Topological spaces

Exercises

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EXERCISES

The exercises are grouped, as a rule, by sections; however, there are sections for which no exercises are given, and in some cases one group is formed of exercises pertaining to two or more consequent closely related sections.

In each group, the arrangement of the exercises is not by the degree of difficulty but mainly according to the sequence (in the main text) of the concepts involved; however, this is by no means a strict rule.

The formulation of the exercises is often concise, and abbreviated expressions are sometimes used which would not be admissible in the main text.

(Section 1)

1. If ϱ is a fibering relation, then $\varrho[X] = \mathbf{E}\varrho$ implies $X \supset \mathbf{D}\varrho$. If $\varrho[X] = \mathbf{E}\varrho$ implies $X \supset \mathbf{D}\varrho$, then there exists a fibering relation $\sigma \subset \varrho$ such that $\mathbf{D}\sigma = \mathbf{D}\varrho$.

2. If A is a class and ϱ is a relation, then there exists a class $B \subset A$ such that $\varrho[(x)] = B$ for no $x \in A$.

3. If σ is a relation, $\emptyset \neq \sigma \neq \mathsf{J}$, then there exists a ϱ with $\sigma \circ \varrho \neq \varrho \circ \sigma$.

4. If ϱ is transitive, then ϱ^{-1} is transitive. If ϱ and σ are transitive, then $\varrho \circ \sigma$ need not be transitive (give an example).

5. A relation ϱ is an equivalence if and only if it is reflexive and $\varrho \circ \varrho^{-1} \subset \varrho$.

6. If ϱ is a reflexive relation and any two fibres $\varrho[(x)]$, $\varrho[(y)]$ either coincide or have no elements in common, then ϱ is an equivalence.

7. A relation ϱ is single-valued if and only if $\varrho \circ \varrho^{-1} \subset \mathsf{J}$. If ϱ is single-valued, then $\varrho^{-1} \circ \varrho$ is an equivalence but not conversely.

8. A relation ϱ is one-to-one if and only if $\varrho \circ \varrho^{-1} \subset \mathsf{J}$, $\varrho^{-1} \circ \varrho \subset \mathsf{J}$.

9. The class of all sets is non-comprisable.

10. Let φ be a single-valued relation such that $\mathbf{D}\varphi$ consists of all sets and every φX is a set. Then $\mathbf{E}\{\varphi X \mid \varphi X \notin X\}$ is non-comprisable.

(Section 3)

1. If $\mathscr{X}$ is a monotone class of bijective relations, then $\bigcup\mathscr{X}$ is bijective.

2. Let ϱ be a relation and let $\mathscr{M}$ be a class of sets; consider the class $\mathscr{M}^* = \varrho[[\mathscr{M}]]$ of all $\varrho[M]$, $M \in \mathscr{M}$. If A is the largest set in $\mathscr{M}$, then $\varrho[A]$ is the largest set in $\mathscr{M}^*$ (but not conversely, even if $\mathbf{D}\varrho \supset \bigcup\mathscr{M}$). If B is a maximal set in $\mathscr{M}$, $\mathbf{D}\varrho \supset \bigcup\mathscr{M}$, then $\varrho[B]$ need not be maximal in $\mathscr{M}^*$. If ϱ is a fibering relation, then $\varrho[B]$ is maximal in $\mathscr{M}^*$ whenever $B \subset \mathbf{D}\varrho$ and B is maximal in $\mathscr{M}$.

3. If ϱ is a relation and $\mathscr{M}$ is a monotone class of sets, then $\varrho[[\mathscr{M}]]$ is monotone.

4. Let $\{X_a \mid a \in A\}$ be an indexed class of sets. Let $\mathscr{B}$ be a monotone class of subsets of A. Then $\mathbf{E}\{\bigcup\{X_a \mid a \in B\} \mid B \in \mathscr{B}\}$ is monotone.

5. Let A be a class and let φ be a single-valued relation on $A \times A$ into A. There exists exactly one single-valued relation Φ on A into A^{N} such that the following holds: if $a \in A$ and $\Phi a = \{a_n \mid n \in \mathsf{N}\}$, then $a_0 = a$, $a_{n+1} = \varphi\langle a_n, a\rangle$ for every $n \in \mathsf{N}$.

6. If A is a class and Z is the class of all multiplets of elements from A (see 3 F.9), then $Z \times Z \subset Z$, $Z - (Z \times Z) \subset A$.

7. An n-multiplet ($n \in \mathsf{N}$, $n \geqq 1$) of elements of a class A is defined as follows. Consider the class M of all finite relations ϱ with $\mathbf{D}\varrho \subset \mathsf{N} - (0)$ such that $x \in \varrho[(1)]$ implies $x \in A$, and if $x \in \varrho[(n)]$, $n > 1$, then there exist p, q, u and v such that $u \in \varrho[(p)]$, $v \in \varrho[(q)]$, $p + q = n$ and $x = \langle u, v\rangle$. If $n \in \mathsf{N}$, $n \geqq 1$, then an element x is called an n-multiplet of elements from A if $x \in \varrho[(n)]$ for some $\varrho \in M$.

Prove that an element x is a multiplet of elements of A (cf. 3 F.9) if and only if it is an n-multiplet (as just introduced) for some n. [Hint: It follows by induction that each n-multiplet is a multiplet. On the other hand, if $\varrho \in M$, $\sigma \in M$, $\langle m, \alpha\rangle \in \varrho$, $\langle n, \beta\rangle \in \sigma$, then $\varrho \cup \sigma \cup (\langle m + n, \langle\alpha, \beta\rangle\rangle) \in M$; hence $\langle\alpha, \beta\rangle$ is an $(m + n)$ multiplet if α is an m-multiplet and β is an n-multiplet. Thus $T \times T \subset T$ where T is the class of all k-multiplets, $k \in \mathsf{N}$, $n \geqq 1$.]

(Section 4)

1. Every monotone class is additive and multiplicative. The class of all N_k, $k \in \mathsf{N}$, is monotone and completely multiplicative but is not monotonically additive.

2. The class of all single-valued relations is completely multiplicative and monotonically additive but is not additive.

3. Let $\mathscr{A}$ be a non-void completely additive and completely multiplicative class of sets such that $X \in \mathscr{A}$, $Y \in \mathscr{A} \Rightarrow X - Y \in \mathscr{A}$. Then there exists a class B and a fibering relation ϱ such that $\mathscr{A}$ consists of all sets of the form $\varrho[X]$, $X \subset B$.

4. Let $\mathscr{A}$ be a class of sets. Then there exists a (uniquely determined) smallest additive class of sets $\mathscr{B}$ containing $\mathscr{A}$; if $\mathscr{A}$ is comprisable, then $\mathscr{B}$ is comprisable. The assertion remains valid if "additive" is replaced by "multiplicative" or "additive and multiplicative" or " completely additive" and so on.

5. Prove, on the base of axioms from Section 1, that

(a) Theorem 4 B.2 and the Axiom of Choice are equivalent.

(b) The proposition obtained from Theorem 4 B.2 by inserting "comprisable relation" instead of "relation" is equivalent with Theorem 4 C.1 as well as with the assertion that every non-empty collection of sets which is of finite character (see 4 C.5) contains a maximal set. [Hints: If 4 C.1 holds and ϱ is comprisable, consider the collection of all single-valued $\varphi \subset \varrho$; if φ is maximal, then $\mathbf{D}\varphi = \mathbf{D}\varrho$. If the assertion on classes of finite character holds, let $\mathscr{M}$ satisfy the suppositions of 4 C.1. Consider the collection $\mathbf{B}$ of all monotone $\mathscr{X} \subset \mathscr{M}$ such that $X \in \mathscr{X} \Rightarrow X \supset A$. Then $\mathbf{B}$ is of finite character. Let $\mathscr{A} \in \mathbf{B}$ be maximal. Then $\bigcup\mathscr{A}$ is maximal in $\mathscr{M}$.]

6. Let X be a minimally non-comprisable class. If φ is a single-valued relation, then $\varphi[X]$ is either a set or a minimally non-comprisable class; in particular, every subclass of X is either a set or a minimally non-comprisable class.

7. If A is a minimally non-comprisable class, then $\exp A$ is also a minimally non-comprisable class. [Hint: if $A = \bigcup\mathscr{X}$, $\mathscr{X}$ monotone, then $\exp A = \bigcup\{\exp X \mid X \in \mathscr{X}\}$.]

8. Any two minimally non-comprisable classes are equipollent. [See 9 A.5.]

9. If X, Y are minimally non-comprisable classes, then both $X \cup Y$ and $X \times Y$ are minimally non-comprisable.

10. A relation ϱ is minimally non-comprisable if and only if either both $\mathbf{D}\varrho$ and $\mathbf{E}\varrho$ are minimally non-comprisable or one of them is minimally non-comprisable and the other is a set.

11. We shall say that a class of sets $\mathscr{A}$ is closed with respect to accessibility if (1) if $X \in \mathscr{A}$ and Y is a set equipollent with a subset of X, then $Y \in \mathscr{A}$, (2) if $\mathscr{X} \in \mathscr{A}$, $\mathscr{X} \subset \mathscr{A}$, then $\bigcup\mathscr{X} \in \mathscr{A}$, (3) if $X \in \mathscr{A}$, then $\exp X \in \mathscr{A}$. Prove that if a class of sets $\mathscr{A} \neq \emptyset$ is closed with respect to accessibility and is monotonically additive, then $\mathscr{A}$ consists of all sets. [Hint: using 4 A.7 with a suitable φ prove that there exists a non-comprisable monotone $\mathscr{B} \subset \mathscr{A}$; apply 4 D.5.] Show that a monotonically additive non-void class of sets $\mathscr{A}$ satisfying (1) either contains all sets or is of the form $\mathbf{E}\ \{X \mid \operatorname{card} X \leqq n\}$ with $n \in \mathrm{N}$.

12. Let $\mathscr{A}$ be a class of sets. A set X will be called accessible from $\mathscr{A}$ if it is contained in every class of sets $\mathscr{Y} \supset \mathscr{A}$ which is closed with respect to accessibility. If $\mathscr{A}$ is a singleton (A) we shall also say that X is accessible from the set A; a set accessible from a countable set will be called simply "accessible".

Prove that the class of all sets accessible from a given class is closed with respect to accessibility.

13. Let A be the class of all accessible sets. Let A^* be the class of all multiplets of elements of A (see 3 F.9). Clearly, (1) if $x \in A^*$, $y \in A^*$, then $\langle x, y\rangle \in A^*$ and, conversely, if $\langle x, y\rangle \in A^*$, then $x \in A^*$, $y \in A^*$. Prove that (2) every singleton belongs to A^*; if ϱ is is a relation and every $\varrho[(x)]$ belongs to A^*, then $\varrho[X] \in A^*$ whenever X is a class and $X \in A^*$; if $X \in A^*$ and X is a class, then the class of all $Y \subset X$ such that $Y \in A^*$ is equal to $\exp X$ and belongs to A^*. [Hint: prove that a set belongs to A^* if and only if it belongs to A.]

Remark. The above assertions show that, roughly speaking, the property "$x \in A^*$" may serve as an "interpretation" of the property "x is an element" giving rise to an "internal model" (in the sense of mathematical logic) of the axiomatic system presented in this book.

(Section 6)

1. Let σ be a composition on a class X. If $a \in X$, then the least stable class $Y \subset X$ containing a is countable. If σ is associative and Y is infinite, then $\langle Y, \sigma\rangle$ is isomorphic with $\langle \mathsf{N} - (0), +\rangle$. If $\langle X, \sigma\rangle$ is a group and Y is finite, then $\langle Y, \sigma\rangle$ is a group.

2. Let X, σ, a, Y be as above. Give an example where Y is infinite and $\langle Y, \sigma\rangle$ is not isomorphic with $\langle \mathsf{N} - (0), +\rangle$, and an example where σ is associative and Y is finite without being a group.

3. Let $\mathscr{S}$ consist of all finite sequences; let σ denote the composition on $\mathscr{S}$ described in 6 B.2. For any non-empty $X \subset \mathscr{S}$ let $H(X)$ be the smallest stable class containing X. Every sub-semi-group $G \neq \emptyset$ of $\langle \mathscr{S}, \sigma\rangle$ with $x \in G, x \sigma y \in G \Rightarrow y \in G$ is isomorphic with some $H(X)$ where X consits of one-element sequences (and the void sequence).

4. With the above notation, no $\alpha \in \mathscr{S}$ (except $\emptyset$) is invertible; however, every $\alpha \in \mathscr{S}$ is virtually invertible.

5. If A is a class, let $\mathscr{S}(A)$ denote the class of all finite sequences of elements from A; let $\mathscr{S}(A)$ be endowed with the composition described in 6 B.2. If G is a semi-group and $X \subset G$ generates G, then there exists a homomorphism-relation φ on $\mathscr{S}(B)$ onto G where B is a class equipollent with X.

6. Let $\mathscr{G} = \langle G, \mu\rangle$ be a semi-group; we shall write xy instead of $x\mu y$. Let $\mathscr{R} = \langle R, +, .\rangle$ be a ring. Let $\Phi = \mathscr{R}(\mathscr{G})$ denote the set of all $\varphi \in R^G$ such that $\varphi g = 0$ for all $g \in G$ with finitely many exceptions. Consider the following compositions on $\mathscr{R}(\mathscr{G})$: if $\varphi \in \Phi$, $\psi \in \Phi$, then $\varphi + \psi = \{\varphi g + \psi g \mid g \in G\}$ and $\varphi . \psi = \{\chi g \mid g \in G\}$ where χg is equal to the sum of all $\varphi h . \psi k$ with $h \in G$, $k \in G, hk = g$, $\varphi h \neq 0 \neq \psi k$. Prove that Φ endowed with these compositions is a ring.

Remark: This ring, denoted e.g. by $\mathscr{R}(\mathscr{G})$, is called the $\mathscr{R}$-ring over $\mathscr{G}$. An element $\varphi \in \mathscr{R}(\mathscr{G})$ which assigns α_i to g_i, $i = 1, \ldots, n$, and 0 to each $g \in G$ distinct from all g_i, is often denoted by $\alpha_1 g_1 + \ldots + \alpha_n g_n$ and called a "formal combination" of elements $g_1, \ldots, g_n$ of G with coefficients $\alpha_1, \ldots, \alpha_n$ from R.

7. If $\langle G, \sigma\rangle$, $\langle H, \mu\rangle$ are semi-groups and, for any $x \in G$, $y \in G$, there exists a (σ, μ)-homomorphism-relation φ such that $\varphi x \neq \varphi y$, then, for some set B, there exists a one-to-one (σ, μ^B)-homomorphism-relation on G into H^B. An analogous proposition is valid for groups, semi-rings, rings and modules.

(Section 8)

1. Let $\mathbf{S}$ denote the class of all comprisable algebraic structs of a given type $\mathfrak{t}$. If $\{\mathscr{X}_a\}$ is a family of structs from $\mathbf{S}$, put $\mathscr{X} = \Pi \mathscr{X}_a$ (see 8 B.8) and denote by π_a the projection of $\mathscr{X}$ onto $\mathscr{X}_a$. If $\mathscr{Y} \in \mathbf{S}$ and $\varphi_a \in \text{Hom}(\mathscr{Y}, \mathscr{X}_a)$, then there is exactly one $\psi \in \text{Hom}(\mathscr{Y}, \mathscr{X})$ with $\varphi_a = \pi_a \circ \psi$. This condition characterizes $\Pi\mathscr{X}_a$ up to natural isomorphism. – Give an exact formulation and prove.

2. With $\mathbf{S}$ as above, let $\mathbf{S}_0 \subset \mathbf{S}$. If $\mathscr{X}_a \in \mathbf{S}$ (usually we have $\mathscr{X}_a \in \mathbf{S}_0$), then $\mathscr{X} \in \mathbf{S}_0$ is called a "free $\mathbf{S}_0$-product" (or a "free product in $\mathbf{S}_0$") of $\{\mathscr{X}_a\}$ if there are homomorphisms $\lambda_a \in \text{Hom}(\mathscr{X}_a, \mathscr{X})$ such that, for any $\mathscr{Y} \in \mathbf{S}_0$ and $\varphi_a \in \text{Hom}(\mathscr{X}_a, \mathscr{Y})$, there exists exactly one $\psi \in \text{Hom}(\mathscr{X}, \mathscr{Y})$ with $\varphi_a = \psi \circ \lambda_a$. Prove that any two "free $\mathbf{S}_0$-products" of $\{\mathscr{X}_a\}$ are isomorphic. (Remark: the term "free sum" seems more appropriate.)

3. Every family of groups (semi-groups, abelian groups, commutative semi-groups) has a free product (in the corresponding class). Every family of (commutative) semi-groups has a "free (commutative) product with unit", i.e. a free $\mathbf{S}_0$-product, $\mathbf{S}_0$ being the class of all semi-groups (or commutative semi-groups, as the case may be) containing a neutral element. [Hint (for semi-groups): given $\langle G_a, \sigma_a\rangle$, $\{G_a\}$ disjoint, consider the semi-group H of all non-void finite sequences of elements of $\bigcup G_a$; let λ_a consist of all pairs $\langle g, h\rangle$ where $g = \{x, y\}$, $x \in G_a$, $y \in G_a$, for some a, and h is the one-element sequence $\{x\sigma_a y\}$; consider the smallest congruence on H containing (as subsets) all λ_a.]

4. No non-trivial family of fields has a free product (in the class of fields).

5. With $\mathbf{S}$ and $\mathbf{S}_0$ as in exercise 2, a struct $\mathscr{X} \in \mathbf{S}_0$ is called $\mathbf{S}_0$-free if, for any $\mathscr{Y} \in \mathbf{S}_0$ and any surjective $\varphi \in \text{Hom}(\mathscr{Y}, \mathscr{X})$, there exists a $\psi \in \text{Hom}(\mathscr{X}, \mathscr{Y})$ such that $\varphi \circ \psi = \mathsf{J} : \mathscr{X} \to \mathscr{X}$. – A semi-group is free if and only if it is a free product of a family of semi-groups isomorphic to $\langle \mathsf{N} - (0), +\rangle$.

6. Let X_a, $a \in A$, be disjoint semi-groups isomorphic to $\langle \mathsf{N} - (0), +\rangle$; let X_a consist of elements $x_a, x_a^2, \ldots$ Let R be a commutative ring. Denote by $R(\{x_a\})$ the R-ring over the commutative free product with unit of the $\{X_a\}$; every $p \in R(\{x_a\})$ is called a "polynomial in x_a, $a \in A$, with coefficients in R". – If $\xi_a \in R$ for each $a \in A$, then there is exactly one homomorphism Θ of $R(\{x_a\})$ into R such that $\Theta(r \,.\, x_a) = r\xi_a$ for each $a \in A$, $r \in R$. If p is a polynomial as described above, then Θp is called the value of p for $x_a = \xi_a$, $a \in A$.

