# 21 Dispersions of the $\varkappa$-th kind; $\varkappa = 1, 2, 3, 4$

We have already encountered, in § 18.1, the concept of a dispersion of the $\kappa$-th kind, $\kappa = 1, 2, 3, 4$.

The theory of general dispersions given in § 20 above contains, as a special case, a theory of dispersions of the $\kappa$-th kind with which we are now concerned and so gives additional information about the latter. We shall see, however, that besides this new information, there will also emerge some entirely fresh aspects as a consequence of the special character of these dispersions.

In the following we shall consider an oscillatory differential equation (q) in the interval $j = (a, b)$, assuming that $q < 0$ and $q \in C_2$ for $t \in j$.

## 21.1 Introduction

*By dispersions of the first, second, third, or fourth kind of the differential equation* (q) *we mean respectively a general dispersion of the differential equations* (q), (q); ($\hat{\mathrm{q}}_1$), ($\hat{\mathrm{q}}_1$); ($\hat{\mathrm{q}}_1$), (q); (q), ($\hat{\mathrm{q}}_1$).

Every dispersion $\zeta$ of the $\kappa$-th kind can be constructed as a general dispersion using appropriate initial values $t_0$, $T_0$ and generator $\boldsymbol{p}$ (§ 20.2). Depending on the value of $\kappa$, $\boldsymbol{p}$ is a linear mapping of the integral space $r$ of (q) or of the integral space $r_1$ of ($\hat{\mathrm{q}}_1$) onto $r$ or $r_1$. According as $\kappa = 1, 2, 3, 4$, these mappings are $r \to r$, $r_1 \to r_1$, $r \to r_1$, $r_1 \to r$ respectively. We shall begin our further studies with some remarks about linear mappings in the above cases.

Let $\boldsymbol{p}$ be a linear mapping of the integral space $r$ or $r_1$ onto the integral space $r$ or $r_1$ and let $\boldsymbol{P}$ be the projection (§ 1.9) of the integral space $r$ on $r_1$. Moreover, in each of the following cases

$$r \to r; \quad r_1 \to r_1; \quad r \to r_1; \quad r_1 \to r \quad (\boldsymbol{p}) \tag{21.1}$$

let $y \in r$ or $y_1 \in r_1$ be arbitrary integrals of the differential equation (q) or ($\hat{\mathrm{q}}_1$) respectively and $Y \in r$ or $Y_1 \in r_1$ its image under $\boldsymbol{p}$, thus

$$y \to Y; \quad y_1 \to Y_1; \quad y \to Y_1; \quad y_1 \to Y \quad (\boldsymbol{p}).$$

Then in each of these cases we can define a linear mapping $\boldsymbol{p}_0$ of the integral space $r$ onto itself by means of the appropriate one of the following formulae

$$y \to Y; \quad \boldsymbol{P}^{-1}y_1 \to \boldsymbol{P}^{-1}Y_1; \quad y \to \boldsymbol{P}^{-1}Y_1; \quad \boldsymbol{P}^{-1}y_1 \to Y \quad (\boldsymbol{p}_0).$$

This mapping $\boldsymbol{p}_0$ we call the *kernel* of $\boldsymbol{p}$ and we write $\boldsymbol{p}_0 = \mathrm{K}\boldsymbol{p}$.

Every linear mapping $\boldsymbol{p}$ of the integral space $r$ or $r_1$ onto $r$ or $r_1$ has therefore-precisely one kernel $\boldsymbol{p}_0$; this is a linear mapping of the integral space $r$ onto itself. Conversely, every linear mapping $\boldsymbol{p}_0$ of the integral space $r$ onto itself in each of the above cases (1) represents the kernel of precisely one linear mapping $\boldsymbol{p}$, and indeed the linear mapping $y \to Y\,(\boldsymbol{p}_0)$ is the kernel of the following linear mappings:

$$y \to Y; \quad \frac{y'}{\sqrt{-q}} \to \frac{Y'}{\sqrt{-q}}; \quad y \to \frac{Y'}{\sqrt{-q}}; \quad \frac{y'}{\sqrt{-q}} \to Y \quad (\boldsymbol{p}).$$

## 21.2 Determination of dispersions of the $\kappa$-th kind, $\kappa = 1, 2, 3, 4$

We have the following theorems:

1. All dispersions $X_1$, $X_2$, $X_3$, $X_4$ of the first, second, third, and fourth kinds of the differential equation (q) are determined by the following formulae:

$$\alpha(X_1) = \bar{\alpha}(t); \quad \beta(X_2) = \bar{\beta}(t); \quad \beta(X_3) = \bar{\alpha}(t); \quad \alpha(X_4) = \bar{\beta}(t); \qquad (21.2)$$

in which $\alpha$, $\bar{\alpha}$ denote arbitrary or appropriate first phases and similarly $\beta$, $\bar{\beta}$ second phases of the differential equation (q).

*Proof.* We shall confine ourselves to the proof of the first two statements.

The relation (2) formed with two arbitrary first phases $\alpha$, $\bar{\alpha}$ of (q) determines (using § 20.4) a general dispersion $X_1$ of the (coincident) differential equations (q), (q). This consequently represents a dispersion of the first kind of the differential equation (q). Conversely, every dispersion $X_1$ of the first kind of the differential equation (q), is a general dispersion of the differential equations (q), (q) (§ 21.1) and consequently satisfies the first relation (2) with some appropriate first phases $\alpha$, $\bar{\alpha}$ of (q).

