20 General dispersions of the differential equations (q), (Q)

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20 General dispersions of the differential equations (q), (Q)

We now come to the theory of general dispersions of differential equations (q), (Q). We shall continue to use the above concepts and notation.

20.1 Fundamental numbers and fundamental intervals of the differential equation (q)

Let $t_0 \in j$ be arbitrary and let t_ν be the ν-th 1-conjugate number with t_0, that is to say $t_\nu = \phi_\nu(t_0)$; $\nu = 0, \pm 1, \pm 2, \ldots$. The numbers t_ν are therefore the zeros of every integral v of (q) which vanishes at the point t_0: t_ν coincides with t_0 or represents the ν-th zero of v following or preceding this number, according as $\nu = 0$ or > 0 or < 0.

We call t_ν the ν-th *fundamental number of the differential equation* (q) *with respect to* t_0, more briefly the ν-th fundamental number.

The intervals $j_\nu = [t_\nu, t_{\nu+1})$ and $\bar{j}_\nu = (t_{\nu-1}, t_\nu]$ we call respectively the *right* and *left fundamental intervals of the differential equation* (q) *with respect to* t_0, more briefly the ν-th right and left fundamental intervals. Obviously, the fundamental intervals j_ν and $\bar{j}_\nu$ have only the number t_ν in common; the interior of j_ν coincides with the interior of $\bar{j}_{\nu+1}$.

Every number $t \in (a, b)$ lies in a determinate right fundamental interval j_ν and in a determinate left fundamental interval $\bar{j}_\mu$; if $t = t_\nu$ we have $\mu = \nu$, if $t \neq t_\nu$ we have $\mu = \nu + 1$. In particular, every zero of an arbitrary integral y of (q) lies in determinate fundamental intervals j_ν and $\bar{j}_\mu$; conversely, every fundamental interval j_ν or $\bar{j}_\mu$ contains precisely one zero of y.

20.2 The concept of general dispersion

We now associate with every normalized linear mapping $\boldsymbol{p}$ of the integral space r of (q) onto the integral space R of (Q) a certain function X defined in the interval $j = (a, b)$.

Let $\boldsymbol{p}$ be a normalized linear mapping of the integral space $\boldsymbol{r}$ onto the integral space R with respect to the (arbitrarily chosen) numbers $t_0 \in j$, $T_0 \in J$. We denote by t_ν, T_ν the fundamental numbers of the differential equations (q), (Q) with respect to t_0, T_0; we further denote by j_ν, J_ν the right and by $\bar{j}_\nu$, $\bar{J}_\nu$ the left fundamental intervals of the differential equations (q), (Q) with respect to t_0, T_0; $\nu = 0, \pm 1, \pm 2, \ldots$.

Let $t \in (a, b)$ be arbitrary and y an integral of the differential equation (q) vanishing at the point t; the number t lies in a determinate right ν-th fundamental interval j_ν.

Now we define the value $X(t)$ of the function X at the point t, according as $\chi\boldsymbol{p} > 0$ or $\chi\boldsymbol{p} < 0$, as follows:

In the case $\chi\boldsymbol{p} > 0$, $X(t)$ is the zero of the integral $\boldsymbol{p}y$ of (Q) which lies in the right ν-th fundamental interval J_ν.

In the case $\chi\boldsymbol{p} < 0$, $X(t)$ is the zero of the integral $\boldsymbol{p}y$ of (Q) which lies in the left $-\nu$-th fundamental interval $\bar{J}_{-\nu}$.

This function X is called the *general dispersion of the differential equations* (q), (Q) (in this order) *with respect to the numbers* t_0, T_0 *and the linear mapping* $\boldsymbol{p}$; more briefly: the general dispersion. The numbers t_ν, T_ν are the *fundamental numbers* and in particular t_0, T_0 are the *initial numbers of the general dispersion* X; the linear mapping $\boldsymbol{p}$ is called the *generator* of X. In the case $\chi\boldsymbol{p} > 0$ we call the general dispersion X *direct*, in the case $\chi\boldsymbol{p} < 0$ we call it *indirect*. In the case $Q = q$ we speak more briefly of the *general dispersion* X *of the differential equation* (q).