7. Prove that, for a given $\{\xi_a\}$ and an ideal $T \subset R$, the set of those polynomials p the value of which for $x_a = \xi_a$ lies in T, is an ideal of $R(\{x_a\})$.

8. Let $\mathscr{R} = \langle R, \sigma, \mu, \alpha, \beta \rangle$ be a module over $\mathscr{A} = \langle \mathbf{D}\mu, \alpha, \beta \rangle$; let $\mathscr{G} = \langle G, \tau \rangle$ be a semi-group. For $g \in G$, $h \in G$, $x \in R^G$, put $x_g h = x(g\tau h)$; let ϱ consist of all $\langle g, x, x_g \rangle$, $g \in G$, $x \in R^G$. Then $\langle R^G, \sigma^G, \langle \mu, \alpha, \beta \rangle^G, \varrho, \tau \rangle$ is a module-like struct of the type $\langle \langle 1, 2 \rangle, \mathscr{A}, \mathscr{G} \rangle$.

9. Let A be a set, and let $\mathscr{F}$ be a proper ideal (8 D.4, 8 D.9, convention) under $\bigcap_{\exp A}$ (in other words, a proper filter of sets on A, see Section 12). For every $a \in A$, let $\mathscr{X}_a$ be an algebraic struct of a given type $\mathfrak{t}$ and let λ_a be a congruence on $\mathscr{X}_a$. For elements $x = \{x_a\}$, $y = \{y_a\}$ of $\mathscr{X} = \Pi \mathscr{X}_a$, put $x \lambda y$ if and only if there is an $S \in \mathscr{F}$ with $a \in S \Rightarrow x_a \lambda_a y_a$. Then λ is a congruence on $\mathscr{X}$.

10. Maximal ideals in $\mathscr{X}^A$ where $\mathscr{X}$ is a field are precisely the sets T of the following form: $\mathscr{F}$ is an ultrafilter (see 12 C.1) on A; T consists of all $x = \{x_a\} \in \mathscr{X}^A$ such that $\mathbf{E}\{a \mid x_a = 0\}$ belongs to $\mathscr{F}$.

(Section 9)

1. For any infinite cardinal b, there exist arbitrarily large cardinals x with $x^b > x$ and arbitrarily large cardinals y with $y^b = y$.

2. For any infinite cardinal x denote by $\log x$ the least y such that $x \leqq \exp y$. There exist arbitrarily large cardinals x with $\log x = x$. If $\log x = x$, then there is no greatest element in the set of all $z < x$.

3. For any infinite cardinals x, y, we have $\log(xy) = \log x + \log y$, $\log x^y = y \log x$.

4. Let X be an infinite set, card $X = x$. Let $b(x)$ be the least cardinality of a set $B \subset \mathsf{N}^X$ such that, for any $f \in \mathsf{N}^X$, there is a $g \in B$ with $fz \leqq gz$ for all $z \in X$. Then $x < b(x) \leqq \exp x$ (it is not known whether e.g. $b(\aleph_0) = \exp \aleph_0$).

(Section 10)

1. Every comprisable order is an intersection of monotone orders.

2. The product of two quasi-ordered classes $\langle A, \sigma \rangle \times \langle B, \tau \rangle$ cannot be monotonically quasi-ordered unless $\sigma = A \times A$ or $\tau = B \times B$.

3. Let $\langle A, \leqq \rangle$ be an ordered set. For every $a \in A$, let $\mathscr{X}_a = \langle X_a, \sigma_a \rangle$ be a quasi-ordered set. If $\langle a, x \rangle \in \Sigma X_a$, $\langle b, y \rangle \in \Sigma X_a$, put $\langle a, x \rangle \, \sigma \langle b, y \rangle$ if and only if either $a < b$ or $a = b$, $x \sigma_a y$. Then $\langle \Sigma X_a, \sigma \rangle$ will be denoted by $\Sigma \langle \{\mathscr{X}_a \mid a \in A\}, \leqq \rangle$ or simply $\Sigma \mathscr{X}_a$ and will be called the sum of $\{\mathscr{X}_a\}$ under the order $\leqq$. If $\leqq$ is equal to J_A, then we shall speak of the discrete sum (or simply sum) of $\{\mathscr{X}_a\}$.

Prove: if $\mathscr{X}_a = \mathscr{X}$ for all a, then $\Sigma\mathscr{X}_a$ is isomorphic with $\langle A, \leqq\rangle \times_{\text{lex}} \mathscr{X}$, and the discrete sum $\Sigma\mathscr{X}_a$ is isomorphic with $\langle A, \mathsf{J}\rangle \times \mathscr{X}$; $\Sigma\mathscr{X}_a$, with all $\mathscr{X}_a$ non-void, is monotone if and only if $\langle A, \leqq\rangle$ and also all $\mathscr{X}_a$ are monotone.

4. An ordered class $\mathscr{A} = \langle A, \sigma\rangle$ is monotone if and only if, for any ordered $\mathscr{B} = \langle B, \tau\rangle$ and any surjective order-preserving $f : \mathscr{A} \to \mathscr{B}$ there exists an order-preserving $g : \mathscr{B} \to \mathscr{A}$ with $f \circ g = \mathsf{J} : \mathscr{B} \to \mathscr{B}$.

5. Let f be surjective for a quasi-ordered set $\mathscr{A} = \langle A, \varrho\rangle$ and a set B. Let σ be the intersection of all quasi-orders τ on B such that $f : \mathscr{A} \to \langle B, \tau\rangle$ is order-preserving. Then σ is a quasi-order and $f : \mathscr{A} \to \langle B, \sigma\rangle$ is order-preserving. We denote σ by ϱ/f and call it the quotient of ϱ under f; $\langle B, \varrho/f\rangle$ will be denoted by $\langle A, \varrho\rangle/f$ and will be called the quotient of $\langle A, \varrho\rangle$ under f. A mapping of the form $f : \langle A, \varrho\rangle \to \to \langle A, \varrho\rangle/f$ will be called a quotient mapping (for quasi-ordered sets).

6. Every quotient of a monotonically quasi-ordered set is monotonically quasi-ordered. A quotient of an ordered set need not be ordered.

7. Every ordered set is a quotient of some $D \times (0, 1)$ with D discrete, i.e. endowed with J_D.

8. If $x \in \mathsf{N}^{\mathsf{N}}$, $y \in \mathsf{N}^{\mathsf{N}}$, put $x\sigma y$ if and only if there exists a number $p \in \mathsf{N}$ such that $n \in \mathsf{N}$, $n > p \Rightarrow xn \leqq yn$. Then σ is a reflexive quasi-order on N^{N}. In $\langle \mathsf{N}^{\mathsf{N}}, \sigma\rangle$, every countable set is bounded.

9. The collection of all left-saturated left-cofinal subsets of a quasi-ordered class is multiplicative.

10. Let $\leqq$ be a monotone order on a set A; for every $a \in A$, let $\mathscr{X}_a = \langle X_a, \sigma_a\rangle$ be an ordered set. If $x \in \Pi X_a$, $y \in \Pi X_a$, let $x\sigma y$ if and only if either $x = y$ or there exists an element $a \in A$ such that (1) $b \in A$, $b \leqq a \Rightarrow (xb)\, \sigma_b (yb)$, (2) for some $c \in A$, $c \leqq a$, $xc \neq yc$. Then σ is an order on ΠX_a; it will be denoted by $\Pi_{\text{lex}}\{\sigma_a\}$ or $\Pi_{\text{lex}}\langle\{\sigma_a \mid a \in A\}, \leqq\rangle$. The ordered set $\langle \Pi X_a, \sigma\rangle$ will be denoted by $\Pi_{\text{lex}}\mathscr{X}_a$ or $\Pi_{\text{lex}}\langle\{\mathscr{X}_a \mid a \in A\}, \leqq\rangle$ and called the lexicographical product of $\{\mathscr{X}_a\}$. It is monotone whenever A is well-ordered (11 A.1) and all $\mathscr{X}_a$ are monotone; conversely, if $\langle \Pi X_a, \sigma\rangle$ is monotone, all X_a are non-void and contain, with finitely many exceptions, more than one element, then A is well-ordered and all $\mathscr{X}_a$ are monotone.

11. Let $\langle A, \leqq\rangle$ be a monotone ordered set; let $\mathscr{X}_a$, $a \in A$, be complete ordered sets containing more than one point. Then $\Pi_{\text{lex}}\{\mathscr{X}_a \mid a \in A\}$ is complete if and only if A is well-ordered.

12. Let A be an infinite set. If $\mathscr{X} \subset \exp A$, $\mathscr{Y} \subset \exp A$, put $\mathscr{X}\sigma\mathscr{Y}$ if and only if $\mathscr{Y}$ majorizes $\mathscr{X}$ in $\langle \exp A, \subset\rangle$. Then σ is a quasi-order on $\exp\exp A$. Put $U = = \mathsf{E}\{\mathscr{X} \mid \mathscr{X} \subset \exp A, \bigcup\mathscr{X} = A\}$. The cardinality of the collection of all left filters in U is $\exp\exp\exp a$ where $a = \operatorname{card} A$. [Hint: if Θ is a set of free ultrafilters in A, let V_Θ consist of all those $\mathscr{X} \subset \exp A$ which intersect every $\mathscr{F} \in \Theta$. Then V_Θ is a filter; if $\Theta \neq \Theta'$, then $V_\Theta \neq V_{\Theta'}$. Apply 12 C.7.]

13. The following collections of sets are lattices (under inclusion): the collection of all subgroups of a given group; of all subrings of a given ring; of all finite subsets of a given set; of all congruences (see 8 C.10) on a given algebraic struct; of all equivalences on a given set. Each of these collections, with one exception, is also complete.

14. The ordered set $\langle \mathsf{R}, \leqq \rangle^{\mathsf{R}}$ is boundedly complete. Let Y consist of those $f \in \mathsf{R}^{\mathsf{R}} = X$ which satisfy the following condition:

(*) $f(\lambda_1 x_1 + \lambda_2 x_2) \geqq \lambda_1(fx_1) + \lambda_2(fx_2)$ whenever $\lambda_1 \geqq 0$, $\lambda_2 \geqq 0$, $\lambda_1 + \lambda_2 = 1$. Let $f_1 x = -|x|$, $f_2 x = -|x - 1|$. Then $\sup_X(f_1, f_2) \neq \sup_Y(f_1, f_2)$.

15. Let S be a class; put $A = \exp\exp S$, $B = \mathsf{E}\{\exp X \mid X \subset S\}$. Then $\langle A, \subset \rangle$ is complete, B is meet-complete, completely meet-stable, completely meet-preserving, and monotonically join-complete, but it is neither finitely join-complete nor join-stable.

16. Let $\mathscr{A} = \langle A, \sigma \rangle$ be a quasi-ordered class; suppose that no finite non-void $X \subset A$ has more than one join. If a class $B \subset A$ is finitely join-complete and join-preserving, then it is join-stable.

17. Let a quasi-ordered class $\langle A, \sigma \rangle$ be finitely join-complete. If $B \subset A$ is join-stable, then it is join-preserving.

18. Let $A = \mathsf{N} \cup (\alpha_1) \cup (\alpha_2)$, α_i non $\in \mathsf{N}$, $\alpha_1 \neq \alpha_2$. For $x \in A$, $y \in A$, put $x\sigma y$ if either $x \in \mathsf{N}$, $y \in \mathsf{N}$, $x \leqq y$, or $x = y$ or else $y = \alpha_i$, $x \in \mathsf{N}$. Put $B = (0) \cup (\alpha_1) \cup (\alpha_2)$, $B_p = (A - \mathsf{N}_p) \cup (0)$. Then the B_p are meet-preserving; $B = \bigcap B_p$ is meet-stable, but is not meet-preserving.

19. Let $\mathscr{A} = \langle A, \sigma \rangle$ be a quasi-ordered class. The union of a monotone collection of join-preserving subsets is join-preserving, and similarly for meet- and lattice-preserving sets.

20. A quasi-ordered class $\langle A, \sigma \rangle$ is called (countably) join-complete if every non-void (countable) set $X \subset A$ has a join, monotonically join-complete if every non-void $X \subset A$ which is monotonically ordered (under σ) has a join.

A second definition is obtained if "join" is replaced by "meet". Finally, $\langle A, \sigma \rangle$ is called countably (monotonically) complete if it is both countably (monotonically) join-complete and countably (monotonically) meet-complete.

Let A be an uncountable set. Let B_1, B_2, B_3 consist, respectively, of all finite $X \subset A$; all countable $X \subset A$ and all $X \subset A$ of cardinality $> \exp \aleph_0$; all countable $X \subset A$ and all their complements. Then B_1 is meet-complete and finitely join-complete, but neither countably nor monotonically join-complete, B_2 is countably join-complete, but there is a countable monotone $\mathscr{X} \subset B_2$ with no meet (provided card $A > \exp \aleph_0$); B_3 is countably complete, but neither monotonically join-complete nor monotonically meet-complete.

21. Let A be a class. In $\langle \exp A, \subset \rangle$, a class $\mathscr{B} \subset \exp A$ is completely join- (meet-) stable if and only if it is completely additive (completely multiplicative).

22. A monotonically complete and finitely complete (**10 F**.5) quasi-ordered class is complete.

23. Let $\mathscr{A} = \langle A, \varrho \rangle$ be a quasi-ordered class; let $B \subset A$ be finitely stable. If, for any monotone non-void $X \subset B$, every join (in $\mathscr{A}$) of X belongs to B, then B is completely join-stable.

(Section 11)

1. Every ordered set $\langle A, \leqq \rangle$ such that

$$(*) \qquad \text{all } \mathbf{E}\{x \mid x \leqq a\} \text{ are finite}$$

can be embedded into $\langle B, \supset \rangle$ where B is the collection of all finite subsets of a given set. Every infinite monotonically ordered set with property (*) is isomorphic to $\langle \mathsf{N}, \leqq \rangle$.

2. Let A and $\mathscr{X}_a$, $a \in A$, be ordered sets; let $\mathscr{X}_a$ contain more than one point. The lexicographical product $\prod_{\text{lex}\, a \in A} \mathscr{X}_a$ (see 10 ex. 10) is well-ordered if and only if A and all $\mathscr{X}_a$ are well-ordered.

3. Give an example of a monotonically ordered set which possesses the property indicated in **11** A.5, but is not well-ordered.

4. Let A be a well-ordered class such that every A_x is a set; let a be the first element of A. Let M be a class; let $b \in M$. Let $S \subset \bigcup_{x \in A} M^{A_x}$ be such that (1) $(\langle a, b \rangle) \in S$, (2) if $x \in A$, $x \neq a$, $A_x = \bigcup \{A_y \mid y < x\}$, $f \in M^{A_x}$ and $f \mid A_y \in S$ for every $y < x$, then $f \in S$. Let φ be a single-valued relation such that $\mathbf{D}\varphi \supset S$, $\mathbf{E}\varphi \subset M$ and if $f \in S$, $x \in A$, $x \neq a$, $f \in M^{A_x}$, then $f \cup (\langle x, \varphi f \rangle) \in S$.

Then there exists exactly one $g \in M^A$ such that (1) $ga = b$, (2) if $x \in A$, $x \neq a$, then the restriction $g(x)$ of g to A_x belongs to S, and $gx = \varphi g(x)$.

5. If $\langle A, \leqq \rangle$ as well as $\mathscr{X}_a = \langle X_a, \sigma_a \rangle$, $a \in A$, are well-ordered sets, then the sum $\Sigma\{\mathscr{X}_a\} = \Sigma\langle\{\mathscr{X}_a \mid a \in A\}, \leqq\rangle$ (defined as the set ΣX_a endowed by the order consisting of all pairs $\langle\langle a, x\rangle, \langle a, y\rangle\rangle$ with $x\sigma_a y$, $a \in A$, and all pairs $\langle\langle a, x\rangle, \langle b, y\rangle\rangle$ with $a < b$) is well-ordered. If ord $X_a = \xi_a$, we put $\sum_{a \in A} \xi_a = \text{ord}\, \Sigma\, \{X_a\}$. Show that this definition is correct and that the composition $\xi + \eta$ obtained in this way is associative, but is not commutative.

6. If $\langle A, \leqq \rangle$ is a finite monotonically ordered set and if X_a, $a \in A$, are well-ordered sets, then the lexicographical product $\Pi_{\text{lex}}\{X_a\}$ (see 10 ex. 10) is well-ordered. If ord $X_a = \xi_a$, put $\prod_{a \in A} \xi_a = \text{ord}\, \Pi\{X_a\}$. Show that this definition is correct and that the composition $\xi \,.\, \eta$ obtained in this way is associative, but is not commutative. (Remark: most authors denote by $\eta \,.\, \xi$ the number denoted here by $\xi \,.\, \eta$.)

7. If A is well-ordered, ord $A = \alpha$, ξ is an ordinal and $\xi_a = \xi$ for every $a \in A$, then $\sum_{a \in A} \xi_a = \alpha \,.\, \xi$.

(Section 12)

1. Let A be an infinite set, card $A = a$. Let $\mathbf{F}$ be the set of all finite covers (collections) of A. Then card $\mathbf{F} = \exp a$. Consider sets $\mathbf{M} \subset \mathbf{F}$ such that $\{(x) \mid x \in A\}$ is a meet of $\mathbf{M}$ (under the quasi-order described in 12 A.4). Then the least power of such a set $\mathbf{M}$ is equal to $\log a$ (see 9 ex. 2).

2. For any cover $\mathscr{X}$, St $\mathscr{X}$ refines St $(\mathscr{X}, \mathscr{X})$, St $(\mathscr{X}, \mathscr{X})$ refines St (St $\mathscr{X}$).

3. If St $\mathscr{X}$ refines $\mathscr{X}$, then $\mathscr{X}$ is refined by a disjoint cover of $\bigcup\mathscr{X}$. The converse does not hold.

4. Put $A = (0, 1, 2)$, $\mathscr{X} = ((0, 1), (0, 2))$, $\mathscr{Y} = ((0, 1), (1, 2))$. Then $\mathscr{Z} = ((0),(1),(2))$ is a meet of $\mathscr{X}$ and $\mathscr{Y}$, but no meet of St $\mathscr{X}$ and St $\mathscr{Y}$ refines St $\mathscr{Z}$.

