Bc$ be arbitrary elements of Bc and w the Wronskian of y, $\bar{y}$:

$$w = y\bar{y}' - y'\bar{y}. \quad (15.6)$$

At every point $t \in j$ the function w has the derivative

$$w' = (\bar{q} - q)y\bar{y}. \tag{15.7}$$

Clearly, the functions w, w' vanish at every node c_ν $(= \phi_\nu(c))$ of Bc, $(\nu = 0, \pm 1, \pm 2, \ldots; c_0 = c)$, i.e.

$$w(c_\nu) = 0, \qquad w'(c_\nu) = 0. \tag{15.8}$$

Further, from (13.5) and (7) we have

$$w\phi_\nu = w, \tag{15.9}$$

$$(\bar{q}\phi_\nu - q\phi_\nu)\phi_\nu'^2 = \bar{q} - q. \tag{15.10}$$

Now we define the function p in the interval j by:

$$p(t) = \begin{cases} \dfrac{\bar{y}(t)}{y(t)} & \text{for } \; t \neq c_\nu; \quad \nu = 0, \pm 1, \pm 2, \ldots; \quad c_0 = c, \\ \dfrac{\bar{y}'(c)}{y'(c)} & \text{for } \; t = c_\nu, \end{cases} \tag{15.11}$$

and the remainder of this paragraph is devoted to studying the properties of the function p.

First, from (13.5), we have the following relationship

$$p\phi_\nu(t) = p(t), \tag{15.12}$$

holding in the interval j; further, the function p is either always positive or always negative, according as $\bar{y}'(c)/y'(c) > 0$ or < 0. For let us assume, for definiteness, that $\bar{y}'(c)/y'(c) > 0$. Then both the functions y, $\bar{y}$ are positive or both are negative in the interval $(c, \phi(c))$, and consequently p is positive. Thus $p(t) > 0$ for $t \in [c, \phi(c)]$ and consequently, from (12), for all $t \in j$.

Also, the function p is continuous in the interval j. This follows from the fact that by definition it is continuous at every point $t \neq c_\nu$ and for $t \to c_\nu$ it tends to the limit $p(c_\nu)$ $(= p(c))$.

The function p belongs, indeed, to the class C_2. Obviously it is twice continuously differentiable at every point $t \neq c_\nu$, since

$$p' = \frac{w}{y^2} \tag{15.13}$$

$$p'' = (\bar{q} - q)p - 2\frac{y'}{y}p'. \tag{15.14}$$

Further, applying L'Hôpital's rule,

$$\lim_{t \to c_\nu} p'(t) = 0, \qquad \lim_{t \to c_\nu} p''(t) = \frac{1}{3}[\bar{q}(c_\nu) - q(c_\nu)]p(c_\nu), \tag{15.15}$$

whence the functions p, p' possess derivatives $p'(c_\nu)$, $p''(c_\nu)$ at the point c_ν which are equal to the limiting values (15) and our statement is proved.

Moreover, from (4) and (11), we have

$$\int_t^{\phi(t)} \left[\frac{1}{p^2(\sigma)} - \frac{1}{p^2(c)}\right] \frac{d\sigma}{y^2(\sigma)} = 0 \tag{15.16}$$

in the interval j. Finally we note that the integrand in (16) is continuous in the interval j if we define its value at each point c_ν by the following limit:—

$$\lim_{\sigma \to c_\nu} \left[\frac{1}{p^2(\sigma)} - \frac{1}{p^2(c)}\right] \frac{1}{y^2(\sigma)} = -\frac{p''(c_\nu)}{p^3(c_\nu)y'^2(c_\nu)}. \tag{15.17}$$

Coordinating the above facts, the function p has the following properties:—

$$\left.\begin{aligned} &1.\ p \neq 0 \quad \text{for} \quad t \in j;\\ &2.\ p\phi(t) = p(t) \quad \text{for} \quad t \in j;\\ &3.\ p \in C_2;\\ &4.\ p'(c) = 0;\\ &5.\ \int_c^{\phi(c)} \left[\frac{1}{p^2(\sigma)} - \frac{1}{p^2(c)}\right] \frac{d\sigma}{y^2(\sigma)} = 0. \end{aligned}\right\} \tag{15.18}$$

15.6 Relations between carriers with the same fundamental dispersion ϕ

We consider arbitrary elements $q, \bar{q}$ of the set $Q\phi$; thus the differential equations (q) ($\bar{\text{q}}$) have the same fundamental dispersion ϕ. For brevity we write $\Delta = \bar{q} - q$.