If we consider the general dispersion of the differential equations (q), (Q) with respect to a mapping $c\boldsymbol{p}$ ($c \neq 0$) (which is linearly dependent upon $\boldsymbol{p}$), and to the same numbers t_0, T_0 then this general dispersion coincides with X, since the linear mapping $c\boldsymbol{p}$ is still normalized with respect to t_0, T_0 and the zeros of the images $\boldsymbol{p}y$, $c\boldsymbol{p}y$ of every integral y of (q) are the same.

The general dispersion X is thus uniquely determined by its initial numbers t_0, T_0 and the generator $\boldsymbol{p}$.

Obviously we have, according as $\chi\boldsymbol{p} > 0$ or $\chi\boldsymbol{p} < 0$ the formulae

$$X(t_\nu) = T_\nu \qquad \text{or} \qquad X(t_\nu) = T_{-\nu}, \tag{20.1}$$

and moreover at every point $t \in (t_\nu, t_{\nu+1})$ we have

$$X(t) \in (T_\nu, T_{\nu+1}) \qquad \text{or} \qquad X(t) \in (T_{-\nu-1}, T_{-\nu}) \tag{20.2}$$
$$(\nu = 0, \pm 1, \pm 2, \ldots).$$

20.3 Properties of general dispersions

Fundamental for the theory of general dispersions is the following theorem

Theorem. Let X be the general dispersion with initial numbers t_0, T_0 and generator $\boldsymbol{p}$. Moreover, let (α, A) be a canonical phase basis of $\boldsymbol{p}$ with respect to the numbers t_0, T_0. Then the general dispersion X satisfies in the interval j the functional equation

$$\alpha(t) = \mathrm{A}(X(t)). \tag{20.3}$$

Proof. Let $x \in j$ be arbitrary. This must lie in a determinate interval j_ν and we thus have, since $\alpha(t_0) = 0$,

$$\nu\pi \leqslant \alpha(x) < (\nu + 1)\pi \qquad \text{or} \qquad -(\nu + 1)\pi < \alpha(x) \leqslant -\nu\pi, \tag{20.4}$$

according as $\operatorname{sgn} \alpha' = +1$ or -1.

Let $y \in r$ be an integral of the differential equation (q) vanishing at the point x; this is given by the formula (19.4) using appropriate constants k_1, k_2. Since $y(x) = 0$ we have

$$\alpha(x) + k_2 = n\pi, \qquad n \text{ integral}. \tag{20.5}$$

The integral $Y = \boldsymbol{p}y \in R$ of (Q) is given by (19.4) using the same constants k_1, k_2. Now, by our definition of X the number $X(x)$ is a zero of the integral Y, hence

$$\mathrm{A}(X(x)) + k_2 = N\pi, \qquad N \text{ integral.} \tag{20.6}$$

From the relations (5), (6) it follows that

$$\alpha(x) - \mathrm{A}(X(x)) = m\pi, \qquad m \text{ integral.} \tag{20.7}$$

We now distinguish two cases, according as $\chi\boldsymbol{p} > 0$ or $\chi\boldsymbol{p} < 0$.

In the case $\chi\boldsymbol{p} > 0$, the number $X(x)$ lies in the interval J_ν. We then have, taking account of the fact that $\mathrm{A}(T_0) = 0$,

$$\nu\pi \leq \mathrm{A}(X(x)) < (\nu + 1)\pi \qquad \text{or} \qquad -(\nu + 1)\pi < \mathrm{A}(X(x)) \leq -\nu\pi, \tag{20.8}$$

according as $\operatorname{sgn} \dot{\mathrm{A}} = +1$ or -1.

Now since (from (19.5)) $\operatorname{sgn} \alpha' \operatorname{sgn} \mathrm{A} = +1$, either the first or the second of the inequalities (4) and (8) holds. In both cases, we obtain from these inequalities

$$-\pi < \alpha(x) - \mathrm{A}(X(x)) < \pi. \tag{20.9}$$

This gives, together with (7), $\alpha(x) - \mathrm{A}(X(x)) = 0$.