5. Let $\mathscr{X}^{(1)} = \{X_a^{(1)}\}$, $\mathscr{X}^{(2)} = \{X_b^{(2)}\}$ be covers; put $X^{(i)} = \bigcup\mathscr{X}^{(i)}$. Let π_i denote the projection of $X = X^{(1)} \times X^{(2)}$ onto $X^{(i)}$. If $\mathscr{Z}$ is a cover, $Z = \bigcup\mathscr{Z}$, $f_i : Z \to X^{(i)}$ are mappings, and the f_i-image of $\mathscr{Z}$ refines $\mathscr{X}^{(i)}$, then there exists a $h : Z \to X$ such that the h-image of $\mathscr{Z}$ refines $\{X_a^{(1)} \times X_b^{(2)}\}$ and $f_i = \pi_i \circ h$. – This property of $\{X_a^{(1)} \times Y_b^{(2)}\}$ is characteristic for every cover of X which both refines and is refined by $\{X_a^{(1)} \times X_b^{(2)}\}$. Give an exact formulation and prove.

6. Let $\mathscr{X} = ((0, 1, 2), (2, 3))$, $\mathscr{Y} = ((0, 1), (1, 2, 3))$. Then there is no meet of $\mathscr{X}$ and $\mathscr{Y}$ under the quasi-order defined in 12 A.13.

7. If both covers $\{X_a\}$ and $\{Y_b\}$ are point-finite (or star-finite), then $\{X_a \times Y_b\}$ is point-finite (star-finite).

8. If $\mathscr{X}$ is a point-finite cover and f is a mapping of $\bigcup\mathscr{X}$ onto a set Y such that all inverse fibres $f^{-1}[(y)]$ are finite, then the f-image of $\mathscr{X}$ is point-finite.

9. The union of a directed (under $\subset$) set of centred collections of sets is centred.

10. Let A be a set; let $\mathscr{S} \subset \exp A$ and let $\mathscr{T}$ be the smallest additive and completely multiplicative collection containing $\mathscr{S}$. If $\bigcap\mathscr{X} \neq \emptyset$ for every centred $\mathscr{X} \subset \mathscr{S}$, then also $\bigcap\mathscr{X} \neq \emptyset$ for every centred $\mathscr{X} \subset \mathscr{T}$.

11. Let $\mathscr{X}$ be a collection of sets such that $(*)$ $\bigcap\mathscr{Y} \neq \emptyset$ for every centered subcollection $\mathscr{Y}$. Then, for every domain-full fibering correspondence $f : \bigcup\mathscr{X} \to Z$, the collection $f[[\mathscr{X}]]$ also possesses property $(*)$.

12. For any mapping $f : A \to B$, the f^{-1}-transform of any free filter on B is free.

(Section 13)

1. Let $\langle \Phi, \sigma \rangle$ be the categoroid of homomorphisms of groups described in 13 A.1, example (D). Let Z be the class of those homomorphisms $\varphi = \varphi : \mathscr{G} \to \mathscr{H}$ from Φ for which $\varphi[\mathscr{G}]$ contains only the neutral element (of $\mathscr{H}$). Then $\langle Z, \sigma_Z \rangle$ is a categoroid but is not a subcategoroid of $\langle \Phi, \sigma \rangle$.

2. Consider the category $\mathscr{M}$ of all sets. – (a) If $f : X \to Y$ is a morphism of $\mathscr{M}$, put $Ff = f \times f : X \times X \to Y \times Y$. Then $F : \mathscr{M} \to \mathscr{M}$ is a covariant functor; the associated relation assigns $X \times X$ to X. – (b) For any morphism $f : X \to Y$ let Ff be the mapping of $\exp X$ into $\exp Y$ assigning $f[Z]$ to $Z \subset X$. Then $F : \mathscr{M} \to \mathscr{M}$ is a covariant functor; the associated relation (13 B.7) assigns $\exp X$ to X. – (c) For any morphism $f : X \to Y$ let Ff be the mapping of $\exp Y$ into $\exp X$ which assigns $f^{-1}[Z]$ to $Z \subset Y$. Then $F : \mathscr{M} \to \mathscr{M}$ is a contravariant functor.

3. Let $\mathscr{K}$ be the product of categoroids $\mathscr{K}_i = \langle \Phi_i, \sigma_i \rangle$, $i = 1, 2$. Then $\langle \varphi_1, \varphi_2 \rangle \in \Phi_1 \times \Phi_2$ is a monomorphism (epimorphism, strong monomorphism, etc.; cf. 13 B.9, remark) if and only if both φ_1 and φ_2 possess the property in question.

4. Let $\mathscr{K}_i = \langle \Phi_i, \sigma_i \rangle$, $i = 1, 2$, be categoroids. Put $\mathscr{K} = \mathscr{K}_1 \times \mathscr{K}_2$. If $\varphi = \langle \varphi_1, \varphi_2 \rangle \in \Phi_1 \times \Phi_2$, put $F_i\varphi = \varphi_i$. Then $F_i : \mathscr{K} \to \mathscr{K}_i$ are covariant functors. If $\mathscr{L}$ is a categoroid and $G_i : \mathscr{L} \to \mathscr{K}_i$ are covariant, then there exists exactly one covariant functor $H : \mathscr{L} \to \mathscr{K}$ such that $G_i = F_i \circ H$.

5. Let $\mathscr{K}$ be a category. Let a_1, a_2 be objects of $\mathscr{K}$. Suppose that a is an object, $\varphi_i \in \operatorname{Hom}(a, a_i)$, and for any $\psi_i \in \operatorname{Hom}(x, a_i)$ there exists exactly one $\psi \in \operatorname{Hom}(x, a)$ with $\psi_i = \varphi_i \circ \psi$. Then a will be called a product of objects a_1, a_2 in $\mathscr{K}$. – Prove that any two products of a_1, a_2 are isomorphic. Prove that in the category $\mathscr{M}$ of all sets, $X_1 \times X_2$ is a product of objects (i.e. sets) X_1, X_2. Consider products of two objects in various categories introduced in Section 13. Show that, in the category of all monotonically ordered sets containing more than one point, no two objects possess a product.

6. Let $\mathscr{K}$ be a category. Let a_1, a_2 be objects of $\mathscr{K}$. Suppose that a is an object, $\varphi_i \in \operatorname{Hom}(a_i, a)$, and for any $\psi_i \in \operatorname{Hom}(a_i, x)$ there exists exactly one $\psi \in \operatorname{Hom}(a, x)$ with $\psi_i = \psi \circ \varphi_i$. Then the object a will be called a sum of objects a_1, a_2 in $\mathscr{K}$. Prove that any two sums of a_1, a_2 (in a given category) are isomorphic. Prove that, in the category $\mathscr{M}$, $((1) \times X_1) \cup ((2) \times X_2)$ is a sum of sets X_1, X_2. Consider sums of two objects in various categories introduced in Section 13.

7. Extend the above definitions from the case of two objects to that of a family of objects.

8. Let $\mathscr{K}_1, \mathscr{K}_2, \mathscr{L}$ be categoroids. A mapping $F = F : \mathscr{K}_1 \times \mathscr{K}_2 \to \mathscr{L}$ will be called (1) "covariant relative to both factors" if it is a covariant functor in the sense of 13A.10, (2) "covariant relative to the first and contravariant relative to the second factor" if $F : \mathscr{K}_1 \times \overline{\mathscr{K}}_2 \to \mathscr{L}$ is covariant ($\overline{\mathscr{K}}_i$ denotes the category contragredient to $\mathscr{K}_i$), etc. – Consider the underlying categoroid $\langle \Phi, \sigma \rangle$ of the category of all sets. If $f \in \Phi$, $g \in \Phi$, denote by $\mathscr{H}(f, g)$ the mapping of $\operatorname{Hom}(\mathbf{E}f, \mathbf{D}g)$ into $\operatorname{Hom}(\mathbf{D}f, \mathbf{E}g)$ assigning $g \circ \varphi \circ f$ to φ. Then $\{\langle f, g \rangle \to \mathscr{H}(f, g)\}$ determines a functor contravariant in the first and covariant in the second factor.

9. Let A be a non-void class. Let Φ consist of all non-void finite sequences of elements of A. If $\varphi = \{a_0, \ldots, a_m\} \in \Phi$, $\psi = \{b_0, \ldots, b_n\} \in \Phi$ and $b_n = a_0$, let $\varphi r \psi$ be equal to $\{b_0, \ldots, b_n, a_1, \ldots, a_m\}$; otherwise, $\varphi r \psi$ is not defined. If $a \in A$, $b \in A$, $\varphi \in \Phi$, $\varphi = \{c_0, \ldots, c_k\}$, then $\langle a, b \rangle \kappa \varphi$ if and only if $a = c_0$, $b = c_k$. – The quadruple $\langle \Phi, r, A, \kappa \rangle$ satisfies conditions (1)–(4) from 13 B.3; it satisfies condition (5) if and only if A is comprisable.

10. In the category of all rings, $\mathsf{J} : \mathsf{Z} \to \mathsf{Q}$ is an epimorphism.

11. Let H be a group, G a subgroup of H. If $\mathsf{J} : G \to H$ is an epimorphism (in the category of all groups), then $G = H$. [Hint: If $G \neq H$, consider two replicas H_1, H_2 of H and their "free product" (see 8 ex. 2) S; let φ_i be an isomorphism of H onto H_i (considered as a subgroup of S). Let λ be the least congruence on S containing all $\langle \varphi_1 g, \varphi_2 g \rangle$ with $g \in G$. Let φ be the mapping of S onto S/λ assigning $\lambda[(x)]$ to x. Then $\varphi \circ \varphi_i : H \to S/\lambda$, $i = 1, 2$, are distinct but coincide on G.]

Remark: The above assertion easily implies that, in the category of all groups, every bimorphism is an isomorphism.

12. Let $\mathscr{A}$ be the class of all separated uniformizable topological spaces (see 27 A.1, 24 A.1); let Φ be the class of all continuous mappings $f : X \to Y$ where $\mathscr{X} \in \mathscr{A}$, $Y \in \mathscr{A}$. Consider the category $\langle \Phi, \circ, \mathscr{A}, \kappa \rangle$ with κ defined in the obvious way. Prove that, in this category, epimorphisms coincide with continuous mappings onto dense (see 22 A.1) subsets, monomorphisms coincide with injective continuous $f : X \to Y$, strong epimorphisms coincide with quotient mappings (relative to $\mathscr{A}$) of X onto Y, strong monomorphisms coincide with embeddings $f : X \to Y$ such that $f[X]$ is closed in Y. Consider these and other kinds of monomorphisms, epimorphisms, etc., for various categories of spaces (topological, proximal, uniform).

(Section 14)

1. If P is a set and if int is a single-valued relation on exp P ranging in exp P and satisfying conditions (int i), $i = 1, 2, 3$ of 14 A.11, then $u = \{X \to P - \operatorname{int}(P - X) \mid X \subset P\}$ is a closure operation for P and int $= \operatorname{int}_u$.

2. Let P be a set and let ϱ be a single-valued relation on exp P ranging in exp P and satisfying the following two conditions: $\varrho\emptyset = \emptyset$, $\varrho(X \cup Y) = \varrho X \cup \varrho Y$ for each $X \subset P$, $Y \subset P$. Then $u = \{X \to X \cup \varrho X \mid X \subset P\}$ is a closure operation for P and the derivative of X is contained in ϱX for each $X \subset P$. If $x \notin \varrho(x)$ for each $x \in P$, then ϱX is the derivative of X in $\langle P, u \rangle$ for each $X \subset P$. [Hint: if $x \in \varrho X$, then $x \in \varrho(X - (x)) \cup \varrho(x)$ and by the last condition this implies that $x \in \varrho(X - (x))$.] If $x \in \varrho(x)$, then $\varrho(x)$ is not the derivative of (x) in $\langle P, u \rangle$.

3. A family $\{X_a \mid a \in A\}$ is said to be hereditarily closure-preserving in a closure space $\mathscr{P}$ if each family $\{Y_a \mid a \in B\}$, where $B \subset A$, $Y_a \subset X_a$, is closure-preserving in $\mathscr{P}$. Every locally finite family is hereditarily closure-preserving, and in an accrete space each family is hereditarily closure-preserving; thus a hereditarily closure-preserving family need not be locally finite.

4. Let $\{X_a \mid a \in A\}$ be a family of subsets of a space, $\{A_c \mid c \in C\}$ be a family in $\exp A$ and $Y_c = \bigcup\{X_a \mid a \in A_c\}$. If $\{A_c\}$ is point-finite and $\{X_a\}$ is locally finite then $\{Y_c\}$ is locally finite. If $\bigcup\{A_c\} = A$, $\{Y_c\}$ is locally finite and each $\{X_a \mid a \in A_c\}$ is locally finite, then $\{X_a \mid a \in A\}$ is locally finite.

5. In a generalized ordered space $\langle P, \leqq, u\rangle$ the order-closed intervals which are neighborhoods of a given point x form a local base at x.

6. If $\langle P, \leqq\rangle$ is order-dense (i.e., $x < y \Rightarrow \rrbracket x, y \llbracket \neq \emptyset$), no point of the ordered space $\langle P, \leqq, u\rangle$ is isolated. No point of R is isolated.

(Section 15)

1. A subset X of a T-space is the intersection of- a closed set with an open set if and only if $\overline{X} - X$ is a closed set. [Hint: I. $X = \overline{X} \cap (P - (\overline{X} - X))$. – II. If $X = F \cap G$, F closed, G open, then $\overline{X} = \overline{F \cap G} \cap F$ and hence $\overline{X} - X = F \cap (\overline{F \cap G} - G)$.]

2. Let $\langle P, u\rangle$ be a topological space. For each $X \subset P$ let $\mathscr{A}_X$ be the smallest collection of subsets of P such that $X \in \mathscr{A}_X$ and $Y \in \mathscr{A}_X$ implies that $uY \in \mathscr{A}_X$, $P - Y \in \mathscr{A}_X$. Then each collection $\mathscr{A}_X$ has at most 14 elements. Find a subset X of the space R of reals such that $\mathscr{A}_X$ has 14 elements.

3. If $\langle P, \leqq, u\rangle$ is an order-complete discrete ordered space (i.e. $\langle P, \leqq\rangle$ is order-complete, u is the order closure for $\langle P, \leqq\rangle$ and u is a discrete closure), then P is finite. [Hint: if x_n is an increasing (decreasing) sequence in $\langle P, u\rangle$, then $\sup\{x_n\}$ ($\inf\{x_n\}$) is a limit point of $\{x_n\}$.] If $\langle P, \leqq, u\rangle$ is a boundedly order-complete discrete ordered space, then $\langle P, \leqq, u\rangle$ is countable (each interval $\llbracket x, y \rrbracket$ is finite). On the other hand, for each cardinal m there exists a discrete ordered space $\langle P, \leqq, u\rangle$ such that the cardinal of P is m (if I is the ordered set of integers and T is a segment of ordinals, then the order closure for the lexicographic product of T and I is discrete).

4. Let $\mathscr{U}$ be a local sub-base at a point x in a space $\mathscr{P}$. If a net $\mathscr{N}$ ranging in $\mathscr{P}$ is eventually in each $U \in \mathscr{U}$, then $\mathscr{N}$ converges to x. A similar result for accumulation points is not true; e.g. consider a point x in a space such that two sets X and Y form a local sub-base and $X \cap Y = (x)$.

5. (a) If $\langle P, u\rangle$ is a closure space, $x \in uX$ and $\mathscr{U}$ is a local base at x in $\langle P, u\rangle$, then $\supset$ directs $\mathscr{U}$ and there exists a net $\langle\{x_U \mid U \in \mathscr{U}\}, \supset\rangle$ which ranges in X and converges to x in $\langle P, u\rangle$ (choose $x_U \in X \cap U$). Thus any space $\mathscr{P}$ can be described by means of convergence of nets, the ordered domains of which are local bases of points of $\mathscr{P}$.

(b) If $\{U_n \mid n \in \mathrm{N}\}$ is a local base at x and a sequence $S = \{S_i \mid i \in \mathrm{N}\}$ converges to x, then there exists a subsequence $\{S_{i_n}\}$ of S with $S_{i_n} \in U_n$ for each n.

(c) If a space is of a countable local character at x and a sequence $\{S_n\}$ converges to x, then there exists a monotone local base $\{U_n\}$ at x with $S_n \in U_n$ for each n; in addition, if the space is topological then the sets U_n may be taken open.

6. Ultranets. A net $\mathcal{N}$ is said to be an ultranet if the following condition is fulfilled: if $\mathcal{N}$ ranges in the union of two sets X and Y, then $\mathcal{N}$ is eventually either in X or in Y. It is almost self-evident that:

(a) If x is an accumulation point of a ultranet $\mathcal{N}$ in a space $\mathcal{P}$, then x is a limit point of $\mathcal{N}$ in $\mathcal{P}$.

(b) Every directed net has a generalized directed subnet which is an ultranet. [Hint: Let $\mathcal{N} = \langle N, \leqq \rangle$ be a directed net and let $\mathcal{V}$ be the collection consisting of all subsets Y of $\mathbf{E}N$ such that $N^{-1}[Y]$ is residual in $\langle \mathbf{D}N, \leqq \rangle$, and let us choose an ultrafilter $\mathcal{U}$ on $\mathbf{E}N$ containing $\mathcal{V}$. The generalized subnet $\mathcal{M}$ of $\mathcal{N}$ constructed in the proof of 15 B.22 is an ultranet; indeed, $\mathcal{M}$ is eventually in each element of $\mathcal{U}$ and therefore, if $\mathcal{M}$ ranges in $X \cup Y$, then $X \cap \mathbf{E}N \in \mathcal{U}$ or $Y \cap \mathbf{E}N \in \mathcal{U}$ because $\mathcal{U}$ is an ultrafilter and hence $\mathcal{M}$ is eventually in X or in Y.]

(c) If $x \in uX$, where $\langle P, u \rangle$ is a closure space, then there exists an ultranet ranging in X which converges to x in $\langle P, u \rangle$.

(d) A directed net $\mathcal{N}$ converges to x in a space $\mathcal{P}$ if and only if each generalized directed subnet of $\mathcal{P}$, which is an ultranet, converges to x in $\mathcal{P}$.

7. Convergence of filters. A proper filter base $\mathcal{X}$ on a space $\mathcal{P}$ is said to be convergent to a point x if each neighborhood of x contains an element of $\mathcal{X}$. Thus a proper filter $\mathcal{X}$ on $\mathcal{P}$ converges to x if and only if $\mathcal{X}$ contains the neighborhood system of x in $\mathcal{P}$. A point x is said to be a cluster point of a proper filter base $\mathcal{X}$ in a space $\mathcal{P}$ if each neighborhood of x intersects each element of $\mathcal{X}$, or equivalently, if $x \in \bigcap\{\overline{X} \mid X \in \mathcal{X}\}$. x is a limit point of $\mathcal{X}$ if $\mathcal{X}$ converges to x.