The second relation (2) formed with arbitrary first phases $\beta$, $\bar{\beta}$ of the differential equation ($\hat{q}_1$) determines (by § 20.4) a general dispersion $X_2$ of the differential equations ($\hat{q}_1$), ($\hat{q}_1$) (§ 5.11). Consequently this represents a dispersion of the second kind of the differential equation (q). Similarly it can be seen that every dispersion $X_2$ of the second kind of the differential equation (q), satisfies the second relation (2) with appropriate second phases $\beta$, $\bar{\beta}$ of (q).

2. Let $X_\kappa$ be a dispersion of the $\kappa$-th kind of the differential equation (q) and $\boldsymbol{p}_\kappa$ its generator ($\kappa = 1, 2, 3, 4$). By means of suitable variation of the linear mapping $\boldsymbol{p}_\kappa$ we can establish the following relations in the interval $j$ between the integrals $y$ and $Y = \mathrm{K}\boldsymbol{p}_\kappa y$ of (q) and their derivatives $y'$, $Y'$:—

$$\left.\begin{aligned} &\frac{YX_1(t)}{\sqrt{|X_1'(t)|}} = y(t); \quad \frac{1}{\sqrt{|X_2'(t)|}} \cdot \frac{Y'X_2(t)}{\sqrt{-qX_2(t)}} = \frac{y'(t)}{\sqrt{-q(t)}}; \\ &\frac{1}{\sqrt{|X_3'(t)|}} \cdot \frac{Y'X_3(t)}{\sqrt{-qX_3(t)}} = y(t); \quad \frac{YX_4(t)}{\sqrt{|X_4'(t)|}} = \frac{y'(t)}{\sqrt{-q(t)}}. \end{aligned}\right\} \qquad (21.3)$$

This follows from the theorem of § 20.3, 9.

## 21.3 Determination of the central dispersion of the $\kappa$-th kind; $\kappa = 1, 2, 3, 4$

The dispersions of the $\kappa$-th kind of the differential equation (q) naturally include the central dispersions of the corresponding kind. The latter are, therefore, determined by formulae such as (2), in which the phases which occur satisfy appropriate conditions. On this topic we have the following theorems:

1. The dispersion $X_1$ of the first kind of the differential equation (q) determined by the first formula (2) is a central dispersion of the first kind of (q) if and only if the phases $\alpha$, $\bar{\alpha}$ belong to the first phase system of the same basis of the differential equation (q), and in that case $X_1 = \phi_\nu$ ($\nu = 0, \pm 1, \pm 2, \ldots$) if and only if $\bar{\alpha} = \alpha + \nu\pi \operatorname{sgn} \alpha'$.

The dispersion of the second kind $X_2$ of the differential equation (q) determined by the second formula (2) is a central dispersion of the second kind of (q) if and only if the phases $\beta$, $\bar{\beta}$ belong to the second phase system of the same basis of the differential equation (q), and in this case $X_2 = \psi_\nu$ ($\nu = 0, \pm 1, \pm 2, \ldots$) if and only if $\bar{\beta} = \beta + \nu\pi \operatorname{sgn} \beta'$.

The dispersion of the third kind $X_3$ of the differential equation (q) determined by the third formula (2) is a central dispersion of the third kind of (q) if and only if the phases $\bar{\alpha}$, $\beta$ belong to the first and second phase systems of the same basis of the differential equation (q), and in that case $X_3 = \chi_\rho$ ($\rho = \pm 1, \pm 2, \ldots$) if and only if

$$-\frac{1}{2}[1 - (2\rho - \operatorname{sgn} \rho)\varepsilon]\pi < \bar{\alpha} - \beta < \frac{1}{2}[1 + (2\rho - \operatorname{sgn} \rho)\varepsilon]\pi$$
$$(\varepsilon = \operatorname{sgn} \bar{\alpha}' = \operatorname{sgn} \beta').$$

The dispersion of the fourth kind, $X_4$ of the differential equation (q) determined by the fourth formula (2) is a central dispersion of the fourth kind of (q) if and only if the phases $\alpha$, $\bar{\beta}$ belong to the first and second phase systems of the same basis of the differential equation (q) and in this case $X_\rho = \omega_\rho$ ($\rho = \pm 1, \pm 2, \ldots$) if and only if

$$-\frac{1}{2}[1 - (2\rho - \operatorname{sgn} \rho)\varepsilon]\pi < \bar{\beta} - \alpha < \frac{1}{2}[1 + (2\rho - \operatorname{sgn} \rho)\varepsilon]\pi$$
$$(\varepsilon = \operatorname{sgn} \alpha' = \operatorname{sgn} \bar{\beta}').$$

These theorems follow from the Abel functional equations (§ 13.7).

2. Let $\zeta$ be a dispersion of the $\kappa$-th kind of the differential equation (q) with initial numbers $t_0$, $T_0$ and generator $\boldsymbol{p}$; $\kappa = 1, 2, 3, 4$. Then, and only then, the following relations hold in the interval $j$:

| | | | | |
|---|---|---|---|---|
| (a) $\zeta = \phi_\nu$, | if | $T_0 = \phi_\nu(t_0)$ | and | $\boldsymbol{p} = c\boldsymbol{e}$; |
| (b) $\zeta = \psi_\nu$, | if | $T_0 = \psi_\nu(t_0)$ | and | $\boldsymbol{p} = c\boldsymbol{e}$; |
| (c) $\zeta = \chi_\rho$, | if | $T_0 = \chi_\rho(t_0)$ | and | $\boldsymbol{p} = c\boldsymbol{P}$; |
| (d) $\zeta = \omega_\rho$, | if | $T_0 = \omega_\rho(t_0)$ | and | $\boldsymbol{p} = c\boldsymbol{P}$; |

where $\boldsymbol{e}$ denotes the identity mapping of the integral space $r$ or $r_1$ onto itself, $\boldsymbol{P}$ the projection from $r$ on $r_1$ and $c$ ($\neq 0$) a constant; $\nu = 0, \pm 1, \pm 2, \ldots$; $\rho = \pm 1, \pm 2, \ldots$.