In the case $\chi\boldsymbol{p} < 0$, the number $X(x)$ lies in $\bar{J}_{-\nu}$. We then have, by similar reasoning to that above,

$$-(\nu + 1)\pi < \mathrm{A}(X(x)) \leqslant -\nu\pi \qquad \text{or} \qquad \nu\pi \leqslant \mathrm{A}(X(x)) < (\nu + 1)\pi, \tag{20.10}$$

according as $\operatorname{sgn} \dot{\mathrm{A}} = +1$ or -1.

Now, from (19.5), $\operatorname{sgn} \alpha' \operatorname{sgn} \dot{\mathrm{A}} = -1$, and consequently there hold either the first inequality of (4) and the second inequality of (10), or the second inequality of (4) and the first of (10). In each case, we again obtain from these inequalities the formulae (9) and hence $\alpha(x) - \mathrm{A}(X(x)) = 0$. This completes the proof.

The importance of the functional equation (3) lies in the fact that we can apply it to obtain properties of general dispersions from properties of phases of the differential equations (q), (Q).

We consider a general dispersion X. Let t_0, T_0 be its initial numbers, and $\boldsymbol{p}$ its generator. Moreover, let (α, A) be a canonical phase basis of the linear mapping $\boldsymbol{p}$ with respect to the numbers t_0, T_0. From the above theorem, the functional equation (3) therefore holds in the interval j.

1. For $t \in j$ we have

$$X(t) = \mathrm{A}^{-1}\alpha(t); \tag{20.11}$$

where A^{-1} naturally denotes the function inverse to the phase A. This follows immediately from the functional equation (3).

2. The function X increases in the interval j from A to B or decreases in this interval from B to A, according as it is direct or indirect.

For if, for instance, the general dispersion X is direct, then (by (19.5)), we have $\operatorname{sgn} \alpha' \operatorname{sgn} \dot{\mathrm{A}} = +1$, so the functions α, A either increase in the intervals j, J from $-\infty$ to ∞ or decrease from ∞ to $-\infty$. Then (11) gives the corresponding part of our statement.

3. The function X^{-1} inverse to the general dispersion X is the general dispersion $x(T)$ of the differential equations (Q), (q) generated by the mapping $\boldsymbol{p}^{-1}$ inverse to $\boldsymbol{p}$, with the initial numbers T_0, t_0.

Proof. Formula (11) shows that the function X^{-1} inverse to X is the following:

$$X^{-1}(T) = \alpha^{-1}\mathrm{A}(T).$$

But (A, α) is a canonical phase basis of the mapping $\boldsymbol{p}^{-1}$ inverse to $\boldsymbol{p}$ with respect to the numbers T_0, t_0 (§ 19.8) hence $X^{-1}(T) = x(T)$.

4. The general dispersion X is three times continuously differentiable in the interval j, and the following formulae hold at any two homologous points $t \in j$, $X \in J$; that is to say, two points linked by the relations $X = X(t)$, $t = X^{-1}(X)$:

$$\left.\begin{aligned} X'(t) &= \frac{\alpha'(t)}{\dot{\mathrm{A}}(X)}; & X''(t) &= \frac{1}{\dot{\mathrm{A}}^3(X)}\,[\alpha''(t)\dot{\mathrm{A}}^2(X) - \alpha'^2(t)\ddot{\mathrm{A}}(X)], \\ \dot{\mathrm{A}}(X) &= \frac{\alpha'(t)}{X'(t)}; & \ddot{\mathrm{A}}(X) &= \frac{1}{X'^3(t)}\,[\alpha''(t)X'(t) - X''(t)\alpha'(t)]. \end{aligned}\right\} \tag{20.12}$$

The first part of this statement follows immediately from (11), since the functions α, A are three times continuously differentiable in the intervals j, J and $\dot{\mathrm{A}}$ is always non-zero. The second part is obtained by differentiating the functional equation (3) twice.

5. The first derivative X' of the general dispersion X is always positive or always negative in the interval j according as the latter is direct or indirect: sgn X' = sgn $\chi\boldsymbol{p}$.