Prove: (a) Each limit point is a cluster point, and a cluster point of an ultrafilter is a limit point.

(b) If $\mathcal{X}$ and $\mathcal{Y}$ are proper filter bases on a space $\mathcal{P}$ and $\mathcal{X} \subset \mathcal{Y}$, then each limit point of $\mathcal{X}$ is limit point of $\mathcal{Y}$, and each cluster point of $\mathcal{Y}$ is a cluster point of $\mathcal{X}$.

(c) Let $\mathcal{N}$ be a directed net ranging in a space $\mathcal{P}$ and let $\mathcal{X}$ be the set of all subsets X of $\mathcal{P}$ such that $\mathcal{N}$ is eventually in X. Then $\mathcal{X}$ is a proper filter of sets on $\mathcal{P}$, and x is a limit (accumulation) point of $\mathcal{N}$ if and only if x is a limit (cluster) point of $\mathcal{X}$ in $\mathcal{P}$.

(d) Let $\mathcal{X}$ be a proper filter on a space $\mathcal{P}$. If $\mathcal{X}$ converges to a point x and a net $\mathcal{N}$ ranging in $\mathcal{P}$ is eventually in each element of $\mathcal{X}$, then $\mathcal{N}$ converges to x, in particular, if $\{x_X \mid X \in \mathcal{X}\}$ is a family such that $x_X \in X$ for each X in $\mathcal{X}$, then the net $\langle\{x_X \mid X \in \in \mathcal{X}\}, \supset\rangle$ converges to x. If $\mathcal{X}$ does not converge to x, then some net $\mathcal{N} = = \langle\{x_X \mid X \in \mathcal{X}\}, \supset\rangle$, where $x_X \in X$, does not converge to x, and moreover, x is not an accumulation point of $\mathcal{N}$.

(e) If x is a cluster point of a proper filter $\mathcal{X}$ on a space $\mathcal{P}$, then x is a limit point of an ultrafilter $\mathcal{Y} \supset \mathcal{X}$, and x is a limit point of a net $\mathcal{N}$ which is eventually in each $X \in \mathcal{X}$.

8. All nets are assumed to be directed. Let $\mathcal{N} = \langle N, \leqq \rangle$ be a net in $\langle \exp X, \subset \rangle$ where X is a set and let $\mathcal{A}$ be the collection of all residual subsets of $\langle \mathbf{D}N, \leqq \rangle$.

Then

$$\limsup \mathcal{N} = \bigcap\{\bigcup\{N_a \mid a \in A\} \mid A \in \mathscr{A}\}$$
$$\liminf \mathcal{N} = \bigcup\{\bigcap\{N_a \mid a \in A\} \mid A \in \mathscr{A}\}\,.$$

9. Let $\mathcal{N} = \langle N, \leqq \rangle$ be a net in an ordered set $\langle P, \prec \rangle$. If $\mathcal{M}$ is a generalized subnet of $\mathcal{N}$, then

$$\liminf \mathcal{N} \prec \liminf \mathcal{M} \prec \limsup \mathcal{M} \prec \limsup \mathcal{N}$$

provided the elements in question exist; if $\lim \mathcal{N}$ exists then $\lim \mathcal{M}$ exists and $\lim \mathcal{N} = \lim \mathcal{M}$.

10. Let $\mathcal{N} = \langle N, \leqq \rangle$ be a net ranging in $\exp |\mathscr{P}|$ where $\mathscr{P}$ is a closure space. The topological upper (lower) limit of $\mathcal{N}$ in $\mathscr{P}$, denoted by $T_{\mathscr{P}} \limsup \mathcal{N}$ ($T_{\mathscr{P}} \liminf \mathcal{N}$), is the set of all points x of $\mathscr{P}$ such that, for each neighborhood U of x in $\mathscr{P}$, the set of all $a \in \mathbf{D}N$ such that $U \cap N_a \neq \emptyset$ is cofinal (residual) in $\langle \mathbf{D}N, \leqq \rangle$. We have $T_{\mathscr{P}} \limsup \mathcal{N} \supset T_{\mathscr{P}} \liminf \mathcal{N}$. If the topological upper limit and lower limit coincide, then the set $T_{\mathscr{P}} \limsup \mathcal{N}$ is called the topological limit of $\mathcal{N}$ in $\mathscr{P}$ and is denoted by $T_{\mathscr{P}} \lim \mathcal{N}$. Prove:

(a) If $\mathcal{M}$ is a generalized subnet of $\mathcal{N}$, then

$$T_{\mathscr{P}} \limsup \mathcal{N} \supset T_{\mathscr{P}} \limsup \mathcal{M} \supset T_{\mathscr{P}} \liminf \mathcal{M} \supset T_{\mathscr{P}} \liminf \mathcal{N}\,,$$

in particular, $T_{\mathscr{P}} \lim \mathcal{N} = T_{\mathscr{P}} \lim \mathcal{M}$ provided that $T_{\mathscr{P}} \lim \mathcal{N}$ exists.

(b) If $\mathcal{N}$ is decreasing (under inclusion), then $T_{\mathscr{P}} \lim \mathcal{N} = \bigcap\{\overline{N}_a \mid a \in \mathbf{D}N\}$. If $\mathcal{N}$ is increasing then $T_{\mathscr{P}} \lim \mathcal{N}$ is the closure of $\bigcup\{N_a \mid a \in \mathbf{D}N\}$.

(c) $T_{\mathscr{P}} \limsup \mathcal{N} = \bigcap\{\overline{\bigcup\{N_a \mid \alpha \leqq a\}} \mid \alpha \in \mathbf{D}N\}$.

(d) If $\mathscr{P}$ is topological then the sets $T_{\mathscr{P}} \limsup \mathcal{N}$ and $T_{\mathscr{P}} \liminf \mathcal{N}$ are closed and $T_{\mathscr{P}} \limsup \mathcal{N} = T_{\mathscr{P}} \limsup \overline{\mathcal{N}}$, $T_{\mathscr{P}} \liminf \mathcal{N} = T_{\mathscr{P}} \liminf \overline{\mathcal{N}}$, where $\overline{\mathcal{N}} = \langle\{\overline{N}_a \mid a \in \mathbf{D}N\}, \leqq\rangle$.

(e) If $\langle M, \leqq \rangle$ is a net in $\mathscr{P}$ and $\mathcal{M} = \langle\{(M_a) \mid a \in \mathbf{D}M\}, \leqq\rangle$, then $T_{\mathscr{P}} \limsup \mathcal{M}$ is the set of all accumulation points of $\langle M, \leqq \rangle$.

(f) If $\langle\{M_a\}, \leqq\rangle$ and $\langle\{N_a\}, \leqq\rangle$ are nets in $\exp |\mathscr{P}|$, then

$$T_{\mathscr{P}} \limsup \{M_a \cup N_a\} = T_{\mathscr{P}} \limsup \{M_a\} \cup T_{\mathscr{P}} \limsup \{N_a\}\,,$$
$$T_{\mathscr{P}} \liminf \{M_a \cup N_a\} \supset T_{\mathscr{P}} \liminf \{M_a\} \cup T_{\mathscr{P}} \liminf \{N_a\}\,,$$

and hence

$$T_{\mathscr{P}} \lim \{M_a \cup N_a\} = T_{\mathscr{P}} \lim \{M_a\} \cup T_{\mathscr{P}} \lim \{\mathcal{N}_a\}$$

provided both limits on the right side exist.

11. A monotone ordered set is boundedly order-complete if and only if each interval-like set is an interval. [Hint: For "if", given X, consider the interval-like sets $\leqq[X]$ and $\leqq^{-1}[X]$. For "only if" consider separately the case when $\inf X$ or $\sup X$ exists.]

12. βX is topological and the sets $\overline{Y}$, $Y \subset X$, form an open base for βX.

(Section 16)

1. A mapping f of a closure space $\langle P, u\rangle$ into a closure space $\langle Q, v\rangle$ is continuous if and only if $\operatorname{int}_u f^{-1}[X] \subset f^{-1}[\operatorname{int}_v X]$ for each $X \subset Q$.

2. Describe continuity by means of cluster points of sets.

3. Let f be a mapping of a closure space $\mathscr{P}$ into another one $\mathscr{Q}$, $x \in |\mathscr{P}|$ and let $\mathscr{U}$ be a local sub-base at fx in $\mathscr{Q}$. Then f is continuous at x if and only if $f^{-1}[U]$ is a neighborhood of x for each U in $\mathscr{U}$. [Hint: $f^{-1}[\bigcap\{U_a\}] = \bigcap\{f^{-1}[U_a]\}$.]

4. Let f be a mapping of a space $\mathscr{P}$ into a topological space $\mathscr{Q}$ and let $\mathscr{B}$ be an open sub-base of $\mathscr{Q}$. Then f is continuous if and only if $f^{-1}[B]$ is open in $\mathscr{P}$ for each B in $\mathscr{B}$.

5. A mapping f of a space $\mathscr{P}$ into a space $\mathscr{Q}$ is continuous at $x \in |\mathscr{P}|$ if and only if the filter base $\mathbf{E}\{f[U] \mid U \in \mathscr{U}\}$ converges to fx, where $\mathscr{U}$ is the neighborhood system of x in $\mathscr{P}$.

6. Let $\langle N, \leqq\rangle$ be a net ranging in a closure space $\mathscr{P}$ and let $x \in |\mathscr{P}|$. Let $\mathscr{A}$ be the collection of all residual sets in $\langle \mathbf{D}N, \leqq\rangle$. Then $\mathscr{A}$ is a proper filter on $\mathbf{D}N$. Let $Q = \mathbf{D}N \cup (\mathscr{A})$, and let v be the closure for Q such that each point of $\mathbf{D}N$ is isolated and $(\mathscr{A}) \cup [\mathscr{A}]$ $(= \mathbf{E}\{(\mathscr{A}) \cup A \mid A \in \mathscr{A}\})$ is the neighborhood system at $\mathscr{A}$ in $\langle Q, v\rangle$. Let us consider the mapping f of $\langle Q, v\rangle$ into $\mathscr{P}$ such that N is a restriction of $\operatorname{gr} f$ and $f\mathscr{A} = x$. Then f is continuous if and only if the net $\langle N, \leqq\rangle$ converges to x in $\mathscr{P}$.

7. Let u be a closure for a set P and let $\{u_\alpha \mid \alpha \text{ is an ordinal}\}$ be a family of single-valued relations on $\exp P$ ranging in $\exp P$ such that $u_0 = u$, $u_{\alpha+1}X = uu_\alpha X$ for each α and $u_\alpha X = \bigcup\{u_\beta X \mid \beta < \alpha\}$ if α is a limit ordinal. Then each u_α is a closure operation, and there exists an α such that $u_\alpha = u_\beta$ for all $\beta > \alpha$, whereupon u_α is the topological modification of u.

8. Let f be an order-preserving mapping of an ordered space $\langle P, \leqq, u\rangle$ into another one $\langle Q, \prec, v\rangle$. The mapping f need not be continuous, but if f is completely lattice-preserving then f is continuous.

9. Let f be a continuous mapping of a space $\mathscr{P}$ into a space $\mathscr{Q}$. If $\{X_a\}$ is a locally finite family in $\mathscr{Q}$ then $\{f^{-1}[X_a]\}$ is a locally finite family in $\mathscr{P}$; on the other hand, if $\{X_a\}$ is closure-preserving or hereditarily closure-preserving, then $\{f^{-1}[X_a]\}$ need not have the corresponding property. [Hint: for $\mathscr{Q}$ take an accrete space.]

(Sections 17 – 18)

1. Let $\mathscr{Q}_1$ and $\mathscr{Q}_2$ be subspaces of $\mathscr{P}$, the closure of $|\mathscr{Q}_1|$ in $\mathscr{P}$ be contained in $|\mathscr{Q}_2|$ and $\mathscr{X}$ be a family in $\exp |\mathscr{Q}_1|$. If $\mathscr{X}$ is locally finite in $\mathscr{P}$ then $\mathscr{X}$ is locally finite in $\mathscr{Q}_1$. If $\mathscr{X}$ is locally finite in $\mathscr{Q}_2$ then $\mathscr{X}$ is locally finite in $\mathscr{P}$.

2. Box-product. Let m be an infinite cardinal. If $\{\mathscr{P}_a \mid a \in A\}$ is a family of spaces and $P = \Pi\{|\mathscr{P}_a|\}$, then the m-box-product of $\{\mathscr{P}_a\}$ is the space $\langle P, u\rangle$ such that, for each $x \in P$, the sets of the form $\bigcap\{\pi_b^{-1}[U_b] \mid b \in B\}$ form a local base at x, where $\pi_b = \mathrm{pr}_b \cap (P \times |\mathscr{P}_b|)$, U_b is a neighborhood of $\pi_b x$ in $\mathscr{P}_b$, $B \subset A$ and card $B < m$. If card $A < m$ then $\langle P, u\rangle$ is termed the box-product of $\{\mathscr{P}_a\}$. The box-product closure operation is the finest closure for P such that the embeddings $f_{ya} : \mathscr{P}_a \to \langle P, u\rangle$ are continuous and the boxes (i.e. sets of the form $\Pi\{X_a\}$) form local bases, where $\pi_a f_{ya} x = x$, $\pi_b f_{ya} x = \pi_b y$ for $b \neq a$. The projections of m-box-products are continuous and carry all neighborhoods into neighborhoods. $\aleph_0$-box-products coincide with products. The closures of finite sets with respect to the product closure and the box-product closure coincide.

3. Local characters. (a) Let $\mathscr{Q}$ be a subspace of a space $\mathscr{P}$. The local character at $x \in \mathscr{Q}$ in $\mathscr{Q}$ is at most the local character at x in $\mathscr{P}$. The local character of $\mathscr{Q}$ is at most the local character of $\mathscr{P}$. The total character of $\mathscr{Q}$ is at most the total character of $\mathscr{P}$ ($\mathscr{P}$ is assumed to be topological).

(b) The local character of a sum is the supremum of the local characters. The total character of a sum is the sum of the total characters.

(c) Let $x \in \mathscr{P} = \Pi\{\mathscr{P}_a \mid a \in A\}$, card $A \geqq \aleph_0$. The local character at x is less than or equal to the supremum of card A and all the local characters at $\mathrm{pr}_a x$, $a \in A$, in $\mathscr{P}_a$; if $\mathscr{P}_a$ is the only neighborhood of $\mathrm{pr}_a x$ for no a then equality holds. If the local character of each $\mathscr{P}_a$ at $\mathrm{pr}_a x$ is 1, then the local character of $\mathscr{P}$ at x is card A. Discuss total characters similarly.

(d) If $\mathscr{P}$ is a discrete space, card $|\mathscr{P}| > 1$ and card $A \geqq \aleph_0$, then the local character of $\mathscr{P}^A$ is card A and the total character is card $|\mathscr{P}|$. card A.

4. Domain-extensions of continuous mappings. Let f be a mapping of a dense subspace $\langle P, u\rangle$ of $\langle Q, v\rangle$ into $\langle R, w\rangle$. Let g be an extension of f to a mapping F of $P \cup (x)$ into $\langle R, w\rangle$. Each of the following two conditions is necessary and sufficient for F to be continuous at x : (a) If a net N ranges in P and converges to x in $\langle Q, v\rangle$ then $f \circ N$ converges to Fx; (b) If $\mathscr{U}$ is a local base at x in $\langle Q, u\rangle$ then the filter base $f[[U] \cap P]$ converges to Fx. If f is continuous and F is continuous at x, then F need not be continuous; if, in addition, (x) is closed or the closure of (x) is disjoint with P, then F is continuous.

5. A space is said to be compact (countably compact) if each directed net (sequence) has an accumulation point. An ordered space is compact if and only if it is order-complete. In particular, each closed interval of reals and each finite space are compact. The ordered space T_{ω_1} of countable ordinals is not compact (it is not order-complete), but it is countably compact (each sequence is bounded in T_{ω_1}). Each of the following conditions is necessary and sufficient for a space $\mathscr{P}$ to be compact (countably compact):

(a) Each proper filter (with a countable base) has a cluster point (see 15 ex. 7).

(b) Each interior cover (countable interior cover) contains a finite cover.

[Hint: Equivalence of (a) and (b) follows from de Morgan formula; for necessity and sufficiency of (a) see 15 ex. 7.]

A space $\mathscr{P}$ is countably compact if and only if each locally finite family of non-void sets if finite.

Any closed subspace of a compact (countably compact) space is compact (countably compact). The sum of a family $\{\mathscr{P}_a \mid a \in A\}$ of non-void compact spaces is compact if and only if A is finite. In 29 B.5 and 41 A.12 we shall prove that the product of compact spaces is compact. For finite families the proof is quite elementary. Let $\mathscr{P} = \Pi\{\mathscr{P}_a \mid a \in A\}$ with A finite and $\mathscr{P}_a$ compact and let $\mathscr{N}$ be a net in $\mathscr{P}$; we may assume $A = (0, 1, \ldots, n)$. Since $\mathscr{P}_0$ is compact there exists a generalized subnet $\mathscr{N}_0$ of $\mathscr{N}$ such that $\mathrm{pr}_0 \circ \mathscr{N}_0$ converges to a point x_0 in P_0 and, by induction, a generalized subnet $\mathscr{N}_{k+1}$ of $\mathscr{N}_k$, $k + 1 \leqq n$, such that $\mathrm{pr}_{k+1} \circ \mathscr{N}_{k+1}$ converges to a point x_{k+1} in $\mathscr{P}_{k+1}$. Clearly $\mathscr{N}_n$ is a generalized subnet of $\mathscr{N}$ and $\mathscr{N}_n$ converges to $\{x_a \mid a \in A\}$.

If F is closed and bounded (i.e. coordinates are bounded) in R^n then F is compact. Any bounded net in R^n has an accumulation point.

If f is a surjective continuous mapping and $\mathbf{D}^*f$ is compact then $\mathbf{E}^*f$ is a compact space.

Any βX is compact. [Prove: Any interior cover of βX can be refined by an open cover $\mathscr{X}$ consisting of sets of the form $\overline{Y}$, $Y \subset X$. The fact that $\mathscr{X}$ is cover is equivalent to the statement that each ultrafilter contains a set Y with $\overline{Y}$ in $\mathscr{X}$. Assuming that the set $\mathscr{Y}$ of all Y with $\overline{Y}$ in $\mathscr{X}$ contains no finite cover of X we can find a ultrafilter which contains no $Y \in \mathscr{Y}$; consider a ultrafilter containing complements (in X) of all finite unions of elements of $\mathscr{Y}$.]