Let $c \in j$ be arbitrary. We shall be concerned with the number of zeros of the function Δ lying between two neighbouring nodes of the integral strip Bc; the possibility of an infinite number of such zeros is not excluded.

Formula (10) shows that the relation $\Delta(c) = 0 \Rightarrow \Delta(c_\nu) = 0$ for every node c_ν of Bc. In other words, if the carriers $q, \bar{q}$ take equal values at the point c, then they take equal values at every node of Bc.

We next show that the number of zeros of Δ lying between two neighbouring nodes of Bc is always the same. For, let c, $\phi(c)$ and c', $\phi(c')$ be two neighbouring nodes of Bc and suppose, for definiteness, that $c < c'$. Then we have $c' = \phi_\nu(c)$, $\phi(c') = \phi_\nu\phi(c)$ for some positive index ν. Since the function ϕ_ν is increasing, it maps the interval $(c, \phi(c))$ simply onto $(c', \phi(c'))$. But formula (10) shows that in this mapping zeros of Δ are mapped onto zeros, and the result follows.

Clearly, the number of points lying between two neighbouring nodes of the integral strip Bc at which the carriers $q, \bar{q}$ take the same value is always the same, and consequently does not depend upon the choice of these nodes.

We now wish to prove the following important theorem:

Theorem. The number of the zeros of the function Δ lying in an arbitrary interval $[c, \phi(c))$, $c \in j$, is always at least four.

Proof. Let $y, \bar{y} \in Bc$ be arbitrary elements of Bc and p the function defined in the interval j by means of the formula (11).

First, the relations (18), 1° and 5° show that the function p takes the value $p(c)$ at a point $x \in (c, \phi(c))$. Consequently, the function p assumes the same value $p(c)$ at the points c, x and $\phi(c)$ which are such that $c < x < \phi(c)$. Hence its derivative p' vanishes at least at two points $x_1' \in (c, x)$, $x_2' \in (x, \phi(c))$. Now, from (13), x_1' and x_2' are zeros of the function w. Consequently, w vanishes at points x_1', x_2' such that $c < x_1' < x_2' < \phi(c)$; its derivative w' therefore vanishes at least at three points $x_1 \in (c, x_1')$, $x_2 \in (x_1', x_2')$, $x_3 \in (x_2', \phi(c))$.

From (7) the numbers x_1, x_2, x_3 are zeros of the function Δ. This completes the first part of our proof.

In the second part we show that: if the function $\Delta \neq 0$ at the point c, and consequently also at $\phi(c)$, then it has not less than four zeros in the interval $(c, \phi(c))$.

We assume that $\Delta(c) \neq 0$, and that there are precisely three zeros $x_1 < x_2 < x_3$ of Δ lying in the interval $(c, \phi(c))$. Then we have

$$c < x_1 < x_2 < x_3 < \phi(c) < \phi(x_1) < \phi(x_2) < \phi(x_3)$$

and between any two neighbouring elements in this sequence the function Δ is not zero. Consider now the integral strip Bx_1. Clearly, between two neighbouring nodes x_1, $\phi(x_1)$ of Bx_1 there lie precisely two zeros x_2, x_3 of Δ. This completes the proof.

This result may also be formulated as follows:

Two arbitrary carriers q, $\bar{q}$ with the same fundamental dispersion of the first kind ϕ take the same value at not less than four points in each interval $[c, \phi(c))$, $c \in j$.

We shall soon show (§ 15.8) that this theorem is best-possible in the sense that *there are carriers q, $\bar{q}$ with the same fundamental dispersion ϕ, for which the lower bound* 4 *in the above theorem is obtained.*

15.7 Explicit formula for carriers with the same fundamental dispersion ϕ

Let q be the carrier of an oscillatory differential equation (q) in the interval $j = (a, b)$ and ϕ its fundamental dispersion of the first kind. Let $c \in j$ be arbitrary and $y \in Bc$ an arbitrary element of the integral strip Bc of the set (Qϕ). Then y is an integral of the differential equation, belonging to the set (Qϕ), vanishing at the point c; its zeros, which are therefore the nodes of Bc, are $c_\nu = \phi_\nu(c)$; $\nu = 0, \pm 1, \pm 2, \ldots$, $c_0 = c$.