This follows from the above Theorem 2 and the first formula (12).

6. For $t \in j$ we have the formula

$$X[\phi_\nu(t)] = \Phi_{\nu\cdot\operatorname{sgn}X'}[X(t)] \quad (\nu = 0, \pm 1, \pm 2, \ldots); \tag{20.13}$$

in which ϕ_ν, Φ_ν denote the ν-th central dispersions of the differential equations (q), (Q).

Proof. The functional equation (3) gives the following relation holding at the point $\phi_\nu(t)$:

$$\alpha\phi_\nu(t) = \mathrm{A}X\phi_\nu(t),$$

and on using (13.21), we have

$$\mathrm{A}X(t) + \nu\pi \cdot \operatorname{sgn}\alpha' = \mathrm{A}X\phi_\nu(t).$$

The function on the left hand side of this equation may obviously be written in the form

$$\mathrm{A}X(t) + (\nu \cdot \operatorname{sgn}\alpha' \cdot \operatorname{sgn}\dot{\mathrm{A}})\pi \cdot \operatorname{sgn}\dot{\mathrm{A}}$$

and since sgn α' sgn $\dot{\mathrm{A}}$ = sgn X', it may also be expressed as $\mathrm{A}\Phi_{\nu\cdot\operatorname{sgn}X'}[X(t)]$.

We have therefore

$$\mathrm{A}\Phi_{\nu\cdot\operatorname{sgn}X'}[X(t)] = \mathrm{A}X\phi_\nu(t),$$

from which the formula (13) follows.

7. The general dispersion X represents, in the interval j, a solution of the non-linear third order differential equation

$$-\{X, t\} + Q(X)X'^2 = q(t). \qquad \text{(Qq)}$$

Proof. Let $t \in j$ be arbitrary. From the theorem of § 20.3, the functional equation (3) holds at the point t and in its neighbourhood. If we take the Schwarzian derivative of both sides, then use (1.17) and the first formula (12) we get

$$\{X, t\} + [\{\mathbf{A}, X\} + \dot{\mathbf{A}}^2(X)]X'^2 = \{\alpha, t\} + \alpha'^2(t).$$

and, by (5.16), this is precisely the equation (Qq).

The relationship (Qq) we call the *differential equation of the general dispersions of the differential equations* (q), (Q), more briefly the general dispersion equation.

8. The function x inverse to the general dispersion X represents in the interval J a solution of the non-linear third order differential equation

$$-\{x, T\} + q(x)\dot{x}^2 = Q(T). \qquad \text{(qQ)}$$

This is an obvious consequence of the Theorems 3 and 7 above.

9. We now consider, alongside the linear mapping $\boldsymbol{p}$, a linear mapping $\boldsymbol{P}$ of the integral space R on the integral space $\bar{R}$, normalized with respect to the numbers T_0, Z_0 (Z_0 arbitrary); moreover, let $(\mathbf{A}, \bar{\mathbf{A}})$ be a canonical phase basis of $\boldsymbol{P}$ with respect to T_0, Z_0.

We know that the composed linear mapping $\boldsymbol{Pp}$ of the integral space r on the integral space $\bar{R}$ is normalized with respect to t_0, Z_0 and that it admits of the canonical phase basis $(\alpha, \bar{\mathbf{A}})$ with respect to t_0, Z_0 (§ 19.8).

Let X, $\bar{X}$ be two general dispersions of the differential equations (q), (Q) and (Q), $(\bar{Q})$ generated by the linear mappings $\boldsymbol{p}$, $\boldsymbol{P}$ with initial numbers t_0, T_0 and T_0, Z_0. We shall show that the composed function $\bar{X}X$ is the general dispersion $\bar{\bar{X}}$ of the differential equations (q), $(\bar{Q})$ generated by the linear mapping $\boldsymbol{Pp}$ with initial numbers t_0, Z_0.