6. A pseudometric space is a metric space if and only if each net (sequence) has at most one limit point. In a semi-metric space a sequence may have many limit points.

7. If d is a semi-pseudometric for a set P such that $d\langle x, z\rangle \leqq 2\max(d\langle x, y\rangle, d\langle y, z\rangle)$, then $d\langle x_0, x_n\rangle \leqq 4\sum_{i=1}^{n} d\langle x_{i-1}, x_i\rangle - 2d\langle x_0, x_1\rangle - 2d\langle x_{n-1}, x_n\rangle$, and hence $4D\langle x, y\rangle \geqq d\langle x, y\rangle$ where D is the greatest pseudometric which is smaller than d. Consequently, d is Lipschitz equivalent with a pseudometric.

8. Let u be the closure induced by a semi-pseudometric d for a set P. The function

$$f = d : \mathrm{ind}\,(\langle P, u\rangle \times \langle P, u\rangle) \to \mathsf{R}$$

is continuous (i.e., d is inductively continuous or "separately continuous" on $\langle P, u\rangle \times \langle P, u\rangle$) if and only if each open sphere is an open set and each closed sphere is a closed set. This follows from the following two statements: f is lower (or upper) semi-continuous if and only if each closed (open, respectively) sphere is a closed set (an open set, respectively).

9. (a) A closure space $\langle P, u\rangle$ is semi-pseudometrizable if and only if there exists a family $\{X_{x,n} \mid x \in P,\ n \in \mathsf{N}\}$ such that $\{X_{x,n} \mid n \in \mathsf{N}\}$ is a local base at x for each $x \in P$, and if $y \in X_{x_n,n}$ for $n \in \mathsf{N}$, then $\{x_n\}$ converges to y.

(b) Let $P = \mathsf{R} \times \mathsf{R}$ and let u be a closure for P such that the neighborhoods of points $\langle x, y\rangle$ with $y \neq 0$ or $x \in \mathsf{Q}$ coincide with those in the product space $\mathsf{R} \times \mathsf{R}$, and that the sets $S(z, c) = \mathbf{E}\{w \mid d\langle z, w\rangle + a\langle z, w\rangle < c\}$, $c > 0$, form a local base at $z = \langle x, 0\rangle$, $x \in \mathsf{R} - \mathsf{Q}$, where d is the usual metric for $\mathsf{R} \times \mathsf{R}$ and $a\langle z, w\rangle$ is the smallest non-negative angle (in radians) formed by the line $\mathsf{R} \times (0)$ and the line containing z and w. The space $\langle P, u\rangle$ is topological and semi-metrizable. It will be shown in 22 ex. 7 that $\langle P, u\rangle$ cannot be semi-metrized in such a manner that each open sphere be open.*)

10. If an interval $I = [\![a, b]\!]$ in R does not contain 0, then the mapping $\{x \to x^{-1}\} : I \to \mathsf{R}$ is Lipschitz continuous; hence $\{x \to x^{-1}\} : \mathsf{R} - (0) \to \mathsf{R}$ is continuous.

11. Let d_1 and d_2 be topologically equivalent metrics for a set P and let d be the greatest pseudometric smaller than both d_1 and d_2. It is an interesting problem to find (necessary, sufficient or both) conditions on the closure operation u induced by d_1 for d to induce u.

(a) Let $P = [\![0, 1]\!]$, A_1 and A_2 be disjoint subsets of P such that the pseudometrics

$$d_i = \{\langle x, y\rangle \to \mu(A_i \cap ([\![x, y]\!] \cup [\![y, x]\!]))\}$$

where μ is the Lebesgue measure, induce the closure structure of the space $[\![0, 1]\!]$ (i.e., $d_i\langle x, y\rangle > 0$ for each $x \neq y$). Using the fact that the density of A_i is 0 at almost all $x \in P - A_i$ one finds without difficulty that $d = 0$.

(b) Another construction of d_i for $[\![0, 1]\!]$. Let Q be the set of all $k/2^n \leqq 1$, $k \in \mathsf{N}$, $n \in \mathsf{N}$. Let $d_i\langle 0, 1\rangle = 1$, and by induction, let $\alpha = d_i\langle k/2^n, (k + 1)/2^n\rangle$ with $0 \leqq k < 2^n$, $n > 0$, be defined as follows: (1) if k is even, then α is

$$d_i\langle \tfrac{1}{2}k/2^{n-1}, \tfrac{1}{2}(k + 2)/2^{n-1}\rangle$$

multiplied by $\frac{1}{4}$ or $\frac{3}{4}$ according as i is 1 or 2; (2) if k is odd then α is $d_i\langle \frac{1}{2}(k - 1)/2^{n-1}, \frac{1}{2}(k + 1)/2^{n-1}\rangle$ multiplied by $\frac{3}{4}$ or $\frac{1}{4}$ according as i is 1 or 2. Clearly, the numbers α are well-defined. If $0 \leqq k < l \leqq 2^n$, then put

$$d_i\left\langle \frac{k}{2^n}, \frac{l}{2^n}\right\rangle = \sum_{i=k}^{l-1} d_i\left\langle \frac{j}{2^n}, \frac{j + 1}{2^n}\right\rangle.$$

These numbers are also well-defined. Finally, putting $d_i\langle x, y\rangle = d_i\langle y, x\rangle$, $d_i\langle x, x\rangle = 0$, we obtain a metric for the set Q. It is easily seen that

$$x \leqq y \leqq z \quad \text{implies} \quad d_i\langle x, y\rangle + d_i\langle y, z\rangle = d_i\langle x, z\rangle,$$

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* This example is due to R. HEATH, Proc. Amer. Math. Soc. *12* (1961), 810–811.

and

$$(\tfrac{1}{4})^n \leqq d_i \left\langle \frac{k}{2^n}, \frac{k+1}{2^n} \right\rangle \leqq (\tfrac{3}{4})^n ,$$

and hence both metrics d_i induce the closure structure of the subspace Q of $[\![0, 1]\!]$. Let d be the greatest pseudometric for Q smaller than both d_1 and d_2. Prove that $d\langle x, y\rangle = 0$ for each $\langle x, y\rangle$. Use the estimate

$$\min_{i=1,2} \left\{ d_i \left\langle \frac{k}{2^n}, \frac{k+1}{2^n} \right\rangle \right\} \leqq (\tfrac{3}{4})^{n/2} \cdot \frac{1}{2^n}$$

which implies

$$d\langle 0, 1\rangle \leqq 2^n \, . \, (\tfrac{3}{4})^{n/2} \frac{1}{2^n} = (\tfrac{3}{4})^{n/2} .$$

It can be shown easily that both metrics d_1 and d_2 admit extensions to metrics for $[\![0, 1]\!]$, inducing the closure of $[\![0, 1]\!]$. Of course the resulting d for the extended d_i is also a zero-relation.

(c) Remark. If $\langle P, d_1\rangle$ is a compact metric space of Hausdorff measure zero and d_2 is any metric topologically equivalent to d_1, then d is a metric; since the space is compact and d is a continuous metric, d is topologically equivalent to d_1 by Theorem 41 C.5. The assumption that the Hausdorff measure is zero can be replaced by the assumption that the space is totally disconnected.

12. Subsequences in pseudometric spaces. Let $\mathscr{P} = \langle P, d\rangle$ be a pseudometric space. A sequence $\{x_n\}$ in $\mathscr{P}$ is defined to be metrically discrete if there exists an $r > 0$ such that $d\langle x_n, x_m\rangle > r$ for each $n \neq m$. A sequence $\{x_n\}$ in $\mathscr{P}$ is said to be a Cauchy sequence if for each $r > 0$ there exists an n in N with $d\langle x_n, x_m\rangle < r$ for each $m \geqq n$.

(α) No Cauchy sequence is metrically discrete.

(β) If a sequence S has a limit point, then S is a Cauchy sequence; the converse is true in complete spaces, see 22 ex. 6.

(γ) Each metrically discrete sequence is locally finite.

(δ) Each sequence in $\mathscr{P}$ has a subsequence which is either metrically discrete or a Cauchy sequence.

[Hint to (δ): Assuming that no subsequence of $\{x_n\}$ is metrically discrete, contsruct a subsequence $\{x_{n_k}\}$ together with a monotone sequence $\{X_k\}$ of infinite subsets of N such that the distance from x_{n_k} to each point x_i, $i \in X_k$, is $(n + 1)^{-1}$ at most; $\{x_{n_k}\}$ is a Cauchy sequence.]

13. Let $\mathscr{B}$ be the collection of all bounded subsets (i.e. with finite diameter, in particular $\emptyset \notin \mathscr{B}$) of a pseudometric space $\langle P, d\rangle$. For each B_1 and B_2 in $\mathscr{B}$ let $D\langle B_1, B_2\rangle$ be the supremum of all the numbers dist (x, B_2), $x \in B_1$ and dist (y, B_1), $y \in B_2$. Prove that D is a pseudometric for $\mathscr{B}$; the space $\langle \mathscr{B}, D\rangle$ is called the Hausdorff hyperspace of $\langle P, d\rangle$ (occasionally this term is used for the subspace of $\langle \mathscr{B}, D\rangle$ consisting of all the closed sets).

(Section 19)

1. Let $\mathscr{U}$ be the collection of all subsets of R containing a set of the form Q $\cap$ $\cap$ $]\!]-r, r[\![$, $r > 0$. It is easy to verify that $\mathscr{U}$ is the neighborhood system at 0 relative to a closure compatible with the additive group structure for R. This group will be denoted by R_1. Prove that $f = \mathsf{J} : R_1 \to R$ is continuous and $\overline{f[U]}$ is a neighborhood of fx provided that U is a neighborhood of x. Each $x \in R_1$ has a neighborhood U such that the interior in R of $f[U]$ is empty.

2. Let L be a finite dimensional linear space over $\mathscr{T}$ where $\mathscr{T} = $ R or $\mathscr{T} = $ C, and let X be a base. *The relation* $\varphi = \{\Sigma r_x x \mid x \in X\} \to \max \{|r_x|\}\}$ *is a norm and the closure operation u induced by* φ *is the unique admissible closure for L such that each singleton* (equivalently, some singleton) *is closed.* Unicity will be proved in a sequence of auxiliary propositions.

(a) *u is the finest closure among all closures admissible for L.* [Let v be any closure admissible for L; $\{r \to rx\} : \mathscr{T} \to \langle L, v\rangle$ is a continuous mapping and $\{r \to rx\}$: : $\mathscr{T} \to \langle L, u\rangle$ is an embedding. Consequently, if L_x is spanned over (x) and u_x or v_x are relativizations of u or v, respectively, to L_x, then v_x is coarser than u_x. The mapping $f = \{\{r_x x \mid x \in X\} \to \Sigma\{r_x x\}\} : \Pi\{\langle L_x, u_x\rangle\} \to \langle L, u\rangle$ is a homeomorphism and the mapping $f : \Pi\{\langle L_x, v_x\rangle\} \to \langle L, v\rangle$ is continuous, because the addition is continuous; hence $\mathsf{J} : \langle L, u\rangle \to \langle L, v\rangle$ is continuous.]

(b) *If a singleton is closed in a topological linear space then each net has at most one limit point.* [Clearly, each singleton is closed. If $x \neq y$ then $x \notin y + U + U$ for some symmetric neighborhood U of 0 and hence $(x + U) \cap (y + U) = \emptyset$.]

(c) *If a net* $\mathscr{N}$ *converges to y in* $\langle L, v\rangle$, *if the singletons are closed in* $\langle L, v\rangle$ *and if* $\varphi \circ \mathscr{N}$ *is bounded, then y is the only accumulation point of* $\mathscr{N}$ *in* $\langle L, u\rangle$. [Since $\varphi \circ \mathscr{N}$ is bounded, $\mathscr{N}$ has an accumulation point in $\langle L, u\rangle$. If z is an accumulation point of $\mathscr{N}$ in $\langle L, u\rangle$ then z is an accumulation point of $\mathscr{N}$ in $\langle L, v\rangle$ and hence $z = y$.]

(d) Assume that $\langle L, v\rangle$ is a topological linear space and v is a strictly coarser than u. There exists a u-neighborhood U of 0 which is not a v-neighborhood of 0. Consequently, there exists a net $\mathscr{N}$ in $L - U$ which converges to 0 in $\langle L, v\rangle$; hence 0 is not an accumulation point of $\mathscr{N}$ in $\langle L, u\rangle$. If $\varphi \circ \mathscr{M}$ is bounded for some subnet $\mathscr{M}$ of $\mathscr{N}$, then the singletons are not closed in $\langle L, v\rangle$ (by (c)). If no $\varphi \circ \mathscr{M}$ is bounded, then the net $\{1/\varphi N_a\}$, where $\mathscr{N} = \langle\{N_a\}, \leqq\rangle$, converges to 0 in $\mathscr{T}$, and hence $\{(1/\varphi N_a)N_a\}$ converges to zero in $\langle L, v\rangle$; on the other hand $\varphi((1/\varphi N_a)\, N_a) = 1$ for each a, and hence the singletons are not closed in $\langle L, v\rangle$ (again by (c)).

3. If $\mathscr{L}$ is a topological linear space over $\mathscr{T} = $ R or $\mathscr{T} = $ C such that the singletons are closed, then $\{r \to rx\} : \mathscr{T} \to \mathscr{L}$ is an embedding for each $x \neq 0$ in $\mathscr{L}$.

4. Neighborhoods in topologized linear spaces. Let L be a linear space over $\mathscr{T}$ where $\mathscr{T} = $ R or $\mathscr{T} = $ C.

(a) A set $X \subset L$ is said to be *absorbing* if for each x in L there exists an $r \geqq 0$ such that $r \geqq r_0$ implies $x \in r \,.\, X$ (or equivalently, $0 < r < r_0$ implies $rx \in X$).

(α) Let u be a closure for L such that the underlying topologized group is inductively continuous. In order that the external multiplication be inductively continuous it is necessary and sufficient that each neighborhood of 0 be absorbing, and if U is a neighborhood of 0 then $r \,.\, U$ be a neighborhood of 0 for each $r > 0$.

(β) Let $\mathscr{U}$ be a filter on L such that

(1) each $U \in \mathscr{U}$ is absorbing (in particular, $0 \in U$);

(2) if $r > 0$ and $U \in \mathscr{U}$ then $r \,.\, U \in \mathscr{U}$ and $-U \in \mathscr{U}$.

There exists a unique closure structure u for L such that $\mathscr{U}$ is the neighborhood system at 0, the underlying topologized group is inductively continuous and the topologized external multiplication is inductively continuous. [Hint: the neighborhood system at x is $x + \mathscr{U}$.]

(b) A set $X \subset L$ is said to be *balanced* if $x \in X$, $|r| \leqq 1 \Rightarrow rx \in X$. Let 0 be the set of all r, $|r| \leqq 1$. For each $X \neq \emptyset$ the set $0.X$ is the smallest balanced set containing X; it is termed the *balanced hull* of X.

(α) If $\langle L, u\rangle$ is a topological linear space, then the collection $\mathscr{U}$ of all balanced neighborhoods of 0 is a local base at 0. $\mathscr{U}$ consists of balanced hulls of neighborhoods of 0, each element of $\mathscr{U}$ is absorbing (see (a)), $U \in \mathscr{U}$, $r > 0 \Rightarrow r.\ U \in \mathscr{U}$, for each U in $\mathscr{U}$ there exists a V in $\mathscr{U}$ with $V + V \subset U$.

(β) Let $\mathscr{U}$ be a filter base on L such that

(1) each element of $\mathscr{U}$ is balanced and absorbing.

(2) If $U \in \mathscr{U}$ then $V + V \subset U$ for some V in $\mathscr{U}$.

Then there exists a unique closure operation admissible for L such that $\mathscr{U}$ is a local base at 0.

(c) A subset X of L is said to be *convex* if $x, y \in X$, $r + s = 1$, $r > 0$, $s > 0 \Rightarrow$ $\Rightarrow rx + sy \in X$. If X and Y are convex then $rX + sY$ is convex for any $r, s \in \mathscr{T}$. Next, if X is convex, then $rX + sX = (r + s)X$ for each $r > 0$, $s > 0$. If $\mathscr{T} = \mathrm{R}$ then a convex set X is balanced if and only if X is symmetric (i.e. $x \in X \Rightarrow -x \in X$). If X is convex then $\sum_{i=1}^{n} r_i x_i \in X$ provided that $r_i \geqq 0$, $\Sigma\{r_i\} = 1$ (such a linear combination is said to be convex).

(α) Let X be a convex and absorbing set (X contains 0). For each $x \in L$ let $A_x =$ $= \mathbf{E}\{r \mid r > 0,\ x \in rX\}$. Clearly $A_x \neq \emptyset$. Let $p = \{x \rightarrow \inf A_x \mid x \in L\}$ (p is said to be the Minkowski functional of X). Prove

$$p0 = 0,\ p(rx) = rpx \text{ if } r > 0,\quad p(x + y) \leqq px + py\,.$$

If X is balanced then

$$p(rx) = |r|\, px$$

for all $x \in \mathscr{T}$, and hence p is a norm. Next

$$x \in X \Rightarrow px \leqq 1, \quad px < 1 \Rightarrow x \in X.$$

If u is a closure admissible for L then

$$p \text{ is continuous} \Leftrightarrow X \text{ is a neighborhood of } 0.$$

(β) Let φ be a norm and $U = \mathbf{E}\{x \mid \varphi x < 1\}$. Then U is a convex, absorbing and balanced set and φ is the Minkowski functional of U. (Hence, if u is an admissible closure for L then φ is continuous if and only if U is open.)

(γ) A space L is said to be *locally convex* if the convex neighborhoods of 0 form a local base at 0. A space L is locally convex if and only if, for any neighborhood U of 0, there exists a continuous norm φ such that $\varphi x < 1 \Rightarrow x \in U$.

(δ) A topological linear space is normable if and only if there exists a convex neighborhood U of 0 such that if V is a neighborhood of 0 then $rV \supset U$ for some r. [Hint: "only if" is evident and to prove "if" consider the Minkowski functional of a bounded, convex and balanced neighborhood $W \subset U$.]

5. The space $S(T)$ (where $T \subset \mathrm{R}^n$ has a positive finite measure). Let S be the real linear space of all measurable functions on T. Then $\varphi = \{x \to \int_T [|x(t)|/1 + |x(t)|] \, dt \mid x \in S\}$ is a norm for the underlying additive group of S, and S endowed with the closure structure induced by φ is a topoiogical linear space.