We now have the following theorem:—

All carriers $\bar{q}$ with the fundamental dispersion ϕ are given by the formula

$$\bar{q} = q + \frac{p''}{p} + 2\frac{y'}{p} \cdot \frac{p'}{y}. \tag{15.19}$$

In this, p is an arbitrary function defined in the interval j with the properties (18), *and the value of the last term is defined at each point c_ν to be $2p''(c_\nu)/p(c)$.*

Proof. (a) Let $\bar{q}$ be an arbitrary carrier with the fundamental dispersion ϕ, so that $q, \bar{q} \in Q\phi$.

Moreover, let $\bar{y}$ be an integral of the differential equation ($\bar{\text{q}}$) contained in the integral strip Bc, and p the function defined in the interval j by the expression (11).

The function p has therefore the properties (18), and at every point $t \in j$ the relation (14) holds; the formula (19) then follows immediately.

(b) Now let $\bar{q}$ be a function defined in the interval j by (19) where of course p is a function with the properties (18) and the value of the last term at each point c_ν is as explained above.

By elementary calculation, we show that the function

$$\bar{y}(t) = p(t)y(t)$$

is a solution of the differential equation ($\bar{q}$), and indeed is the (unique) solution determined by the intial values $\bar{y}(c) = 0$, $\bar{y}'(c) = p(c)y'(c)$. The functions y, $\bar{y}$ obviously have the same zeros $c_\nu = \phi_\nu(c)$ ($\nu = 0, \pm 1, \pm 2, \ldots$; $c_0 = c$) and from the relationship $\bar{y}' = p'y + py'$ it is clear that the ratio $y'/\bar{y}'$ of their derivatives is the same at all these zeros, taking the value $1/p(c)$.

Now let $F(\sigma)$ denote, in the interval j, the integrand of (18), 5°; its value at each point c_ν is specified as the limiting value (17). This function F is continuous in j. Further, from (18) property 2° and (13.5), we see that the following relationship holds for all $t \in j$:

$$F[\phi(t)]\phi'(t) = F(t).$$

It follows that

$$\left[\int_t^{\phi(t)} F(\sigma)\, d\sigma\right]' = 0,$$

and further from (18), property 5°

$$\int_t^{\phi(t)} F(\sigma)\, d\sigma = \int_c^{\phi(c)} F(\sigma)\, d\sigma = 0.$$

Hence for all $t \in j$ we have the relationship

$$\int_t^{\phi(t)} \left[\frac{k}{y^2(\sigma)} - \frac{1}{k}\,\frac{1}{\bar{y}^2(\sigma)}\right] d\sigma = 0 \quad (k = 1/p(c)),$$

and application of the result of § 15.4 shows that the fundamental dispersion of the first kind of the differential equation ($\bar{q}$) coincides with ϕ. This completes the proof.

15.8 Explicit formulae for elementary carriers

In § 8.4 we determined all the elementary carriers in an interval j by means of the formula (6). The theorem of § 15.7 leads to another and perhaps simpler explicit expression for elementary carriers, still of course in the interval $j = (-\infty, \infty)$.

An elementary carrier q in the interval j ($= (-\infty, \infty)$) is characterized by the fact that all its first phases α are elementary—that is, $\alpha(t + \pi) = \alpha(t) + \pi \operatorname{sgn} \alpha'$ $\forall\, t \in j$. We recall, in this context, that differential equations (q) with elementary carriers, and only such equations, have the zeros of all their integrals separated by a distance π.

Clearly the carrier q defined in j is elementary if and only if its fundamental dispersion of the first kind, ϕ, is linear, with $\phi(t) = t + \pi$.

The elementary carriers in the interval j are therefore precisely those carriers whose fundamental dispersion is $\phi(t) = t + \pi$. Among these, naturally, is included the carrier -1; the integral y of this with initial values $y(c) = 0$, $y'(c) = 1$ (which may be assigned at an arbitrary point $c \in j$) is given by the function $y(t) = \sin(t - c)$.

If we apply the result of § 15.7 we obtain the following theorem:

Theorem. The set of all elementary carriers in the interval $j = (-\infty, \infty)$ is given by the following formula

$$\bar{q}(t) = -1 + \frac{p''(t)}{p(t)} + 2\frac{p'(t)}{p(t)}\cot(t - c); \tag{15.20}$$

in which c is an arbitrary number and p a function with the following properties in the interval j

$$\left.\begin{aligned} &1.\ p \neq 0 \quad \text{for} \quad t \in j;\\ &2.\ p(t + \pi) = p(t) \quad \text{for} \quad t \in j;\\ &3.\ p \in C_2;\\ &4.\ p'(c) = 0;\\ &5.\ \int_0^\pi \left[\frac{1}{p^2(\sigma)} - \frac{1}{p^2(c)}\right]\frac{d\sigma}{\sin^2(\sigma - c)} = 0. \end{aligned}\right\} \tag{15.21}$$