For, from the formulae

$$X(t) = \mathbf{A}^{-1}\alpha(t), \qquad \bar{X}(T) = \bar{\mathbf{A}}^{-1}\mathbf{A}(T) \quad (t \in j,\ T \in J)$$

there follows

$$\bar{X}X(t) = \bar{\mathbf{A}}^{-1}\alpha(t).$$

Now $(\alpha, \bar{\mathbf{A}})$ is the canonical phase basis of the linear mapping $\boldsymbol{Pp}$ with respect to the numbers t_0, Z_0, and it follows, by (11), that $\bar{X}X(t) = \bar{\bar{X}}(t)$.

20.4 Determining elements of general dispersions

We know that given the initial numbers and a linear mapping $\boldsymbol{p}$ normalized with respect to them there is determined from these precisely one general dispersion. We now take up the question as to how far general dispersions are characterized by having a given linear mapping $\boldsymbol{p}$ as their generator. We continue to use the above notation.

1. Let p be a linear mapping of the integral space r of (q) onto the integral space R of (Q). By this linear mapping p, there is determined precisely one countable system of general dispersions of the differential equations (q), (Q) with generator p. If $X(t)$ is one general dispersion of this system, then the latter consists precisely of the functions $X\phi_\nu(t)$, $\nu = 0, \pm 1, \pm 2, \ldots$.

Proof. Let X be the general dispersion of the differential equations (q), (Q) with the initial numbers t_0, T_0 and generator p. We consider a canonical phase basis (α, A) of p with respect to the numbers t_0, T_0 so that $\alpha(t_0) = 0$, $\mathrm{A}(T_0) = 0$, and formula (3) holds.

(a) Let Z be a general dispersion of the differential equations (q), (Q) with initial numbers t_0, Z_0 and generator p. Then we have $Z_0 = Z(t_0)$, and the linear mapping p is normalized with respect to the numbers t_0, Z_0 so $Z_0 = \Phi_\nu(T_0)$ with some appropriate index ν.

We now see, from the identity $\mathrm{A}(T) = \mathrm{A}\Phi_{-\nu}\Phi_\nu(T)$ that the function $\mathrm{A}\Phi_{-\nu}(T)$ vanishes at the point Z_0. Consequently (α, $\mathrm{A}\Phi_{-\nu}$) is a canonical phase basis of the linear mapping p with respect to the numbers t_0, Z_0, and from (3) we have for $t \in j$

$$\mathrm{A}X(t) = \alpha(t) = \mathrm{A}\Phi_{-\nu}Z(t).$$

From this relation it follows that $X(t) = \Phi_{-\nu}Z(t)$ and that $Z(t) = \Phi_\nu X(t)$. This formula gives, on taking account of (13), $Z(t) = X\phi_{\pm\nu}(t)$.

(b) We now consider the function $X\phi_\nu$ formed from an arbitrary central dispersion ϕ_ν of the differential equation (q).

Since the linear mapping p is normalized with respect to the numbers t_0, T_0, it is also normalized with respect to the numbers $\phi_{-\nu}(t_0)$, T_0. From the identity $\alpha(t) = \alpha\phi_\nu\phi_{-\nu}(t)$ we see that the function $\alpha\phi_\nu$ vanishes at $\phi_{-\nu}(t_0)$ so ($\alpha\phi_\nu$, A) is a canonical phase basis of the linear mapping p with respect to the numbers $\phi_{-\nu}(t_0)$, T_0. Let Z be the general dispersion of the differential equations (q), (Q) with initial numbers $\phi_{-\nu}(t_0)$, T_0 and generator p. Then we have, from (3), for $t \in j$

$$\alpha\phi_\nu(t) = \mathrm{A}Z(t)$$

and moreover,

$$\mathrm{A}X(t) = \alpha(t) = \mathrm{A}Z\phi_{-\nu}(t).$$

From these relations it follows that $Z(t) = X\phi_\nu(t)$. This completes the proof.

In the second place we show that general dispersions of the differential equations (q), (Q) can be uniquely determined by means of initial conditions of the second order.

2. Let t_0; X_0, X_0' ($\neq 0$), X_0'' be arbitrary numbers. There is precisely one general dispersion X of the differential equations (q), (Q) with the initial conditions

$$X(t_0) = X_0, \quad X'(t_0) = X_0', \quad X''(t_0) = X_0''. \tag{20.14}$$

This general dispersion X is direct or indirect according as $X_0' > 0$ or $X_0' < 0$.