(a) S is not locally convex (and hence, S is not normable). This is an immediate consequence of the following result.

(b) S is the only convex neighborhood of 0. [Let $r > 0$ and let μ be the measure. We shall show that any $x \in S$ is a convex linear combination of elements with norm $< r$. Let $\{X_i \mid 1 \leqq i \leqq n\}$ be a decompoistion of T such that $\mu X_i < r$ for each i, and let x_i be the function which is 0 outside X_i and $x_i t = n \,.\, xt$ if $t \in X_i$. Clearly $x = \sum_{i=1}^{n} 1/n x_i$ and $\varphi x_i \leqq \int X_i \, dt = \mu X_i < r$.]

6. Proof of 18 D.16. Let A be a closed subalgebra of the normed algebra $\mathbf{F}^*(\mathscr{S}, \mathrm{R})$ where $\mathscr{S}$ is any struct, and assume that A contains all constant functions.

(a) If we know that for each $a, b, r \in \mathrm{R}$, $r > 0$, there exists a polynomial function $P = P_{abr}$ such that $|Px - |x|| < r$ for each $x \in [\![a, b]\!]$, then for any $f \in A$ with $|fx| \leqq C$ for each $x \in \mathscr{S}$, we have $\| |f| - P \circ f \| < r$ with $P = P_{-C,C,r}$, and hence $|f| \in A$.

(b) (Alternate proof.) It is well-known and easily proved that $(1 - r)^{1/2} = \sum_{n=1}^{\infty} a_n r^n$ uniformly on $[\![0, 1]\!]$. If $f \in A$, $|fx| \leqq C$ for each $x \in \mathscr{S}$, then $|fx| = (C^2 - (C^2 - (fx)(fx)))^{1/2} = C(1 - (1 - [(f \,.\, f)\, x/C^2]))^{1/2} = C(1 - \sum_{n=1}^{\infty} a_n(1 - [(f \,.\, f)\, x/C^2])^n)$ uniformly.

(Section 20)

1. Boundary. Let $\langle P, u\rangle$ be a closure space and let bd, called the boundary operation associated with u, be the single-valued relation on exp P ranging in exp P which assigns to each set X the boundary of X in P. The following conditions are fulfilled:

(bd 1) $\mathrm{bd}\, \emptyset = \emptyset$, and $X \subset Y$ implies $\mathrm{bd}\, X \subset Y \cup \mathrm{bd}\, Y$.

(bd 2) $\mathrm{bd}\,(X \cup Y) \subset \mathrm{bd}\, X \cup \mathrm{bd}\, Y$ for each $X \subset P$, $Y \subset P$.

(bd 3) $\mathrm{bd}\, X = \mathrm{bd}\,(P - X)$.

If bd is a single-valued relation on exp P ranging in exp P which satisfies conditions (bd 1) and (bd 2), then

$$u = \{X \to X \cup \mathrm{bd}\, X \mid X \subset P\}$$

is a closure operation for P and the boundary of any subset X in $\langle P, u\rangle$ is $(X \cup \cup\, \mathrm{bd}\, X) \cap ((P - X) \cup \mathrm{bd}\,(P - X)) = (\mathrm{bd}\, X \cap \mathrm{bd}\,(P - X)) \cup (\mathrm{bd}\, X \cap (P - X)) \cup \cup\, (X \cap \mathrm{bd}\,(P - X))$, and hence, if bd fulfils also condition (bd 3), then the boundary of any subset X of P is $\mathrm{bd}\, X$.

2. Let bd be the boundary operation associated with the closure structure of a closure space $\mathscr{P}$. Then (a) a family $\{X_a \mid a \in A\}$ of subsets of P is closure-preserving if and only if $\mathrm{bd} \bigcup\{X_a \mid a \in B\} \subset \bigcup\{\mathrm{bd}\, X_a \mid a \in B\}$ for each $B \subset A$ (the equality is not true; e.g. consider any two distinct closed intervals with a common point in the space R of reals); (b) $\mathrm{bd} \bigcap\{X_a\} \subset \bigcup\{\mathrm{bd}\, X_a\}$ for each finite family $\{X_a\}$ in exp P.

3. If $\mathscr{P} = \langle P, \leqq, u\rangle$ is a connected generalized ordered space, then $\mathscr{P}$ is an ordered space.

4. Let $\langle P, u\rangle$ be the subspace of R such that $P = [\![0, 1]\!] \cup]\!] 2, 3 [\![$. Thus u is a generalized ordered space (17 A.22). There exists no order $\prec$ for P such that u is the order closure for $\langle P, \prec\rangle$. [Hint: assuming that u is the order closure for $\langle P, \prec\rangle$ one finds (20 B.3, Corollary c) that $\prec$ agrees with the usual order or with the inverse of the usual order for R on $[\![0, 1]\!]$ as well as on $]\!] 2, 3 [\![$ and therefore (exactly) one of the following possibilities occurs: either 1 is the supremum or the infimum of $]\!] 2, 3 [\![$ in $\langle P, \prec\rangle$ or 0 is the supremum or the infimum of $]\!] 2, 3 [\![$ in $\langle P, \prec\rangle$.]

5. The empty space is connected and $\emptyset$ is the only component.

6. Let $J_n = (n^{-1}) \times [\![0, 1]\!]$, $n = 1, 2, \ldots$, $x = \langle 0, 0\rangle$, $Q = \bigcup\{J_n \mid n = 1, 2, \ldots\}$, $P = Q \cup (x)$, and let us consider the closure u for P such that u agrees with the closure structure of R $\times$ R on Q, the singleton (x) is closed and the sets $\bigcup\{J_n \mid n \geqq k\} - (\mathrm{R} \times (0))$, $k = 1, 2, \ldots$, form a local base at x. Consider the union Y of the locally finite family $\{(\langle n^{-1}, 0\rangle) \mid n = 1, 2, \ldots\}$. The singletons (x) and (y) are separated for each y in Y but the sets (x) and Y are not separated because the closure of each neighborhood of x intersects Y.

7. Homeomorphisms of connected ordered spaces. (a) A point x of a connected ordered space $\langle P, \leqq, u\rangle$, card $P \geqq 2$, is the greatest element of $\langle P, \leqq\rangle$ or the least element of $\langle P, \leqq\rangle$ if and only if the following condition is fulfilled: there exist arbitrarily small neighborhoods U of x such that bd U is a singleton.

(b) Let $\mathscr{P}_i$, $i = 1, 2, 3$, be connected ordered spaces such that card $|\mathscr{P}_i| \geqq 2$, $\mathscr{P}_1$ has the greatest element as well as the least element, $\mathscr{P}_2$ has the greatest element but not the least element, and $\mathscr{P}_3$ has neither the least element nor the greatest element. Then $\mathscr{P}_i$ is not a homeomorph of $\mathscr{P}_j$, $i \neq j$ (Use (a)). E.g. $]\!]\, 0, 1 [\![$ and $[\![\, 0, 1 [\![$ are not homeomorphic.

(c) There exists a one-to-one continuous mapping f of $\Sigma\{[\![\, 0, 1 [\![\mid n \in \mathsf{N}\}$ onto $]\!]\, 0, 1 [\![$.[Hint: f carries $(n) \times [\![\, 0, 1 [\![$ onto $[\![\, (n + 2)^{-1}, (n + 1)^{-1} [\![$.]

(d) Let $\mathscr{P}$ be the sum of a family consisting of a countable infinite number of singletons and of a countable number of open intervals, and $\mathscr{Q}$ be the sum of a family consisting of a countable infinite number of singletons and also of intervals $[\![\, 0, 1 [\![$. Then there exists a one-to-one continuous mapping of $\mathscr{P}$ onto $\mathscr{Q}$ and a one-to-one continuous mapping of $\mathscr{Q}$ onto $\mathscr{P}$, but the two spaces are not homeomorphic.

8. Assume that a one-to-one sequence $\{x_n\}$ converges to an $x \neq x_n$ in R. Consider a closure u for R such that the subset $X = (x) \cup \mathbf{E}\{x_n\}$ is closed, the subspace X is discrete, $\mathsf{R} - X$ is a subspace of R (in the usual closure structure), and U is a neighborhood of $y \in X$ in $\langle \mathsf{R}, u\rangle$ if and only if $(X - (y)) \cup U$ is a neighborhood of y in R. The family $\{x_n\}$ is locally finite in $\langle \mathsf{R}, u\rangle$, the sets (x_n) and (x) are separated for each n, but the sets $\mathbf{E}\{x_n\}$ and (x) are not separated. Another example is given in 29 B.9.

9. A subset C of a metrizable space P is connected if and only if the following condition is fulfilled: if $X \subset P$ and $C \cap X \neq \emptyset \neq C \cap (P - X)$, then $C \cap \operatorname{bd} X \neq \emptyset$. [Use the fact that int X and int $(P - X)$ are semi-separated.]

10. Connected collections of subsets of a set. A collection $\mathscr{X}$ of subsets of a set P is said to be connected if for each X and Y in $\mathscr{X}$ there exists a finite chain from X to Y, i.e. a finite sequence $\{X_i \mid i \leqq n\}$, $n \in \mathsf{N}$, in $\mathscr{X}$ such that $X = X_0$, $Y = X_n$ and $X_{i-1} \cap X_i \neq \emptyset$ for $i \geqq 1$. A component of a collection $\mathscr{X}$ is a maximal connected subcollection $\mathscr{X}_0$ of $\mathscr{X}$, that is, $\mathscr{X}_0$ is a component of $\mathscr{X}$ if $\mathscr{X}_0$ is connected, and whenever $\mathscr{Y}$ is connected and $\mathscr{X}_0 \subset \mathscr{Y} \subset \mathscr{X}$, then $\mathscr{X}_0 = \mathscr{Y}$. Prove:

(a) Every connected subcollection $\mathscr{X}_1$ of a collection $\mathscr{X}$ is contained in a component of $\mathscr{X}$.

(b) If $\mathscr{X}_1$ and $\mathscr{X}_2$ are two components of a collection $\mathscr{X}$, then $\mathscr{X}_1 = \mathscr{X}_2$ or $(\bigcup\mathscr{X}_1) \cap$ $\cap (\bigcup\mathscr{X}_2) = \emptyset$.

(c) If $\mathscr{X}$ is an interior cover of a closure space $\mathscr{P}$ and $\mathscr{X}_0$ is a component of $\mathscr{X}$, then the set $\bigcup\mathscr{X}_0$ is simultaneously closed and open in P.

(d) A closure space $\mathscr{P}$ is connected if and only if each interior cover of $\mathscr{P}$ is connected (see 20 B.12).

(e) The union of a connected collection of connected subsets of a closure space $\mathscr{P}$ is a connected set in $\mathscr{P}$.

11. A collection $\mathscr{X}$ of subsets of a closure space $\mathscr{P}$ is said to be quasi-connected if for each X and Y in $\mathscr{X}$ there exists a finite sequence $\{X_i \mid i \leqq n\}$ such that $X_0 = X$, $X_n = Y$ and no pair of sets X_{i-1}, X_i, $i \geqq 1$, is semi-separated. Prove the following generalization of **10** (e): The union of a quasi-connected collection of connected subsets of a space $\mathscr{P}$ is a connected set. Prove analogues of statements (a), (b), (c) and (d) of **10** with connected collection replaced by quasi-connected collection (define quasi-components!).

(Section 21)

1. Point sets. A point set is defined to be a struct $\mathscr{X} = \langle X, \mathscr{P} \rangle$ where $\mathscr{P}$ is a closure space and X is a subset of $|\mathscr{P}|$. We have often worked with point sets explicitly (e.g. the relation α of 21 A is a class of point sets) or implicitly (we have spoken about the closure, the interior or the boundary of a set not specifying the space in question, e.g. the closure of the union of two sets is equal to the union of closures, a family $\{X_a \mid a \in A\}$ is closure preserving if and only if the closure of $\bigcup\{X_a \mid a \in B\}$ is equal to the union of $\{\overline{X}_a \mid a \in B\}$ for each $B \subset A$). The concept of a point set enables us to give precise formulations of many definitions and theorems. Of course we must give definitions needed for point sets. Roughly speaking, a point set $\mathscr{X} = \langle X, \mathscr{P} \rangle$ has a property $\mathfrak{P}$ if and only if the set X has the property $\mathfrak{P}$ in $\mathscr{P}$, e.g. $\mathscr{X}$ is open or closed if X is open or closed in $\mathscr{P}$, $\mathscr{Y} = \langle Y, \mathscr{Q} \rangle$ is the closure of $\mathscr{X}$ if $\mathscr{P} = \mathscr{Q}$ and Y is the closure of X in $\mathscr{P}$.

Formal definitions. (a) Let T be the class of all point sets. The closure structure of T is the single-valued relation on T ranging in T which assigns to each $\mathscr{X} = \langle X, \langle P, u \rangle \rangle$ the point set $\langle uX, \langle P, u \rangle \rangle$ which is called the closure of $\mathscr{X}$ and denoted by $\overline{\mathscr{X}}$. Similarly we define the interior operation

$$\operatorname{int} = \{\langle X, \langle P, u \rangle \rangle \to \langle \operatorname{int}_u X, \langle P, u \rangle \rangle\},$$

and the boundary operation

$$\operatorname{bd} = \{\langle X, \langle P, u \rangle \rangle \to \langle \operatorname{bd}_u X, \langle P, u \rangle \rangle\}.$$

(Clearly $\mathscr{X}$ is open or closed if and only if $\operatorname{int} \mathscr{X} = \mathscr{X}$ or $\overline{\mathscr{X}} = \mathscr{X}$ respectively.)

(b) Let $\prec$ be the relation

$$\{\langle X, \mathscr{P} \rangle \to \langle Y, \mathscr{P} \rangle \mid X \subset Y\}$$

for T and T; $\prec$ is an order for T which is called the inclusion. It is easily seen that $\langle T, \prec \rangle$ is boundedly-order complete and, given a space $\mathscr{P}$, the relation $\{X \to \langle X, \mathscr{P} \rangle \mid X \subset |\mathscr{P}|\}$ is a bijective order-preserving relation for $\langle \exp |\mathscr{P}|, \subset \rangle$ and the ordered subset of $\langle T, \prec \rangle$ consisting of all $\langle X, \mathscr{P} \rangle$, $X \subset |\mathscr{P}|$. The supremum of a non-void family $\{\langle X_a, \mathscr{P}\rangle\}$ in $\langle T, \prec\rangle$ is the point set $\langle \bigcup\{X_a\}, \mathscr{P}\rangle$ which is called the union of $\{\langle X_a, \mathscr{P}\rangle\}$ and denoted by $\bigcup\{\langle X_a, \mathscr{P}\rangle\}$. The infimum of a $\{\langle X_a, \mathscr{P}\rangle\}$ is the point set $\langle \bigcap\{X_a\}, \mathscr{P}\rangle$ which is called the intersection of $\{\langle X_a, \mathscr{P}\rangle\}$ and denoted by $\bigcap\{\langle X_a, \mathscr{P}\rangle\}$. It is convenient to define $\bigcap\{\langle X_a, \mathscr{P}\rangle \mid a \in \emptyset\} = \langle |\mathscr{P}|, \mathscr{P}\rangle$. Thus the class of all closed sets is join-stable and completely meet-stable in $\langle T, \prec\rangle$, and the class of all open sets is meet-stable and completely join-stable in $\langle T, \prec\rangle$. These two results can be formulated as follows: the intersection of any family of closed point sets is closed, the union of a finite family of closed sets is a closed set, the union of a family of open sets is open, and the intersection of a finite family of open sets is open.

(c) The complement of a point set $\langle X, \mathscr{P}\rangle$ is defined to be the point set $\langle |\mathscr{P}| - X, \mathscr{P}\rangle$. (Clearly $\mathscr{X}$ is open if and only if its complement is closed.)

(d) $\mathscr{X}$ is a neighborhood of $\mathscr{Y}$ if (and only if) $\mathscr{Y} \prec \operatorname{int} \mathscr{X}$. If $\mathscr{U}$ is the collection of all neighborhoods of a point set $\mathscr{X}$, then $\mathscr{U}$ is a filter in $\langle T, \prec\rangle$ and each element of $\mathscr{U}$ contains $\mathscr{X}$. One can define the neighborhood system of a point set, a local base and a local sub-base of a point set, and formulate the relations between the closure structure of T, int, bd and neighborhoods.

(e) The product (the sum) of a family $\{\langle X_a, \mathscr{P}_a\rangle\}$, denoted by $\Pi\{\langle X_a, \mathscr{P}_a\rangle\}$ ($\Sigma\{\langle X_a, \mathscr{P}_a\rangle\}$, respectively) is defined to be the point set $\langle \Pi\{X_a\}, \Pi\{\mathscr{P}_a\}\rangle$ ($\langle \Sigma\{X_a\}, \Sigma\{\mathscr{P}_a\}\rangle$, respectively). Thus the product of connected point sets is a connected point set, the closure of a sum is the sum of closures and the closure of the product is the product of closures; in symbols

$$\overline{\Sigma\{\mathscr{X}_a\}} = \Sigma\{\overline{\mathscr{X}_a}\}, \overline{\Pi\{\mathscr{X}_a\}} = \Pi\{\overline{\mathscr{X}_a}\}.$$

(f) Localization. A relation α for the class of sets and the class of closure spaces such that $\langle X, \mathscr{P}\rangle \in \alpha$ implies $X \subset |\mathscr{P}|$ is a class of point sets. Let α be a class of point sets. A point set $\mathscr{X}$ is said to be an α-set if $\mathscr{X} \in \alpha$. A point set $\mathscr{X} = \langle X, \mathscr{P}\rangle$ is locally an α-set at x if there exist arbitrarily small neighborhoods $\mathscr{U} = \langle U, \mathscr{P}\rangle$ of $\langle (x), \mathscr{P}\rangle$ such that $\mathscr{X} \cap \mathscr{U}$ is an α-set. A point set $\mathscr{X}$ is locally (relatively locally) an α-set if $\mathscr{X}$ is locally an α-set at each $x \in |\mathscr{P}|$ ($x \in X$). Similar definitions may be formulated for feeble localization.

2. *The closure of a locally connected subset need not be locally connected* (see 21 B.3): Let us consider the subset $P = \Sigma\{[\![0, 1]\!] \mid n \in \mathsf{N}\}$ of $\mathsf{R} \times \mathsf{R}$ and put $X = P - (\langle 0, 0\rangle)$. Let u be the closure operation for P such that the relativization of u to X agrees with the relativization of the closure structure of $\mathsf{R} \times \mathsf{R}$, X is open in $\langle P, u\rangle$, and U is a neighborhood of $\langle 0, 0\rangle$ if and only if U is a neighborhood of $\langle 0, 0\rangle$ in the subspace P of $\mathsf{R} \times \mathsf{R}$ and U contains all $\langle n, 0\rangle$ except for a finite number of n's. Consider the set $Y = \Sigma\{]\!] 0, 1]\!] \mid n \in \mathsf{N}\}$. It is easily seen that Y is locally connected in $\langle P, u\rangle$. Evidently $uY = P$ and $\langle P, u\rangle$ is not locally connected at $\langle 0, 0\rangle$.