*Proof.* We shall prove, as an illustration, the result in case (c).

1. Let $\zeta = \chi_\rho$ for $t \in j$. First, we obviously have $T_0 = \zeta(t_0) = \chi_\rho(t_0)$. Now we consider an integral $u \in r$ of the differential equation (q) and let $x$ be a zero of $u$; we have therefore $u(x) = 0$, $u'\zeta(x) = 0$. Moreover, let $u_1 = \boldsymbol{p}u \in r_1$, consequently $u_1 = \bar{u}'/\sqrt{(-q)}$, in which $\bar{u} \in r$. From the definition of $\zeta$ we have $u_1\zeta(x) = 0$, hence

$\bar{u}'\zeta(x) = 0$; the functions $u'$, $\bar{u}'$ have therefore the common zero $\zeta(x)$ so the integrals $u, \bar{u} \in r$ of (q) are linearly dependent, i.e. $\bar{u} = cu$, where $c$ ($\neq 0$) denotes a constant. We have therefore $\boldsymbol{p}u = c\boldsymbol{P}u$.

We next show that the value of the constant $c$ does not depend on the choice of the integral $u \in r$. For, let $v \in r$ be another integral of (q). We first assume that $v$ depends upon $u$, so that $v = ku$, $0 \neq k =$ constant; then we have $k\boldsymbol{p}u = \boldsymbol{p}(ku) = \boldsymbol{p}v = \bar{c}\boldsymbol{P}v = \bar{c}\boldsymbol{P}ku = \bar{c}k\boldsymbol{P}u = (\bar{c}k/c)\boldsymbol{p}u$ ($0 \neq \bar{c} =$ constant) and consequently $\bar{c} = c$. Secondly, let us assume that $v$ is independent of $u$; then by the above reasoning, $\boldsymbol{p}v = \bar{c}\boldsymbol{P}v$ ($0 \neq \bar{c} =$ constant). Now consider the integral $u + v \in r$ of (q); on the one hand, we have $\boldsymbol{p}(u + v) = \boldsymbol{p}u + \boldsymbol{p}v = c\boldsymbol{P}u + \bar{c}\boldsymbol{P}v$, but we also have $\boldsymbol{p}(u + v) = C\boldsymbol{P}(u + v) = C(\boldsymbol{P}u + \boldsymbol{P}v)$ ($0 \neq C =$ constant). From these relations it follows that $(c - C)\boldsymbol{P}u + (\bar{c} - C)\boldsymbol{P}v = 0$ and hence $c = C = \bar{c}$; this shows that $\boldsymbol{p} = c\boldsymbol{P}$.

2. Let $T_0 = \chi_\rho(t_0)$ and $\boldsymbol{p} = c\boldsymbol{P}$.

The fundamental numbers $t_\nu$ of the differential equation (q) with respect to $t_0$ are $t_\nu = \phi_\nu(t_0)$ while the $T_\nu$ are those of the differential equation $(\hat{q}_1)$ with respect to $T_0$: $T_\nu = \chi_\rho\phi_\nu(t_0)$; $\nu = 0, \pm 1, \pm 2, \ldots$. Now let $t \in j$ be arbitrary and $u \in r$ an integral of (q) which vanishes at the point $t$, i.e. $u(t) = 0$. Then $\boldsymbol{p}u = c\boldsymbol{P}u = cu'/\sqrt{(-q)}$. From the definition of $\zeta$, we have $\boldsymbol{p}u\zeta(t) = 0$ and hence $u'\zeta(t) = 0$ so we have $\zeta(t) = \chi_m(t)$ with some appropriate index $m$. We now show that $m = \rho$ independently of the choice of $t$. For, $t$ lies in a certain right fundamental interval $[t_\nu, t_{\nu+1})$. From the relation $\operatorname{sgn} \chi\boldsymbol{p} = \operatorname{sgn} \chi(c\boldsymbol{P}) = +1$ we conclude that $\zeta$ is direct; consequently $\zeta(t)$ lies in the right fundamental interval $[T_\nu, T_{\nu+1})$. At the same time, from $t \in [\phi_\nu(t_0), \phi_{\nu+1}(t_0))$ it follows that $\zeta(t) \in [\chi_m\phi_\nu(t_0), \chi_m\phi_{\nu+1}(t_0))$. Thus $\chi_m\phi_\nu(t_0) = \chi_\rho\phi_\nu(t_0)$, and this relationship gives immediately $m = \rho$. We have therefore $\zeta(t) = \chi_\rho(t)$ for all $t \in j$, and this completes the proof.

## 21.4 The group of dispersions of the first kind of the differential equation (q)

In § 20.7 we have embedded the set $D$ of general dispersions of two differential equations (q), (Q) over the intervals $j = J = (-\infty, \infty)$ in the phase group $\mathfrak{G}$. It was there proved that for arbitary choice of first phases $\alpha$, $\mathrm{A}$ of (q) and (Q) respectively the set $D$ can be represented as

$$D = \mathrm{A}^{-1}\mathfrak{E}\alpha; \tag{21.4}$$

where $\mathfrak{E}$ denotes the fundamental subgroup of $\mathfrak{G}$.