Proof. We first assume that there exists a general dispersion X satisfying the above initial conditions. This is uniquely determined by the initial numbers t_0, X_0 and a linear mapping p of the integral space r on the integral space R which is normalized with respect to these numbers. We choose a canonical phase basis (α, A) of $\boldsymbol{p}$ with respect to the numbers t_0, X_0 in such a manner that

$$\alpha(t_0) = 0, \quad \alpha'(t_0) = 1, \quad \alpha''(t_0) = 0. \tag{20.15}$$

Then the general dispersion X satisfies in the interval j a functional equation such as (3) and the formulae (12) give the values of the functions $\dot{\mathrm{A}}$, $\ddot{\mathrm{A}}$ at the point X_0. In this way we obtain the values

$$\mathrm{A}(X_0) = 0, \quad \dot{\mathrm{A}}(X_0) = 1/X_0', \quad \ddot{\mathrm{A}}(X_0) = -X_0''/X_0'^3, \tag{20.16}$$

by which the first phase A of the differential equation (Q) is uniquely determined. (§ 7.1).

We see that every general dispersion X with the above initial values (14) coincides with that particular one which is determined by the initial numbers t_0, X_0 and the generator $\boldsymbol{p}$. The generator $\boldsymbol{p}$ is determined by the canonical phase basis (α, A) which is given uniquely by the initial values (15), (16). This completes the proof.

The general dispersions of the differential equations (q), (Q) thus form a system which is continuously dependent upon three parameters X_0, X_0', ($\neq 0$), X_0''.

Finally we show that:

3. Given arbitrary phases α, A of the differential equations (q), (Q) there is determined precisely one general dispersion of these differential equations as a solution of the functional equation $\alpha(t) = \mathrm{A}(X(t))$. This dispersion X is the general dispersion of the differential equations (q), (Q) with respect to the zeros t_0, T_0 of the phases α, A and with respect to every linear mapping $\boldsymbol{p}$ with the canonical phase basis (α, A).

Proof. The general dispersion X of the differential equations (q), (Q) with zeros t_0, T_0 of the phases α, A as initial numbers and generator $\boldsymbol{p}$ with the canonical phase basis (α, A) satisfies the functional equation $\alpha(t) = \mathrm{A}(X(t))$ in the interval j (§ 20.3). At the same time, X is the unique solution of the latter: $X(t) = \mathrm{A}^{-1}\alpha(t)$, which proves this result.

20.5 Integration of the differential equation (Qq)

The above results open the way for us to determine all the regular integrals of the non-linear third order differential equation (Qq) in the interval j. By a regular integral X of the differential equation (Qq) we mean a solution whose derivative X' is always non-zero.

We shall prove the following theorem:—

Theorem. The set of all regular integrals of the differential equation (Qq) *defined in the interval j comprises precisely the general dispersions of the differential equations* (q), (Q).

Proof. (a) Let X be a general dispersion of the differential equations (q), (Q). From § 20.3, 5 and 7 this function represents a regular solution of the differential equation (Qq) in j.

(b) Now let X be a solution of the differential equation (Qq) defined in j.

We choose an arbitrary number t_0 and the first phases α, A of the differential equations (q), (Q) determined by the initial values

$$\alpha(t_0) = 0, \quad \alpha'(t_0) = 1, \quad \alpha''(t_0) = 0,$$
$$\mathrm{A}(X_0) = 0, \quad \dot{\mathrm{A}}(X_0) = 1/X_0', \quad \ddot{\mathrm{A}}(X_0) = -X_0''/X_0'^3$$

where X_0, X_0', X_0'' are the values taken by X, X', X'' at the point t_0.