3. A quasi-component which is not connected. Let $\mathscr{P}$ be a subspace of $\mathsf{R} \times \mathsf{R}$ such that $|\mathscr{P}| = (\langle 0, 0\rangle, \langle 0, 1\rangle) \cup \bigcup\{P_n \mid n \in \mathsf{N}\}$ where $P_n = (2^{-n}) \times \times [\![0, 1]\!]$. Each set P_n and also $(\langle 0, 0\rangle)$ and $(\langle 0, 1\rangle)$ are components of $\mathscr{P}$. Clearly each P_n is a quasi-component of $\mathscr{P}$. On the other hand neither $(\langle 0, 0\rangle)$ nor $(\langle 0, 1\rangle)$ is a quasi-component. Indeed, if X is a simultaneously open and closed subset of $\mathscr{P}$ containing $\langle 0, 0\rangle$, then X intersects all P_n except for a finite number of n's, and hence, P_n being connected, $P_n \subset X$ for all P_n except for a finite number of n's. Since X is closed, $\langle 0, 1\rangle$ belongs to X.

4. Totally disconnected spaces. Let α consists of all $\langle X, \mathscr{P}\rangle$ such that X is simultaneously open and closed in $\mathscr{P}$. A space $\mathscr{P}$ is said to be totally disconnected if $\mathscr{P}$ is locally an α-set. (a) Every totally disconnected set is topological. (b) A closure space $\mathscr{P}$ is totally disconnected if and only if $\mathscr{P}$ is topological and sets simultaneously open and closed form an open base for $\mathscr{P}$. (c) The class of all totally disconnected spaces is hereditary and closed under arbitrary products. (d) A space $\mathscr{P}$ with closed singletons is totally disconnected if and only if $\mathscr{P}$ is homeomorphic to a subspace of $2^{\aleph}$ for some cardinal $\aleph$ where 2 denotes the two point set $(0, 1)$ endowed with the discrete closure. [Hint: "If" is evident (see (c)), and "only if" is proved as follows: Let $\mathscr{B}$ be the set of all simultaneously open and closed subsets of $\mathscr{P}$, and for each B in $\mathscr{B}$ let f_B be the mapping of $\mathscr{P}$ into 2 such that $f_B x = 1$ if $x \in B$ and $f_B x = 0$ otherwise. Each f_B is continuous, and the reduced product f of $\{f_B \mid B \in \mathscr{B}\}$ is an embedding provided that $\mathscr{B}$ is an open base for $\mathscr{P}$ and the functions f_B distinguish the points of $\mathscr{P}$.] (e) The following statements are equivalent ($\mathscr{P}$ is totally disconnected): X is an α-set in $\mathscr{P}$; X is locally an α-set in $\mathscr{P}$; X is feebly locally an α-set in $\mathscr{P}$.

5. The collection of all open (closed) sets in a topological space is locally determined.

6. Let α be a relation such that $\langle Y, \mathscr{Q}\rangle \in \alpha$ implies that $\mathscr{Q}$ is a space and $Y \subset |\mathscr{Q}|$. Given a space $\mathscr{P} = \langle P, u\rangle$ and $X \subset P$, let X'' (X', respectively) denote the set of all x such that X is locally (feebly locally) an α-set at x in $\mathscr{P}$. Clearly $X'' \subset X'$. Prove:

(a) If X is an α-set in $\mathscr{P}$, then $X' = P$ but X'' may be empty.

(b) If $\emptyset$ is an α-set in $\mathscr{P}$, then $X' \supset X'' \supset P - uX$; if $\emptyset$ is the only α-set in $\mathscr{P}$, then $X'' = X' = P - uX$.

(c) If $\mathscr{P}$ is topological, then X' is open but X'' need not be; however, if each relatively open subset of an α-set is an α-set (in particular, if α is hereditary), then $X'' = X'$ is also open.

(d) If α is hereditary, and $X \subset Y \subset P$, then $Y' = Y'' \subset X'' = X'$.

(e) Let X^* stand for $P - X'$. If $\mathscr{P}$ is topological, α is hereditary and $\emptyset$ is an α-set in $\mathscr{P}$, then $X \subset Y$ implies $X^* \subset Y^*$, $(X^*)^* \subset X^* = u(X^*) \subset uX$, and if G is open then $G \cap X^* = G \cap (G \cap X)^*$. If, in addition, α is additive (i.e. the union of two α-sets is always an α-set), then $(X \cup Y)^* = X^* \cup Y^*$.

(Section 22)

1. If $\aleph_0 \leqq n \leqq \exp m$, $\mathscr{P}$ is a topological space the total character of which is $\leqq m$, then the density character of $\mathscr{P}^n$ is at most m.

2. The cardinal of any infinite point-finite family of non-void open sets is at most the density character. The cardinal of any disjoint family of non-void open sets is at most the density character.

3. If m is the total character of a space $\mathscr{P}$ then each interior cover of $\mathscr{P}$ contains an interior cover of a cardinal $\leqq m$. In particular, if $\mathscr{P}$ has a countable total character, then each interior cover contains a countable interior cover (a space with the last property is said to be a Lindelöf space).

Let $\mathscr{P}$ be the set of all reals endowed with the closure of right-approximation. Each subspace $\mathscr{R}$ of $\mathscr{P}$ is of a countable density character, and any open cover of $\mathscr{R}$ contains a countable subcover. First prove that if $\mathscr{X}$ is a collection of at least two-point intervals in R, then $\bigcup \mathscr{X} = \bigcup \mathscr{Y}$ for some countable subcollection $\mathscr{Y}$ of $\mathscr{X}$. $\mathscr{P}$ is totally disconnected.

4. A pseudometrizable space is compact if and only if it is countably compact. [Hint: If a pseudometric space $\langle P, d\rangle$ is countably compact then $\langle P, d\rangle$ contains a countable dense set (if each two points of a set X has the distance $\geqq r > 0$ then X is finite), hence $\langle P, d\rangle$ has a countable total character, and hence, each interior cover contains a countable interior cover.]

If a semi-pseudometric space $\langle P, d\rangle$ is countably compact then there exists a countable $X \subset P$ such that each sphere about X is dense.

5. A topological space $\mathscr{P}$ is locally non-meager if and only if $\bigcap \mathscr{U}$ is dense for any countable collection $\mathscr{U}$ of open dense sets. [Hint: If U is open and dense then $|\mathscr{P}| - U$ is nowhere dense.]

6. Complete pseudometric spaces. Let $\langle P, d\rangle$ be a pseudometric space. A Cauchy net is a (directed) net $\langle N, \leqq\rangle$ such that for each $r > 0$ there exists an a such that $b, c \geqq a$ implies $d\langle N_b, N_c\rangle < r$. A proper filter $\mathscr{X}$ is a Cauchy filter if $\mathscr{X}$ contains arbitrarily small sets, i.e. for each $r > 0$ there exists an X in $\mathscr{X}$ such that $d(X) < r$. If x is an accumulation point (cluster point) of a Cauchy net (Cauchy filter) $\mathscr{N}$ then x is a limit point of $\mathscr{N}$. $\langle P, d\rangle$ is said to be complete if the following equivalent conditions are fulfilled:

(a) Each Cauchy filter has a cluster (or equivalently, a limit) point.
(b) Each Cauchy net has an accumulation (or equivalently, a limit) point.
(c) Each Cauchy sequence has an accumulation (or equivalently, a limit) point.

Any closed subspace of a complete pseudometric space is complete.

Each complete pseudometric space is a Baire space. [Hint: Let $\{U_n\}$ be a sequence of open dense sets; choose open V_n such that $\emptyset \neq \overline{V}_{n+1} \subset \bigcap\{U_i \mid i \leqq n\} \cap V_n$, $d(V_{n+1}) < (n + 1)^{-1}$.]

7. The set I of all irrationals is not $\mathbf{F}_\sigma$ in R. [Hint: Each closed $F \subset I$ is nowhere dense and I is not meager because $\mathsf{Q} = \mathsf{R} - I$ is meager and R is non-meager.] There exists an uncountable disjoint family of countable dense sets in R.

The space $\langle P, u \rangle$ in 18 ex. 9 cannot be semi-metrized in such a manner that the open spheres are open. [Hint: Let d semi-metrize $\langle P, u \rangle$; there exist an $r > 0$ and a non-meager $X \subset \mathsf{R} - \mathsf{Q}$ such that, for each z in $X \times (0)$, the r-sphere $U(z, r)$ about z in $\langle P, d \rangle$ is contained in $S(z, 1)$. Let $x \in \mathsf{Q}$ be a cluster point of X; choose $\langle x, y \rangle$ with $d \langle \langle x, 0 \rangle, \langle x, y \rangle \rangle < \frac{1}{2}r$; then $\langle x, 0 \rangle$ is not an interior point of $U(\langle x, y \rangle, \frac{1}{2}r)$.]

8. (a) A normed linear space is said to be complete if the pseudometric given by the norm is complete. The normed space $\mathbf{F}^*(\mathscr{P}, \mathsf{R})$ is complete for any space $\mathscr{P}$. [If $\{f_n\}$ is a Cauchy sequence, then $\{f_n x\}$ is a Cauchy sequence in R for each x; if fx is the unique limit point of $\{f_n x\}$, then f is a limit point of $\{f_n\}$.]

(b) There exists a continuous function on $I = [\![0, 1]\!]$, which does not have a derivative at any point. [Proof. Let C be the subspace of $\mathbf{C}^*(I, \mathsf{R})$ consisting of all f with $f0 = f1$, and for every f in C let f^* be the unique extension of f to R with $f^*(x + 1) = f^*x$ for all x. For each n let F_n be the set of all $f \in C$ such that there exists an x in I with

$$|f^*(x + h) - f^*x|/h \leqq n$$

for all $h > 0$. Each F_n is closed (if $\{f_n\}$ converges to f in C, and f_n fulfils the above inequality at x_n, and if x is an accumulation point of $\{x_n\}$ in I, then f and x satisfy the inequality, and hence $f \in F_n$) and nowhere dense (this is obvious). Since C is locally non-meager (C is closed in $\mathbf{C}^*(I, \mathsf{R})$), $C - \bigcup\{F_n\} \neq \emptyset$.]

9. If $\langle P, d \rangle$ is a semi-pseudometric locally non-meager space such that each open sphere in $\langle P, d \rangle$ is an open set, and each sphere about an $X \subset P$ is dense, then X is dense. The assumption that $\langle P, d \rangle$ is a locally non-meager space cannot be omitted; exhibit an example with X a singleton. Corollary: T_{ω_1} cannot be semi-metrized in such a manner that each open sphere is an open set. (T_{ω_1} is a countably compact Baire space with an uncountable density character, see ex. 4.)

10. Classification of Borel sets. (a) Let P be a set and let ϱ_1 and ϱ_2 be single-valued relations which assign to each countable family in $\exp P$ a subset of P. Assume that $\varrho_i\{X_b \mid b \in (a)\} = X_a$ for each a. Let $\mathscr{X}$ be a collection of subsets of P. If $\mathscr{Y} \subset \exp P$, let $\varrho_i[\mathscr{Y}]$ be the collection of all $\varrho_i\{X_a\}$ with $\{X_a\}$ ranging in $\mathscr{Y}$. Let us define $\mathscr{X}_0 = \mathscr{X}$ and $\mathscr{X}_\alpha = \varrho_i[\bigcup\{\mathscr{X}_\beta \mid \beta < \alpha\}]$, where α, β are ordinals and $i = 1$ if α is odd and $i = 2$ otherwise (each ordinal α can be written in a unique manner in the form $\gamma + n$ where γ is a limit ordinal and $n \in \mathsf{N}$; α is odd if n is odd). Prove that $\bigcup\{\mathscr{X}_\alpha \mid \alpha < \omega_1\} = \mathscr{X}_{\omega_1 + 1} = \mathscr{X}_{\omega_1}$. Hence $\mathscr{X}_{\omega_1}$ is stable under both ϱ_1 and ϱ_2. If α is odd then $\varrho_1[\mathscr{X}_\alpha] = \mathscr{X}_\alpha$, if α is even then $\varrho_2[\mathscr{X}_\alpha] = \mathscr{X}_\alpha$ provided that $\varrho_i \circ \varrho_i \subset \varrho_i$.

(b) If $\mathscr{P}$ is a topological space, $\mathscr{X}$ is the set of all closed sets, $\varrho_1 = \bigcup$ (precisely, $\varrho\{X_a\} = \bigcup\{X_a\}$ for each $\{X_a\}$) and $\varrho_2 = \bigcap$, then $\mathscr{X}_\alpha$ is denoted by $\mathbf{F}_\alpha$ (more precisely

$\mathbf{F}_\alpha(\mathscr{P})$). If $\mathscr{X}$ is the collection of all open sets, $\varrho_1 = \bigcap$, $\varrho_2 = \bigcup$, then $\mathscr{X}_\alpha$ is denoted by $\mathbf{G}_\alpha$ (more precisely, $\mathbf{G}_\alpha(\mathscr{P})$).

(c) If $\mathbf{F}_0(\mathscr{P}) \subset \mathbf{G}_{\omega_1}(\mathscr{P})$ or $\mathbf{G}_0(\mathscr{P}) \subset \mathbf{F}_{\omega_1}(\mathscr{P})$, then $\mathbf{F}_{\omega_1} = \mathbf{G}_{\omega_1}$ is the collection of all Borel sets. If $\mathscr{P}$ is pseudometrizable, then $\mathbf{G}_0 \subset \mathbf{F}_1$ and $\mathbf{F}_0 \subset \mathbf{G}_1$. The families $\{\mathbf{F}_\beta \mid \beta < \omega_1\}$ and $\{\mathbf{G}_\beta \mid \beta < \omega_1\}$ are said to be the Borel classifications of the Borel sets.

11. Let $\mathscr{X}$ be a collection of subsets of a space $\mathscr{P}$, $\mathscr{X}_\sigma(\mathscr{X}_\delta)$ be the collection of all countable unions (countable intersections) of sets in $\mathscr{X}$. The family $\mathscr{X}_\sigma(\mathscr{X}_\delta)$ is countably additive (countably multiplicative), and if $\mathscr{X}$ is multiplicative (additive) then $\mathscr{X}_\sigma$ $(\mathscr{X}_\delta)$ is multiplicative (additive). If $\mathscr{X}$ is closed under locally finite unions then $\mathscr{X}_\sigma$ is closed under locally finite unions, and hence under σ-locally finite unions. If $\mathscr{X}$ is multiplicative and closed under locally finite unions, and $\{X_a\}$ is locally finite in $\mathscr{X}_\delta$, then $\bigcup\{X_a\} \in \mathscr{X}_\delta$ provided $X_a \subset Y_a$ for some locally finite family $\{Y_a\}$ in $\mathscr{X}$.

12. A σ-point-finite family is point-countable and the converse is not true ($\{[\![x, \rightarrow]\!] \mid x \in T_{\omega_1}\}$ is locally countable but not σ-point-finite because each point-finite subfamily is finite.)

(Sections 23 – 25)

1. Adjacent nets. Two nets $\mathscr{N} = \langle N, \prec\rangle$ and $\mathscr{M} = \langle M, \leqq\rangle$ in a semi-uniform space $\mathscr{P} = \langle P, \mathscr{U}\rangle$ are said to be adjacent if $\prec \, = \, \leqq$ (in particular $\mathbf{D}N = \mathbf{D}M$) and the net $\langle\{\langle N_a, M_a\rangle \mid a \in \mathbf{D}N\}, \prec\rangle$ is eventually in each $U \in \mathscr{U}$. A $\mathscr{P}$-adjacent (or $\mathscr{U}$-adjacent) pair is a pair $\langle\mathscr{N}, \mathscr{M}\rangle$ such that the nets $\mathscr{N}$ and $\mathscr{M}$ are adjacent in $\mathscr{P}$. Let $\mathscr{C}(\mathscr{P})$ be the class of all $\mathscr{P}$-adjacent pairs of directed nets. Then

(a) $\mathscr{C}(\mathscr{P})$ is a symmetric relation;

(b) $\mathscr{C}(\mathscr{P})$ is transitive if and only if $\mathscr{P}$ is a uniform space;

(c) $\mathscr{C}(\mathscr{P}) = \mathscr{C}(\mathscr{Q})$ implies $\mathscr{P} = \mathscr{Q}$;

(d) two subsets X and Y of $\mathscr{P}$ are proximal if and only if there exists an $\langle\mathscr{N}, \mathscr{M}\rangle$ in $\mathscr{C}(\mathscr{P})$ such that $\mathscr{N}$ ranges in X and $\mathscr{M}$ ranges in Y;

(e) a net $\mathscr{N}$ converges to x in $\mathscr{P}$ if and only if $\langle\mathscr{N}, \mathscr{M}\rangle \in \mathscr{C}(\mathscr{P})$ where $\mathscr{M}$ ranges in (x) and the ordered domains of $\mathscr{M}$ and $\mathscr{N}$ coincide.

2. Uniform local bases. Let P and A be sets, $A \neq \emptyset$, and let ξ be a relation on $P \times A$ ranging in $\exp P$ such that (1) $x \in \xi_{\langle x,a\rangle}$ for each x, (2) for any a_1 and a_2 in A there exists an a in A such that $\xi_{\langle a,x\rangle} \subset \xi_{\langle a_1,x\rangle} \cap \xi_{\langle a_2,x\rangle}$ for each x, (3) for each a in A there exists a b in A such that $y \in \xi_{\langle b,x\rangle}$ implies $x \in \xi_{\langle a,y\rangle}$. For each a let U_a be the sum of $\{\xi_{\langle a,x\rangle} \mid x \in P\}$. Then the collection of all U_a is a base for a semi-uniformity $\mathscr{U}$. The collection of all $\xi_{\langle a,x\rangle}$, $a \in A$, is a local base at x in $\langle P, \mathscr{U}\rangle$. Discuss interrelations between semi-uniformities and uniform local bases. Find a necessary and sufficient condition for a uniform local base to determine a uniformity.