We now consider the case of coincident differential equations (q), (Q): $q = Q$ for all $t \in j = (-\infty, \infty)$; we here assume only that $q \in C_0$, not necessarily that $q \in C_2$. Then the set $D$ represents the dispersions of the first kind of the differential equation (q). If we choose $\mathrm{A} = \alpha$, then formula (4) gives

$$D = \alpha^{-1}\mathfrak{E}\alpha. \tag{21.5}$$

The right side of this formula does not depend on the choice of the phase $\alpha$, as can easily be seen: all phases of the differential equation (q) form precisely the right coset $\mathfrak{E}\alpha$ of $\mathfrak{E}$ (§ 10.2, 10.3). Consequently every phase $\bar{\alpha}$ of the differential equation (q) has the form $\bar{\alpha} = \xi\alpha$, where $\xi$ is an appropriate element of $\mathfrak{E}$. We have therefore

$\bar{\alpha}^{-1} = \alpha^{-1}\xi^{-1}$ and moreover $\bar{\alpha}^{-1}\mathfrak{E}\bar{\alpha} = \alpha^{-1}(\xi^{-1}\mathfrak{E}\xi)\alpha = \alpha^{-1}\mathfrak{E}\alpha = D$ since obviously $\xi^{-1}\mathfrak{E}\xi = \mathfrak{E}$.

The fundamental subgroup $\mathfrak{E}$ of $\mathfrak{G}$, transformed by an arbitrary phase $\alpha$ of the differential equation (q) in accordance with the right side of (5), obviously represents precisely the set of dispersions of the first kind of the differential equation (q).

It is however known from group theory that under a transformation of this kind a subgroup of $\mathscr{R}$ goes over into another subgroup which is described as conjugate to the first.

*The set of dispersions of the first kind of the differential equation* (q) *forms a subgroup in the phase group* $\mathfrak{G}$ *conjugate with the fundamental subgroup.*

See also p. 237 for some recent results.

## 21.5 Group property of dispersions of the first kind

We now bring out the group property of the set of dispersions of the first kind from their definition, and then study the structure of this group.

To simplify our formulation, a *dispersion* will here always mean a dispersion of the first kind. Correspondingly, those linear mappings which come under consideration will always be linear mappings of the integral space $r$ of the differential equation (q) onto itself, and by fundamental intervals we always mean the fundamental intervals of the differential equation (q); $j$ naturally denotes the interval $(-\infty, \infty)$.

1. Let $\zeta$ be the dispersion determined by arbitrary equal initial numbers $t_0$, $t_0$ and the identically linear mapping $\boldsymbol{e}$. Then we have $\zeta(t) = t$ for $t \in j$.

For, let $t \in j$ be arbitrary. If $t = t_0$, then $\zeta(t) = t_0 = t$, so let us assume that $t \neq t_0$. Let $y \in r$ be an integral of the differential equation (q) vanishing at the point $t$; $t$ lies in a certain $\nu$-th right fundamental interval $j_\nu$ with respect to $t_0$. Since sgn $\chi\boldsymbol{e} = 1$ we conclude that $\zeta$ is direct; consequently $\zeta(t)$ is that zero of the integral $\boldsymbol{e}y = y$ which lies in the same fundamental interval $j_\nu$. We have therefore $\zeta(t) = t$.

2. Let $\zeta$ be the dispersion determined by arbitrary initial numbers $t_0$, $T_0$ and an arbitrary generator $\boldsymbol{p}$. Then the inverse function $\zeta^{-1}$ represents the dispersion determined by the initial numbers $T_0$, $t_0$ and the inverse linear mapping $\boldsymbol{p}^{-1}$; this is direct or indirect according as $\zeta$ is direct or indirect.

*Proof.* We have $\chi\boldsymbol{p} > 0$ or $< 0$ according as $\zeta$ is direct or indirect. The mapping $\boldsymbol{p}^{-1}$ inverse to $\boldsymbol{p}$ is normalized with respect to the numbers $T_0$, $t_0$ (§ 19.7) and has characteristic $1/\chi\boldsymbol{p}$ (§ 19.2).

Let $Z$ be the dispersion determined by the initial numbers $T_0$, $t_0$ and the generator $\boldsymbol{p}^{-1}$; $Z$ is therefore direct or indirect according as $\zeta$ is direct or indirect.

Let $t \in j$ be an arbitrary number. If $t = T_0$, then $\zeta^{-1}(t) = \zeta^{-1}(T_0) = t_0 = Z(T_0) = Z(t)$, and consequently $\zeta^{-1}(t) = Z(t)$.

We now assume $t \neq T_0$. Let $y \in r$ be an integral of the differential equation (q) vanishing at the point $t$. Now $t$ lies in a certain right fundamental interval with respect to $T_0$; let it be the $\nu$-th such interval. It also lies in, say, the $\mu$-th left fundamental interval. Consequently, $\zeta^{-1}(t)$ is the zero of the integral $\boldsymbol{p}^{-1}y \in r$ lying in the $\nu$-th or $-\mu$-th right fundamental interval with respect to $t_0$, according as the dispersion $\zeta$ is direct or indirect. Hence, using the definition of $Z$, $\zeta^{-1}(t) = Z(t)$.

3. Let $\zeta_1$, $\zeta_2$ be the dispersions determined by arbitrary initial numbers $t_0$, $\bar{t}_0$; $\bar{t}_0$, $T_0$ and arbitrary generators $\boldsymbol{p}_1$, $\boldsymbol{p}_2$. Then the composite function $\zeta_2\zeta_1$ represents the dispersion determined by the initial numbers $t_0$, $T_0$ and the composite linear mapping $\boldsymbol{p}_2\boldsymbol{p}_1$. The latter is direct if both dispersions $\zeta_1$, $\zeta_2$ are direct or both are indirect; it is however indirect if one of these dispersions is direct and the other indirect.