Then we have, in the interval j,

$$-\{\tan\alpha, t\} = q(t), \quad -\{\tan \mathrm{A}, X\} = Q(X)$$

and moreover, since the function X satisfies the differential equation (Qq),

$$-\{X, t\} - \{\tan \mathrm{A}, X\} \cdot X'^2 = -\{\tan\alpha, t\}.$$

It then follows, from (1.17), that

$$\{\tan \mathrm{A}(X), t\} = \{\tan\alpha, t\}$$

and further, on taking account of § 1.8, that

$$\tan \mathrm{A}(X) = \frac{c_{11}\tan\alpha(t) + c_{12}}{c_{21}\tan\alpha(t) + c_{22}},$$

where $c_{11}, \ldots, c_{22}$ denote appropriate constants.

Now the initial values of the phases α, A have the consequence that $c_{12} = 0$, $c_{11} = c_{22}$, $c_{21} = 0$ and moreover

$$\alpha(t) = \mathrm{A}(X).$$

Consequently X is the general dispersion of the differential equations (q), (Q) determined by the initial numbers t_0, X_0 and the linear mapping $\boldsymbol{p}$ of the integral space r of (q) on the integral space R of (Q) determined by the phase basis (α, A). This completes the proof.

20.6 Connection between general dispersions and the transformation problem

We consider a general dispersion X of the differential equations (q), (Q) with initial numbers t_0, T_0 and generator $\boldsymbol{p}$: we have therefore $\chi\boldsymbol{p} > 0$ or $\chi\boldsymbol{p} < 0$ according as X is direct or indirect.

1. Let $Y \in R$ be an arbitrary integral of the differential equation (Q) and $y \in r$ its original in the linear mapping $\boldsymbol{p}$, so that $y \rightarrow Y(\boldsymbol{p})$. Then the function $Y(X)/\sqrt{|X'|}$ is an integral of the differential equation (q), and in the interval j we have the relationship

$$\frac{YX(t)}{\sqrt{|X'(t)|}} = \pm\frac{1}{\sqrt{|\chi\boldsymbol{p}|}}\, y(t); \tag{20.17}$$

in which the sign occurring on the right hand side is independent of the choice of the integral Y.

Proof. Let (α, A) be a canonical phase basis of $\boldsymbol{p}$ with respect to the numbers t_0, T_0. Then the relationship $\mathrm{A}X(t) = \alpha(t)$ holds in the interval j (§ 20.3), and there hold also formulae such as (19.4) for the integrals $Y \in R$, $y \in r$. The relationship (17) follows immediately (since $\varepsilon E = \pm 1$).

2. For an appropriate variation $\boldsymbol{p}^*$ of $\boldsymbol{p}$, we have, for every integral $Y \in R$ and its original $y \in r$ $(\boldsymbol{p}^*)$ in the interval j

$$\frac{YX(t)}{\sqrt{|X'(t)|}} = y(t); \tag{20.18}$$

where $\chi\boldsymbol{p}^* = \operatorname{sgn} X'$.

Proof. If in place of the linear mapping $\boldsymbol{p}$ we choose the linear mapping $\boldsymbol{p}^*$, dependent upon $\boldsymbol{p}$, given by the formula $\boldsymbol{p}^* = \varepsilon E\sqrt{|\chi\boldsymbol{p}|}\boldsymbol{p}$ then we obtain the formula (18). From (19.1), we have $\chi\boldsymbol{p}^* = \operatorname{sgn} \chi\boldsymbol{p}$.

3. Let U, V be independent integrals of the differential equation (Q), and W be the Wronskian of (U, V). Then the integrals $UX/\sqrt{|X'|}$ $(= u)$, $VX/\sqrt{|x'|}$ $(= v)$ of the differential equation (q) are also independent, and for the Wronskian w of (u, v) we have the formula $w = W \operatorname{sgn} X'$.

Proof. The first part of this statement follows from the fact that the images under a linear mapping $\boldsymbol{p}$ of two independent integrals of the differential equation (q) are themselves independent, (§ 19.1). The second part is obtained merely by a short calculation.

From Theorem 1 above we see that every general dispersion of the differential equations (q), (Q) represents a transforming function for these differential equations (q), (Q). From § 11.2 we know that every transforming function of the differential equations (q), (Q) in the interval j satisfies the differential equation (qQ), and consequently is a general dispersion of the differential equations (q), (Q) (§ 20.5). We thus have the following theorem:

Theorem. The transforming functions of the oscillatory differential equations (q), (Q) *in the interval j are precisely the general dispersions of these differential equations* (q), (Q).