If we set $p(t) = p(c)\exp f(t)$, then formula (20) gives

$$\bar{q}(t) = -1 + f''(t) + f'^2(t) + 2f'(t)\cdot\cot(t - c), \tag{15.22}$$

in which f denotes a function defined on $(-\infty, \infty)$ with the following properties:—

$$\left.\begin{aligned} &f(t + \pi) = f(t) \quad \text{for} \quad t \in j; \quad f \in C_2; \quad f(c) = f'(c) = 0;\\ &\int_0^\pi \frac{\exp(-2f(\sigma)) - 1}{\sin^2(\sigma - c)}\,d\sigma = 0. \end{aligned}\right\} \tag{15.23}$$

The above formula (22) is due to F. Neuman ([53]). If, in particular, we choose

$$f(t) = -\frac{1}{2}\log\left[1 - \frac{1}{3}\sin 2(t - c)\sin^2(t - c)\right], \tag{15.24}$$

then we obtain the one-parameter system of elementary carriers $q(t|c)$ which we introduced in (8.7).

This result provides also a simple example of carriers $q, \bar{q}$ with the same fundamental dispersion ϕ $(= t + \pi)$, for which the lower limit 4, referred to in the theorem of § 15.6, is in fact attained. This holds, for instance, for the carriers $q(t) = -1$ and $\bar{q}(t) = q(t|0)$. In this case, $\Delta(t) = \bar{q}(t) - q(t) = \sin 4t + \frac{1}{3}\sin^4 t$, and it is easy to verify that for every number $c \in j$ the number of zeros of the function Δ lying in the interval $[c, c + \pi)$ is precisely 4.

15.9 Relations between first phases of differential equations with the same fundamental dispersion ϕ

Let (q), ($\bar{q}$) be oscillatory differential equations in the interval $j = (-\infty, \infty)$ and ϕ, $\bar{\phi}$ their fundamental dispersions of the first kind. Also, let α, $\bar{\alpha}$ be arbitrary (first) phases of (q), ($\bar{q}$) respectively. We then have:

Theorem. The fundamental dispersions ϕ, $\bar{\phi}$ *coincide if and only if the phase functions* $\gamma = \alpha\bar{\alpha}^{-1}$, $\gamma^{-1} = \bar{\alpha}\alpha^{-1}$, (*which are inverse to each other*), *are elementary; that is, if* $\forall t \in j$ *we have*

$$\gamma(t + \pi) = \gamma(t) + \eta\pi, \quad \gamma^{-1}(t + \pi) = \gamma^{-1}(t) + \eta\pi$$
$$(\eta = \operatorname{sgn} \gamma' = \operatorname{sgn} (\gamma^{-1})').$$

Proof. (a) Let $\phi = \bar{\phi}$. Then in the interval j there hold the Abel functional equations

$$\alpha\phi = \alpha + \varepsilon\pi, \quad \bar{\alpha}\bar{\phi} = \bar{\alpha} + \bar{\varepsilon}\pi \quad (\varepsilon = \operatorname{sgn} \alpha', \bar{\varepsilon} = \operatorname{sgn} \bar{\alpha}'),$$

and consequently

$$\bar{\alpha}^{-1}(\bar{\alpha} + \bar{\varepsilon}\pi) = \alpha^{-1}(\alpha + \varepsilon\pi),$$
$$\gamma(\bar{\alpha} + \bar{\varepsilon}\pi) = \alpha + \varepsilon\pi,$$
$$\gamma(t + \bar{\varepsilon}\pi) = \gamma(t) + \varepsilon\pi \quad (t \in j).$$

From the last formula, we have

$$\gamma(t + \pi) = \gamma(t) + \varepsilon\bar{\varepsilon}\pi,$$

and from the fact that $\operatorname{sgn} \gamma' = \varepsilon\bar{\varepsilon}$, it is clear that the phase function γ is elementary. The same, naturally, is true of γ^{-1}.

(b) Let the phase functions γ, γ^{-1} be elementary. Then we have in the interval j

$$\alpha = \gamma\bar{\alpha},$$
$$\alpha\bar{\phi} = \gamma\bar{\alpha}\bar{\phi} = \gamma(\bar{\alpha} + \bar{\varepsilon}\pi) = \gamma\bar{\alpha} + \varepsilon\pi = \alpha + \varepsilon\pi = \alpha\phi;$$

it follows that $\phi = \bar{\phi}$, and the theorem is proved.