*Proof.* We have $\chi\boldsymbol{p}_i > 0$ or $< 0$ according as $\zeta_i$ is direct or indirect ($i = 1, 2$). The linear mapping $\boldsymbol{p} = \boldsymbol{p}_2\boldsymbol{p}_1$ is normalized with respect to the numbers $t_0$, $T_0$ (§ 19.7) and has the characteristic $\chi\boldsymbol{p} = (\chi\boldsymbol{p}_2)(\chi\boldsymbol{p}_1)$ (§ 19.2).

Let $\zeta$ be the dispersion determined by the initial numbers $t_0$, $T_0$ and generator $\boldsymbol{p}$; $\zeta$ is therefore direct or indirect according as $(\chi\boldsymbol{p}_2)(\chi\boldsymbol{p}_1) > 0$ or $< 0$.

Let $t \in j$ be arbitrary. If $t = t_0$ then we have $\zeta_1(t) = \bar{t}_0$, $\zeta_2[\zeta_1(t)] = \zeta_2(\bar{t}_0) = T_0 = \zeta(t_0) = \zeta(t)$ and consequently we have $\zeta_2\zeta_1(t) = \zeta(t)$.

We assume now that $t \neq t_0$. Let $y \in r$ be an integral of the differential equation (q) vanishing at the point $t$; $t$ lies in a certain right fundamental interval—say, the $\nu$-th such interval—with respect to $t_0$. The number $\zeta_1(t)$ is therefore the zero of the integral $\boldsymbol{p}_1 y \in r$ lying in the $\nu$-the right fundamental interval or the $-\nu$-th left fundamental interval with respect to $\bar{t}_0$ according as $\chi\boldsymbol{p}_1 > 0$ or $< 0$.

If $\chi\boldsymbol{p}_1 > 0$, then $\zeta_2[\zeta_1(t)]$ is that zero of the integral $\boldsymbol{p}_2(\boldsymbol{p}_1 y) = \boldsymbol{p}y \in r$ which is contained in the $\nu$-th right or the $-\nu$-th left fundamental interval with respect to $T_0$, according as $\chi\boldsymbol{p}_2 > 0$ or $< 0$.

If $\chi\boldsymbol{p}_1 < 0$, then $\zeta_2[\zeta_1(t)]$ is that zero of $\boldsymbol{p}_2(\boldsymbol{p}_1 y) = \boldsymbol{p}y$ which lies in the $-\nu$-th left or $\nu$-th right fundamental interval with respect to $T_0$, according as $\chi\boldsymbol{p}_2 > 0$ or $< 0$.

Thus the number $\zeta_2\zeta_1(t)$ is that zero of the integral $\boldsymbol{p}y \in r$ which lies in the $\nu$-th right or $-\nu$-th left fundamental interval with respect to $T_0$ according as $(\chi\boldsymbol{p}_2)(\chi\boldsymbol{p}_1) > 0$ or $< 0$. It follows, using the definition of the function $\zeta$, that

$$\zeta_2\zeta_1(t) = \zeta(t).$$

4. Let $t_0$; $\zeta_0$, $\zeta_0'$ $(\neq 0)$, $\zeta_0''$ be arbitrary numbers. There exists precisely one dispersion $\zeta$ satisfying the Cauchy initial conditions

$$\zeta(t_0) = \zeta_0, \quad \zeta'(t_0) = \zeta_0', \quad \zeta''(t_0) = \zeta_0''. \tag{21.6}$$

and this is direct or indirect according as $\zeta_0' > 0$ or $\zeta_0' < 0$.

This theorem is obviously a special case of that in § 20.4, 2. The dispersion $\zeta$ in question is determined uniquely by the first phases $\alpha$, $\mathrm{A}$ of the differential equation (q), which are themselves determined by the initial values

$$\alpha(t_0) = 0, \quad \alpha'(t_0) = 1, \quad \alpha''(t_0) = 0; \quad \mathrm{A}(\zeta_0) = 0, \quad \mathrm{A}'(\zeta_0) = 1/\zeta_0',$$
$$\mathrm{A}''(\zeta_0) = -\zeta_0''/\zeta_0'^3$$

by use of the formula $\mathrm{A}\zeta(t) = \alpha(t)$; $t \in (-\infty, \infty)$.

We see that the dispersions form a system which is continuously dependent upon the parameters $\zeta_0$, $\zeta_0'$ $(\neq 0)$, $\zeta_0''$.

5. The dispersions form a 3-parameter group $\mathfrak{D}$ in which the operation of multiplication is defined as the composition of functions, with unit element $(\underline{1} =)\ t$. The direct, and consequently increasing, dispersions form in the group $\mathfrak{D}$ an invariant subgroup $\mathfrak{P}$ with index 2; the indirect, and consequently decreasing, dispersions form in

the group $\mathfrak{D}$ the coset of $\mathfrak{P}$, and consequently the second element of the factor group $\mathfrak{D}/\mathfrak{P}$.

*Proof.* The first part of this theorem follows immediately from the above results 1–4. Obviously, $\underline{1} \in \mathfrak{P}$ and for arbitrary elements $\zeta_1, \zeta_2 \in \mathfrak{P}$, we have $\zeta_1^{-1} \in \mathfrak{P}$, $\zeta_2\zeta_1 \in \mathfrak{P}$ is a subgroup of $\mathfrak{D}$.