The ordered pairs of functions $[\sqrt{|X'(t)|}, X(t)]$ formed from arbitrary general dispersions X of the differential equations (q), (Q) are therefore precisely the transformations of the differential equation (Q) into the differential equation (q).

20.7 Embedding of the general dispersions in the phase group

We consider in this section differential equations (q), (Q) whose intervals of definition j, J are assumed to coincide with $(-\infty, \infty)$: $j = J = (-\infty, \infty)$.

Let D be the set of general dispersions of the differential equations (q), (Q). We know (§ 20.3), that every general dispersion $X \in D$ is unbounded on both sides, belongs to the class C_3, and that its derivative X' never vanishes. Consequently, X is an unbounded phase function of the class C_3 (§ 5.7) so we conclude that D is a subset of the phase group $\mathfrak{G}$ (§ 10.1); $D \subset \mathfrak{G}$.

Let α, $\mathbf{A}$ be arbitrary (first) phases of the differential equations (q) and (Q). We know, (§§ 10.2, 10.3), that all the phases of the differential equations (q) and (Q) respectively form the right cosets $\mathfrak{E}\alpha$ and $\mathfrak{E}\mathbf{A}$ of $\mathfrak{E}$: $\mathfrak{E}$ denotes naturally the fundamental subgroup of $\mathfrak{G}$.

Now all the functions which are inverse to phases of the set $\mathfrak{E}\mathbf{A}$ form the left coset $\mathbf{A}^{-1}\mathfrak{E}$ of $\mathfrak{E}$ (see [81], page 141). We show that:

The set D of the general dispersions of (q), (Q) *is the product of the left coset* $\mathbf{A}^{-1}\mathfrak{E}$ *with the right coset* $\mathfrak{E}\alpha$. i.e.

$$D = \mathbf{A}^{-1}\mathfrak{E}\alpha.$$

Proof. (a) Let $X(t) \in D$. Then, from § 20.3, 1, for arbitrary choice of the phase $\bar{\mathbf{A}}$ of (Q) there holds the relationship

$$X(t) = \bar{\mathbf{A}}^{-1}\alpha(t). \tag{20.19}$$

Since $\mathbf{A}$, $\bar{\mathbf{A}}$ are phases of the same differential equation (Q) and consequently lie in the same right coset $\mathfrak{E}\mathbf{A} = \mathfrak{E}\bar{\mathbf{A}}$, we have $\bar{\mathbf{A}} = \xi\mathbf{A}$, $\xi \in \mathfrak{E}$. It follows that $\bar{\mathbf{A}}^{-1} = \mathbf{A}^{-1}\xi^{-1}$ and moreover, on taking account of (19)

$$X(t) = \mathbf{A}^{-1}\xi^{-1}\alpha \in \mathbf{A}^{-1}\mathfrak{E}\alpha.$$

We have therefore $D \subset \mathbf{A}^{-1}\mathfrak{E}\alpha$.

(b) Let $X(t) \in \mathbf{A}^{-1}\mathfrak{E}\alpha$. Then we have, for arbitrary choice of $\xi \in \mathfrak{E}$

$$X(t) = \mathbf{A}^{-1}\xi\alpha(t) = (\xi^{-1}\mathbf{A})^{-1}\alpha(t).$$

Now, $\xi^{-1} \in \mathfrak{E}$, and we see that $\bar{\mathbf{A}} = \xi^{-1}\mathbf{A} \in \mathfrak{E}\mathbf{A}$ is a phase of the differential equation (Q); consequently we have

$$X(t) = \bar{\mathbf{A}}^{-1}\alpha(t).$$

Then in view of § 20.4, 3, this relation gives $X(t) \in D$. We have, therefore, $\mathbf{A}^{-1}\mathfrak{E}\alpha \subset D$, and the proof is complete.

On this topic, see also § 29.1 on p. 237.