We now apply the above theorem to obtain a further explicit formula for carriers with the same fundamental dispersion of the first kind.

Let $q, \bar{q} \in Q\phi$ be arbitrary carriers with the same fundamental dispersion ϕ in $j = (-\infty, \infty)$. We choose arbitrary first phases α, $\bar{\alpha}$ of (q), ($\bar{q}$) respectively; then by the above result there is a relationship

$$\bar{\alpha} = \bar{\gamma}\alpha, \tag{15.25}$$

$\bar{\gamma}$ being an appropriate elementary phase function. The latter is obviously of class C_3 and represents a first phase of a differential equation (g). Since $\bar{\gamma}$ is elementary, the carrier $\bar{g}$ is also elementary.

Taking the Schwarz derivative of each side of (25) at an arbitrary point $t \in j$ it follows that

$$\{\tan \bar{\alpha}, t\} = \{\tan \bar{\gamma}, \alpha\}\alpha'^2 + \{\alpha, t\}$$

and hence, using (5.16), (5.18)

$$\bar{q} = q + [1 + \bar{g}\alpha]\alpha'^2.$$

We now apply formula (22), writing t for $t - c$ and $f(t)$ for $f(t + c)$, obtaining

$$\bar{q} = q + [f''\alpha + f'^2\alpha + 2f'\alpha \cdot \cot \alpha]\alpha'^2. \tag{15.26}$$

We have thus the following result:

All carriers $\bar{q}$ with the fundamental dispersion ϕ are given precisely by the formula (26) *in which q is a fixed carrier with the fundamental dispersion ϕ, α is a fixed first phase of the differential equation* (q) *and f is an arbitrary function in the interval j with period π, belonging to the class C_2 and having the properties*

$$f(0) = f'(0) = 0, \qquad \int_0^\pi [\exp(-2f(\sigma)) - 1]\, d\sigma/\sin^2 \sigma = 0$$

15.10 Power of the set $Q\phi$

We now determine the power of the set $Q\phi$. To do this, we go back to the information obtained in § 10, where we studied the algebraic structure of the phase group $\mathfrak{G}$.

As we know, all elementary phases form a sub-group $\mathfrak{H}$ of $\mathfrak{G}$. The power of the set of all carriers, whose first phases lie in one and the same element of the right residue class partition $\mathfrak{G}/_r\mathfrak{H}$ is the same for all elements of $\mathfrak{G}/_r\mathfrak{H}$ and is equal to the power of the continuum $\aleph$.

Now let q be a carrier and ϕ its fundamental dispersion of the first kind, and let α be a first phase of the differential equation (q). From the theorem of § 15.9 it follows that the first phases of the differential equations ($\bar{\text{q}}$) with the fundamental dispersion ϕ are precisely those phases of the form $\bar{\alpha} = \bar{\gamma}\alpha$, in which $\bar{\gamma}$ ranges over all elementary phases; that is, all elements of the sub-group $\mathfrak{H}$. In other words, the first phases with fundamental dispersion ϕ are precisely those elements of $\mathfrak{G}$ lying in the right residue class $\mathfrak{H}\alpha \in \mathfrak{G}/_r\mathfrak{H}$. Consequently, the carriers $\bar{q}$ with the fundamental dispersion ϕ are precisely those carriers whose first phases α lie in the right residue class $\mathfrak{H}\alpha$. From the above, the power of the set of all these carriers $\bar{q}$ is equal to the power of the continuum.

Thus we have the following theorem.

Theorem. The power of the set of all oscillatory differential equations (q) *in the interval $j = (-\infty, \infty)$ with the same fundamental dispersion of the first kind ϕ is independent of the choice of the latter and is equal to the power of the continuum, $\aleph$.*

This result calls for some further comment. In the numerical treatment of differential equations we often have to calculate the zeros of an integral of a particular differential equation (q). Such a calculation naturally depends on the carrier q and in certain circumstances can be very difficult. We now have the possibility of replacing the carrier q by a "representative" $\bar{q}$, that is to say a carrier $\bar{q}$ with the same fundamental dispersion of the first kind as q; then the integrals of the differential equation ($\bar{\text{q}}$) have the same zeros as those of (q). Naturally, we choose the representative $\bar{q}$ in such a way that the calculation of the zeros of its integrals is as simple as possible. The above result ensures that there are always infinitely many representatives $\bar{q}$, (indeed, the set of these has the power $\aleph$) which can replace the given carrier q. This raises the following problem: to develop methods for discovering representatives of a given carrier with advantageous properties for numerical work.