Now let $A \subset \mathfrak{D}$ be the set comprising the indirect dispersions, and consider an arbitrary dispersion $\zeta \in \mathfrak{D}$. From the definition of the left (right) coset $\zeta\mathfrak{P}$ ($\mathfrak{P}\zeta$) of the elements $\zeta \in \mathfrak{D}$ with respect to the subgroup $\mathfrak{P}$, the former constitutes the set comprising the dispersions $\zeta X$ ($X\zeta$) in which $X$ ranges over the elements of $\mathfrak{P}$. But, from 3, $\zeta X \in \mathfrak{P}$ or $\zeta X \in A$ ($X\zeta \in \mathfrak{P}$ or $X\zeta \in A$), according as $\zeta \in \mathfrak{P}$ or $\zeta \in A$. We have therefore $\xi\mathfrak{P} = \mathfrak{P} = \mathfrak{P}\xi$ in the case $\xi \in \mathfrak{P}$ and $\xi\mathfrak{P} = A = \mathfrak{P}\xi$ in the case $\xi \in A$, so that in both cases $\zeta\mathfrak{P} = \mathfrak{P}\zeta$. This shows that the subgroup $\mathfrak{P}$ in $\mathfrak{D}$ is invariant. The factor group $\mathfrak{D}/\mathfrak{P}$ is obviously formed from the two elements $\mathfrak{P}$, $A$. This completes the proof.

We call $\mathfrak{D}$ the *dispersion group of the first kind* of the differential equation (q), or more briefly the dispersion group.

## 21.6 Representation of the dispersion group

1. In the following we shall continue to denote the dispersion group of the first kind of the differential equation (q) by $\mathfrak{D}$. We shall introduce also the following notation:

- $\mathfrak{P}$: The invariant subgroup of $\mathfrak{D}$ comprising the direct dispersions.
- $\mathfrak{C}$: The infinite cyclic group formed from the central dispersions of the first kind of the differential equation (q) (§ 12.5).
- $\mathfrak{S}$: The infinite cyclic group formed from the central dispersions of the first kind with even indices of the differential equation (q).

We have therefore the following inclusion relationships

$$\mathfrak{D} \supset \mathfrak{P} \supset \mathfrak{C} \supset \mathfrak{S} \supset \{\underline{1}\}. \tag{21.7}$$

We now choose a basis $(u, v)$ of the differential equation (q) and denote its Wronskian by $w$. Let $\zeta \in \mathfrak{D}$ be an arbitrary dispersion, and $U$, $V$ be the functions

$$U = \frac{u(\zeta)}{\sqrt{|\zeta'|}}, \qquad V = \frac{v(\zeta)}{\sqrt{|\zeta'|}}. \tag{21.8}$$

From § 20.6, 3, $U$, $V$ are independent integrals of (q) and the Wronskian of the basis of (q) formed from these is $W = w \operatorname{sgn} \zeta'$.

Obviously, the following formulae hold in the interval $j$:

$$\left.\begin{aligned} \frac{u(\zeta)}{\sqrt{|\zeta'|}} &= c_{11}u + c_{12}v, \\ \frac{v(\zeta)}{\sqrt{|\zeta'|}} &= c_{21}u + c_{22}v; \end{aligned}\right\} \tag{21.9}$$

in which $c_{11}$, $c_{12}$, $c_{21}$, $c_{22}$ are appropriate constants. The $2 \times 2$ matrix $C = ||c_{ik}||$ of these constants is determined uniquely by the dispersion $\zeta$.

Now it follows from (9) that $W = |C|w$, and hence

$$|C| = \operatorname{sgn} \zeta';$$

where $|C|$ naturally denotes the determinant of $C$. $C$ is thus a unimodular matrix with determinant $\operatorname{sgn} \zeta'$. From § 20.3, 2 it follows that $|C|$ has the value 1 or $-1$ according as the dispersion $\zeta$ is direct or indirect.

To simplify our notation we shall write the formula (9) in the vector form

$$\frac{u(\zeta)}{\sqrt{|\zeta'|}} = Cu, \tag{21.10}$$

where $u$ denotes the vector formed from the components $u, v$.

2. We now associate with every dispersion $\zeta \in \mathfrak{D}$ the matrix $C$ defined by means of the formula (10). In this way we obtain a mapping $\boldsymbol{d}$ of the group $\mathfrak{D}$ onto the group $\mathfrak{L}$ formed from all unimodular $2 \times 2$ matrices. We are in fact concerned with a mapping *onto* the group $\mathfrak{L}$ as can be seen as follows:

Let $C = ||c_{ik}||$ be an arbitrary element of $\mathfrak{L}$; we have to show the existence of an original $\zeta \in \mathfrak{D}$ of $C$ under the mapping $\boldsymbol{d}$. To accomplish this, we choose an arbitrary zero $t_0$ of the integral $c_{21}u + c_{22}v \in r$ and a zero $T_0$ of $v \in r$, doing this in such a way that

$$\operatorname{sgn} u(T_0) = \operatorname{sgn} (c_{11}u(t_0) + c_{12}v(t_0)) \tag{21.11}$$

(it is easy to show that such a choice is always possible). Let $\zeta$ be the dispersion determined by the initial numbers $t_0$, $T_0$ and the generator $\boldsymbol{p} = [c_{11}u + c_{12}v \to u, c_{21}u + c_{22}v \to v]$. Since $|C| = \operatorname{sgn} |C|$, we have $\chi\boldsymbol{p} = \operatorname{sgn} |C|$. Now the formula (20.17) applied to the integrals $(Y =)\ u, v$ and the dispersion $\zeta$ gives, when we take account of (11), relationships such as (9). Consequently $C$ is the original of the dispersion $\zeta$ in the mapping $\boldsymbol{d}$.

We now wish to show that $\boldsymbol{d}$ is a homomorphic mapping (deformation) of the group $\mathfrak{D}$ onto the matrix group $\mathfrak{L}$.

We therefore consider arbitrary dispersions $\zeta_1, \zeta_2 \in \mathfrak{D}$ and their $\boldsymbol{d}$-images $C_1, C_2 \in \mathfrak{L}$. From the formulae

$$\frac{u(\zeta_1)}{\sqrt{|\zeta_1'|}} = C_1 u, \qquad \frac{u(\zeta_2)}{\sqrt{|\zeta_2'|}} = C_2 u$$

there follows the relationships

$$\frac{u(\zeta_2\zeta_1)}{\sqrt{|\zeta_2(\zeta_1)|}} \cdot \frac{1}{\sqrt{|\zeta_1'|}} = C_2 \frac{u(\zeta_1)}{\sqrt{|\zeta_1'|}} = C_2 C_1 u,$$

and moreover

$$\frac{u(\zeta_2\zeta_1)}{\sqrt{|(\zeta_2\zeta_1)'|}} = C_2 C_1 u,$$

so that we have, indeed $\boldsymbol{d}(\zeta_2\zeta_1) = C_2 C_1$.

3. The unit element of the group $\mathfrak{L}$ is naturally the unit matrix $E = ||\delta_{ik}||$ ($\delta_{11} = \delta_{22} = 1$; $\delta_{12} = \delta_{21} = 0$). We now prove the theorem:

The deformation $\boldsymbol{d}$ of the group $\mathfrak{D}$ onto the group $\mathfrak{L}$ maps onto the unit element $E \in \mathfrak{L}$ precisely the central dispersions of the first kind of the differential equation (q) with even indices while onto the element $-E \in \mathfrak{L}$ it maps precisely the central dispersions of the first kind of the differential equation (q) with odd indices.

*Proof.* Let $\zeta$ be a central dispersion of the first kind of the differential equation (q), so that from (13.10) we have

$$\frac{u(\zeta)}{\sqrt{|\zeta'|}} = u, \quad \text{or} \quad \frac{u(\zeta)}{\sqrt{|\zeta'|}} = -u, \tag{21.12}$$

according as the index of $\zeta$ is even or odd. It follows that $\boldsymbol{d}\zeta = E$ or $\boldsymbol{d}\zeta = -E$. Conversely, let any dispersion $\zeta$ be such that $\boldsymbol{d}\zeta = E$ or $\boldsymbol{d}\zeta = -E$; then the formulae (12) hold and from that and § 3.12 we conclude that $\zeta$ is a central dispersion of the first kind of (q) with an even or an odd index.

If we now apply the first isomorphism theorem for groups (see [81], p. 178), then we obtain the following result:

The group $\mathfrak{S}$ formed from the central dispersions of the first kind of the differential equation (q) with even indices is invariant in the group $\mathfrak{D}$, and the factor group $\mathfrak{D}/\mathfrak{S}$ is isomorphic to the matrix group $\mathfrak{L}$. All the dispersions included in the same element of $\mathfrak{D}/\mathfrak{S}$ are mapped by the deformation $\boldsymbol{d}$ onto the same element of $\mathfrak{L}$.

4. From the formula (20.13) we conclude that every central dispersion of the first kind of the differential equation (q) commutes with each direct dispersion of (q). This implies that the group $\mathfrak{C}$ is a subgroup of the centre of $\mathfrak{P}$. We now show that $\mathfrak{C}$ coincides with the centre, in other words we have the following theorem:

*Theorem. The central dispersions of the first kind of the differential equation* (q) *form the centre of the group* $\mathfrak{P}$ *of the direct dispersions of* (q).

*Proof.* It is clearly sufficient to show that every direct dispersion which commutes with all direct dispersions is a central dispersion of the first kind of (q).

Let $\zeta_0$ be a direct dispersion which commutes with all direct dispersions, and consequently with all elements of $\mathfrak{P}$. Moreover let $\zeta$ be an arbitrary element of $\mathfrak{P}$. Further, let $C_0 = ||c_{ik}^0||$, $C = ||c_{ik}||$ be the $\boldsymbol{d}$-images of $\zeta_0$ and $\zeta$ respectively. We have therefore $C_0 = \boldsymbol{d}\xi_0$, $C = \boldsymbol{d}\xi$; $|C_0| = 1$, $|C| = 1$. From $\zeta_0\zeta = \zeta\zeta_0$ there follows the relationship

$$C_0C = CC_0. \tag{21.13}$$

Since every element of $\mathfrak{L}$ possesses a $\boldsymbol{d}$-original, we conclude from (13) that the matrix $C_0$ commutes with every matrix $C \in \mathfrak{L}$, ($|c| = 1$). Let us now first choose $c_{12} = c_{21} = 0$, $c_{11} \neq c_{22}$, $c_{11}c_{22} = 1$ and then $c_{11} = c_{22} = 0$, $c_{12}c_{21} = -1$; then we obtain $c_{12}^0 = c_{21}^0 = 0$, $c_{11}^0 = c_{22}^0$; $c_{11}^0 c_{22}^0 = 1$. We have therefore $C_0 = E$ or $C_0 = -E$ and we see that $\zeta_0$ is a central dispersion of the first kind of the differential equation (q). This completes the proof.

The above theorem was the motive for our choice of the adjective "central" in the term central dispersion (§ 12.2). We sum up as follows:

*The dispersions of the first kind of the differential equation* (q) *form a* 3*-parameter continuous group* $\mathfrak{D}$. *The direct* (*increasing*) *dispersions form in* $\mathfrak{D}$ *an invariant subgroup* $\mathfrak{P}$ *with index* 2; *the indirect* (*decreasing*) *dispersions form the coset of* $\mathfrak{P}$. *The infinite cyclic group* $\mathfrak{C}$ *formed from the central dispersions of the first kind of the differential equation* (q) *is the centre of* $\mathfrak{P}$. *The central dispersions of the first kind of the differential* *equation* (q) *with even indices form an invariant subgroup* $\mathfrak{S}$ *in* $\mathfrak{D}$, *and the factor group* $\mathfrak{D}/\mathfrak{S}$ *is isomorphic to the group formed from all* $2 \times 2$ *unimodular matrices.*

## 21.7 The group of dispersions of the second kind of the differential equation (q)

We now assume that the differential equation (q) admits of the first associated differential equation $(\hat{q}_1)$ (§ 1.9). Then the differential equation (q) possesses dispersions of the second kind, and these coincide with those of the first kind of $(\hat{q}_1)$. Consequently, the dispersions of the second kind of the differential equation (q) form a continuous 3-parameter group $\mathfrak{D}_1$, whose structure is naturally analogous to that of the group $\mathfrak{D}$, with, of course, central dispersions of the second kind replacing those of the first kind.

## 21.8 The semigroupoid of general dispersions of the differential equations (q), (Q)

Let (q), (Q) be arbitrary oscillatory differential equations in the interval $j = (-\infty, \infty)$.
We consider the non-linear third order differential equations

$$(\mathrm{qq}),\quad (\mathrm{qQ}),\quad (\mathrm{Qq}),\quad (\mathrm{QQ})$$

and denote by

$$G_{11},\quad G_{12},\quad G_{21},\quad G_{22}$$

respectively the sets consisting of all regular integrals of these differential equations in the interval $(-\infty, \infty)$. Thus, for instance, $G_{11}$ is the set of all dispersions of the first kind of the differential equation (q), $G_{12}$ those of the general dispersions of the differential equations (Q), (q), etc.

We know that the function $X(t) = t$ lies in the sets $G_{11}$ and $G_{22}$. We also know that the function $x_{ik}^{-1}$ inverse to any general dispersion $x_{ik} \in G_{ik}$ is a member of the set $G_{ki}$: $x_{ik}^{-1} \in G_{ki}$ $(i, k = 1, 2)$.

Let $x_{ik} \in G_{ik}$, $y_{km} \in G_{km}$ $(i, k, m = 1, 2)$ be arbitrary general dispersions of the corresponding differential equations (q) and (Q). It is easily seen that the function $x_{ik}y_{km}$ formed by composition of these general dispersions (the order is significant) gives a general dispersion contained in the set $G_{im}$; $x_{ik}y_{km} \in G_{im}$; conversely every element $z_{im} \in G_{im}$ is the composition of appropriate general dispersions $x_{ik} \in G_{ik}$, $y_{km} \in G_{km}$: i.e. $x_{ik}y_{km} = z_{im}$. These facts can conveniently be expressed by means of the formula

$$G_{ik}G_{km} = G_{im} \quad (i, k, m = 1, 2)$$

or by the following multiplication table

| | $G_{11}$ | $G_{12}$ | $G_{21}$ | $G_{22}$ |
|---|---|---|---|---|
| $G_{11}$ | $G_{11}$ | $G_{12}$ | — | — |
| $G_{12}$ | — | — | $G_{11}$ | $G_{12}$ |
| $G_{21}$ | $G_{21}$ | $G_{22}$ | — | — |
| $G_{22}$ | — | — | $G_{21}$ | $G_{22}$ |

We now consider the semigroupoid $\Gamma$ constructed from the union set $G_{11} \cup G_{12} \cup G_{21} \cup G_{22}$ with the operation of multiplication defined as composition of functions.

From the above we see that:

The sets $G_{11}$, $G_{12}$ are groups with the common unit element $(\underline{1} =)\ X(t) = t$. The sets $G_{12}$, $G_{21}$ are formed from pairwise inverse elements. Since $G_{11}G_{12} = G_{12}$, $G_{12}G_{22} = G_{12}$, the group $G_{11}$ is a left operator region and the group $G_{22}$ is a right operator region of the set $G_{12}$. Since $G_{22}G_{21} = G_{21}$, $G_{21}G_{11} = G_{21}$ the group $G_{22}$ is a left and the group $G_{11}$ is a right operator region of the set $G_{21}$.

In this way we arrive at the following structure of the semigroupoid $\Gamma$ of the general dispersions of the differential equations (q), (Q):

The semigroupoid $\Gamma$ consists of two groups $G_{11}$, $G_{22}$ with the common unit element $\underline{1}$ and of two further, equivalent, sets $G_{12}$, $G_{21}$. These sets have the two products $G_{12}G_{21}$ and $G_{21}G_{12}$, which coincide with the groups $G_{11}$ and $G_{22}$, and consist of pairwise inverse elements whose product is always the unit element of the groups $G_{11}$, $G_{22}$. The group $G_{11}$ is a left and the group $G_{22}$ is a right operator region of the set $G_{12}$; similarly the group $G_{11}$ is a right and the group $G_{22}$ is a left operator region of the set $G_{21}$.

We conclude this discussion with one remark: the groups $G_{11}$, $G_{22}$ always have in common the group $\{\underline{1}\}$ consisting only of the unit element; in special cases, however their intersection can be larger. This occurs, for instance, when the differential equations (q), (Q) have the same fundamental dispersion $\phi$ of the first kind, for in this case their intersection $G_{11} \cap G_{22}$ contains the cyclic group consisting of all the central dispersions $\phi_\nu$ of the first kind of the differential equations (q), (Q).