3. Let $\langle P, d\rangle$, $P = [\![0, 1]\!]$, be a subspace of the metric space of reals. Let $D\langle y, x\rangle = D\langle x, y\rangle = 1$ if $x = 1/n$, $y = 1/(n + 1)$, $n = 1, 2, \ldots$, and $D\langle x, y\rangle = d\langle x, y\rangle$ otherwise. Evidently D is a semi-metric. Prove:

(a) The mapping $f = \mathrm{J} : \langle P, D\rangle \to \langle P, d\rangle$ is a uniformly continuous proximal homeomorphism, but f is not a uniform homeomorphism. Thus d and D are proximally but not uniformly equivalent.

(b) Both d and D are totally bounded (hence d is proximally coarse), but D is not proximally coarse.

4. (a) The open cover $\mathscr{U} = \{<^{-1}[\alpha] \mid \alpha < \omega_1\}$ of T_{ω_1} is not semi-uniformizable. [Hint: Let V be an inductive neighborhood of the diagonal such that $\{V[x]\}$ refines $\mathscr{U}$. Choose a sequence $\{x_n\}$ such that $x_{n+1} > \sup V[x_n]$ for each n. If x is an accumulation point of $\{x_n\}$ then $x \notin \bigcup\{V[x_n]\}$.]

(b) A semi-uniformizable cover need not be uniformizable. E.g. consider a semi-uniformizable cover $(P - (x), P - (y))$, $x \neq y$, of the space in 14 A.5 (e). See also 28 ex. 11.

(c) A semi-uniformizable vicinity of the diagonal need not be a neighborhood of the diagonal, and hence not a uniformizable vicinity. Exhibit such a vicinity on R.

5. Every interior cover of a pseudometrizable closure space is uniformizable. [Hint: It is sufficient to show that every interior cover of a pseudometrizable space is star-refined by an interior cover. Let $\mathscr{U}$ be any interior cover of a pseudometrizable space $\langle P, d\rangle$; for each x in P let r_x be a positive real and X_x be an element (or a member) of $\mathscr{U}$ such that the r_x-sphere at x is contained in X_x, and let Y_x be the open $r_x/4$-sphere at x. Thus $\mathscr{Y} = \{Y_x \mid x \in P\}$ is a star-refinement of $\mathscr{U}$. In fact, given x, let $r = \sup\{r_y \mid x \in Y_y\}$, and choose a z with $r_z > \frac{3}{4}r$; show that $\operatorname{st}(\mathscr{Y}, x) \subset X_z$.]

6. Two constructions of uniform modification. (a) Let $\mathscr{U}$ be a semi-uniformity for a set P. Define $\mathscr{U}_0 = \mathscr{U}$, $\mathscr{U}_{\alpha+1}$ is a semi-uniformity having the collection of all $U \circ U$, $U \in \mathscr{U}_\alpha$, for a base, $\mathscr{U}_\alpha = \bigcap\{\mathscr{U}_\beta \mid \beta < \alpha\}$ if α is a limit ordinal. There exists an α such that $\mathscr{U}_\alpha = \mathscr{U}_{\alpha+1}$; $\mathscr{U}_\alpha$ is the uniform modification of $\mathscr{U}$.

(b) The collection of all $\bigcup\{U_{\varrho n} \circ \ldots \circ U_{\varrho 0} \mid n \in \mathrm{N}, \varrho \text{ bijective on } \mathrm{N}_{n+1}\}$ with U_i in $\mathscr{U}$ is a base for the uniform modification of $\mathscr{U}$.

(c) If $\{\mathscr{U}_a\}$ is a family of uniformities for a set P, then $\sup\{\mathscr{U}_a\} = \bigcap\{\mathscr{U}_a\}$ need not be a uniformity.

7. Semi-uniformizable modification. For each closure operation u on P there exists a finest semi-uniformizable closure v coarser than u. A mapping $f : \langle P, u\rangle \to \mathscr{Q}$, with $\mathscr{Q}$ semi-uniformizable, is continuous if and only if $f : \langle P, v\rangle \to \mathscr{Q}$ is continuous.

To each closure u for P there exists a coarsest semi-uniformizable closure w for P finer than u; if $\mathscr{Q}$ is semi-uniformizable and if a mapping $f : \mathscr{Q} \to \langle P, u\rangle$ is continuous, then $f : \mathscr{Q} \to \langle P, w\rangle$ need not be continuous.

8. Proximal neighborhoods. Let $\prec$ be a relation for exp P such that (1) $\emptyset \prec X$ for each $X \subset P$; (2) $X \prec Y \Rightarrow X \subset Y$; (3) If $X \subset X_1 \prec Y_1 \subset Y$ then $X \prec Y$; (4) if $X \prec Y_i$, $i = 1, 2$, then $X \prec Y_1 \cap Y_2$; (5) $X \prec Y \Rightarrow (P - Y) \prec (P - X)$. Then there exists a unique proximity p for P such that $X \prec Y$ if and only if Y is a p-proximal neighborhood of X.

9. Let P be an infinite set, $x_0 \in P$. Let XpY if and only if $x_0 \in X \cup Y$ and $X \neq \emptyset \neq Y$ or $X \cap Y \neq \emptyset$; p is a proximity. Let u be the induced closure. A set Y is dense if and only if $x_0 \in Y$ or $P - Y = (y_0)$. Let $d\langle x, y\rangle = 1$ if $x \neq x_0 \neq y$ and $d\langle x, y\rangle = 0$ otherwise. Clearly d induces p.

(a) The constant mapping $\{y \to x_0\} : \mathscr{S} \to \langle P, d\rangle$ is dense in unif $\mathbf{F}(\mathscr{S}, \langle P, d\rangle)$ for each struct $\mathscr{S}$. Hence the uniform limit of a constant net of uniformly continuous mappings need not be continuous.

(b) d is totally bounded but not proximally coarse.

(c) Each interior cover of $\langle P, d\rangle$ is a uniform cover (it contains P). Hence $\langle P, d\rangle$ and the uniform modification of $\langle P, d\rangle$ have the same semi-uniform covers. In particular, two distinct semi-uniform spaces may have the same semi-uniformizable covers.

(d) Let $\mathscr{U}$ be the uniformly finest proximally coarse uniformity for P and let $\mathscr{V}$ consist of all $U \cup X_k \cup X_k^{-1}$, $k \in \mathsf{N}$, $U \in \mathscr{U}$, where $X_k = \{x_n \to x_{n+1} \mid n \geq k\}$ and $\{x_n\}$ is a one-to-one sequence in P; $\mathscr{U}$ is proximally coarse but $\mathscr{V}$ not.

10. A semi-metric for reals. Let $d\langle x, y\rangle = 0$ if $x = y$ and $d\langle x, y\rangle = 1/|x - y|$ otherwise. Evidently d is a semi-metric for R.

(a) A set $X \subset \mathsf{R}$ is open in $\langle \mathsf{R}, d\rangle$ if and only if X contains a set of the form $X(n) = [\![\leftarrow, n]\!] \cup [\![n, \to]\!]$. The space $\langle \mathsf{R}, d\rangle$ is topological.

(b) If a net $\mathscr{N}$ is eventually in each $X(n)$ (or equivalently, $\mathscr{N}$ has no accumulation point in the space R of reals), then $\mathscr{N}$ converges to each point in $\langle \mathsf{R}, d\rangle$.

(c) The function $d : \langle \mathsf{R}, d\rangle \times \langle \mathsf{R}, d\rangle \to \mathsf{R}$ is not continuous.

(d) If $U = P \times P - (0) \times [\![\leftarrow, 0 [\![$, then $\overline{X} \subset U[X]$ for each $X \subset \mathsf{R}$. On the other hand, U is not a semi-neighborhood of the diagonal.

(e) The semi-pseudometric d is totally bounded; however the semi-uniformity is not proximally coarse.

(f) Every bounded mapping (i.e., the range is contained in a bounded interval) of a semi-uniform space into $\langle P, d\rangle$ is a uniform limit of constant mappings. Hence the uniform limit of uniformly continuous mappings into a semi-uniform space need not be continuous.

(g) If $\langle P, d\rangle$ is any semi-metric space, then D defined by $D\langle x, y\rangle = 0$ if $d\langle x, y\rangle = 0$ and $D\langle x, y\rangle = (d\langle x, y\rangle)^{-1}$ otherwise, is a semi-metric for P. Discuss the properties of D.

11. If $\mathscr{P}$ is a uniform space and $\mathscr{Q}$ is a proximity space, then the set $\mathbf{P}(\mathscr{Q}, \mathscr{P})$ is closed in unif $\mathbf{F}(\mathscr{Q}, \mathscr{P})$.

12. A proximally continuous mapping f of a pseudometric P space into a uniform space $\mathscr{Q}$ is uniformly continuous (if d is a uniformly continuous pseudometric on $\mathscr{Q}$ then $f : \mathscr{P} \to \langle |\mathscr{Q}|, d \rangle$ is proximally continuous and hence uniformly continuous).

13. Extensions of uniformly continuous pseudometrics and functions. (a) Any semi-uniform space $\langle Q, \mathscr{V} \rangle$ is a subspace of a semi-uniform space $\langle P, \mathscr{U} \rangle$ such that the relativization of any uniformly continuous pseudometric for $\langle P, \mathscr{U} \rangle$ to $\langle Q, \mathscr{V} \rangle$ is zero. (b) Let $\langle \mathsf{N}, \mathscr{U} \rangle$ be a subspace of R. Since $\mathscr{U}$ is uniformly discrete, each function or pseudometric on $\langle \mathsf{N}, \mathscr{U} \rangle$ is uniformly continuous. If f is a function on $\langle \mathsf{N}, \mathscr{U} \rangle$ such that the sequence $\{|fn - f(n+1)|\}$ is not bounded, then f has no uniformly continuous extension on R. (c) If $d\langle x, y \rangle = |fx - fy|$ with f in (b), then d has no uniformly continuous extension on R.

14. The Urysohn procedure. The fact that in a proximity space $\langle P, p \rangle$ satisfying (prox 5) for each two distant sets X and Y there exists a proximally continuous function f, $0 \leqq f \leqq 1$, such that $f[X] \subset (0)$ and $f[Y] \subset (1)$ can be proved as follows. From (prox 5) we obtain at once that there exists a family X_r, where r varies over all rational dyadic numbers, such that $X_0 = X$, $X_1 = P - Y$, $X_r = P$ if $r > 1$, X_r is a proximal neighborhood of $X_{r-2^{-n}}$, and X_r is distant from $P - X_{r+2^{-n}}$ (where $r = (2p+1)/2^n$). For each $x \in P$ let $fx = \inf \{r \mid x \in A_r\}$. It is easily seen that $f :$ $: \langle P, p \rangle \to \mathsf{R}$ is proximally continuous.

15. The Stone-Weierstrass theorem is not true for complex functions. Let $\mathscr{P}$ be the subspace of the proximity space C with underlying set $\mathbf{E}\{x + iy \mid x^2 + y^2 = 1\}$. $f = \mathsf{J} : \mathscr{P} \to \mathsf{C}$ is a proximal embedding and hence f projectively generates $\mathscr{P}$. The function $g : \{x + iy \to x - iy\} : \mathscr{P} \to \mathsf{C}$ is proximally continuous but $\|g - F\| \geqq$ $\geqq r > 0$ for each polynomial function F. Find r.

16. Linear spaces. A pseudometric d on a module $\mathscr{L}$ is said to be invariant if d is invariant with respect to the underlying group. $\mathscr{L}$ is pseudometrizable if and only if it is induced by an invariant pseudometric. A pseudometrizable $\mathscr{L}$ need not be locally convex (see 19 ex. 5).

17. Let $\mathscr{H}(\mathscr{P}, \mathscr{P})$ denote the set of all uniform homeomorphisms of a uniform space $\mathscr{P}$ onto itself endowed with the group-structure $\circ$ (the restriction of the composition of mappings) and the closure of uniform convergence (i.e. the closure structure is a relativization of the closure structure of unif $\mathbf{F}(\mathscr{P}, \mathscr{P})$).

(a) $\mathscr{H}(\mathscr{P}, \mathscr{P})$ is a topological group (in general not commutative).

(b) If $\mathscr{P} = \mathsf{R}$ then $\mathscr{U}_R \neq \mathscr{U}_L$ (and hence $\mathscr{U}_R \neq \mathscr{U} \neq \mathscr{U}_L$), and the group multiplication $\{\langle f, g \rangle \to f \circ g\}$ is uniformly continuous with respect to no of the group uniformities. [Consider the subgroup consisting of "lines", i.e. mappings of the form $\{x \to ax + h\} : \mathsf{R} \to \mathsf{R}$.]

18. Prove the assertion in the remark of 25 B.18. [Hint: In the notation of Theorem 25 B.18, fix an α in A and take a family $\{\langle P_a, \mathscr{V}_a \rangle\}$ of uniform spaces such that $\mathscr{V}_a = \mathscr{U}_a$ for $a \neq \alpha$ and that $\mathscr{V}_\alpha$ is proximally equivalent to $\mathscr{U}_\alpha$. Since the intersection of two proximal vicinities, one of which is finite square, is a proximal vicinity, it follows easily that the product uniformities $\prod\{\mathscr{U}_a\}$ and $\prod\{\mathscr{V}_a\}$ are proximally equivalent.]

(Section 26)

1. Let $\langle P, u\rangle$ be a closure space. Each of the following conditions is necessary and sufficient for a closure v for the set P to be the quasi-discrete modification of u:

(a) v is the coarsest quasi-discrete closure operation finer than u;

(b) a mapping f of a quasi-discrete space $\mathscr{Q}$ into $\langle P, u\rangle$ is continuous if and only if the mapping $f : \mathscr{Q} \to \langle P, v\rangle$ is continuous.

2. The quasi-discrete modification (as a relation) commutes with the topological modification, that is, the quasi-discrete modification of the topological modification of a space $\mathscr{P}$ coincides with the topological modification of the quasi-discrete modification of $\mathscr{P}$. The quasi-discrete modification also commutes with the operation of forming subspaces.

3. Every quasi-discrete space is locally connected. [The star of a connected set X is connected as the union of connected sets (y, x), $y \in X$, $y \in \overline{(x)}$, each of which intersects a connected set, namely X.]

4. A quasi-discrete semi-uniformizable space $\mathscr{P}$ is connected if and only if the topological modification of $\mathscr{P}$ is an accrete space. More generally, the topological modification of a quasi-discrete semi-uniformizable space is a homeomorph of the sum of a family of accrete spaces.

5. A closure space $\mathscr{P}$ is semi-uniformizable and its quasi-discrete modification is topological if the following condition is fulfilled: if U is a neighborhood of a point x and $y \in \overline{(x)}$, then U is a neighborhood of y. The condition is also necessary provided that $\mathscr{P}$ is quasi-discrete.

6. A quasi-discrete space is discrete if and only if it is semi-separated.

7. Coarse semi-separated closures (see 26 B.8).

(a) A mapping f of a closure space $\mathscr{P}$ into a coarse semi-separated space $\mathscr{Q}$ is continuous if and only if the sets $f^{-1}[y]$, $y \in |\mathscr{Q}|$, are closed.

(b) Two coarse semi-separated spaces are homeomorphic if and only if they are equipollent.

8. (a) If f is a continuous mapping of an infinite coarse semi-separated space $\mathscr{P}$ onto a semi-separated space $\mathscr{Q}$, then card $|\mathscr{Q}| = 1$ or card $|\mathscr{P}| =$ card $|\mathscr{Q}|$. [Hint: either all inverse fibres of f are finite or f is constant.]

(b) If f is an embedding of an infinite coarse semi-separated space $\mathscr{P}$ into a product space $\Pi\{\mathscr{P}_a\}$ such that all $\mathscr{P}_a$ are semi-separated, then card $|\mathscr{P}| \leq$ card $|\mathscr{P}_a|$ for some a. [Hint: consider the mapping $\mathrm{pr}_a \circ f : \mathscr{P} \to \mathscr{P}_a$ and apply (a).]

(c) The set Q in 26 B.10 can be chosen such that card Q = card P (this follows from the proof 26 B.10) but cannot be chosen such that card Q < card P (this follows from (b).

(d) Let $\mathcal{Q}$ be an infinite coarse semi-separated space and let $\aleph$ be an infinite cardinal. A topological semi-separated space $\mathcal{P}$ admits an embedding into $\mathcal{Q}^{\aleph}$ provided that card $|\mathcal{P}| \leq$ card $|\mathcal{Q}|$ and the total character of $\mathcal{P}$ is at most $\aleph$.

9. Let $\mathcal{P}$ be a coarse semi-separated space. The density character of $\mathcal{P}$ is min ($\aleph_0$, card $|\mathcal{P}|$).The total character of $\mathcal{P}$ is card $|\mathcal{P}|$. The local character of $\mathcal{P}$ is 0 or 1 (if $|\mathcal{P}|$ is finite) or card $|\mathcal{P}|$ (if $|\mathcal{P}|$ is infinite).

10. In an infinite coarse semi-separated space, a set X is dense whenever its interior is non-void.

11. Every continuous function on an infinite coarse semi-separated space is constant.

12. Every coarse semi-separated space is compact in the sense of 17 ex. 5.

13. The box-product of any family of quasi-discrete spaces is a quasi-discrete space.

(Section 27)

1. Prove the necessity of condition (a) of 27 A.7 without Theorem 27A.6. [For each a_1 and a_2 in A let $D_{a_1a_2} = \mathbf{E}\{x \mid \pi_{a_1}x = \pi_{a_2}x\}$. Prove that each $D_{a_1a_2}$ is closed, and D is the intersection of all $D_{a_1a_2}$, $a_i \in A$.]

2. If f is the identity mapping of a dense subspace $\mathcal{Q}$ of a separated space $\mathcal{P}$ onto $\mathcal{Q}$, then f has no continuous domain-extension on any subspace $\mathcal{R}$ of $\mathcal{P}$, $\mathcal{R} \neq \mathcal{Q}$. [If g is a continuous domain-extension on $\mathcal{R}$, then the mapping $h = \{x \to g_x\} : \mathcal{R} \to \mathcal{R}$ is identity on a dense subspace of $\mathcal{R}$, namely on $\mathcal{Q}$, and hence h is the identity mapping. Thus $\mathcal{R} = \mathcal{Q}$.]

3. A closure space $\mathcal{P}$ is said to be strongly separated if for each two distinct points x and y there exist neighborhoods U of x and V of y such that $\overline{U} \cap \overline{V} = \emptyset$. Prove: every regular separated space is strongly separated and every strongly separated space is separated. A separated space which is not strongly separated: Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be disjoint dense subspaces of a separated space $\mathcal{R}$ and let $P = |\mathcal{Q}_1| \cup ((0) \times |\mathcal{Q}_2|) \cup ((1) \times |\mathcal{Q}_2|)$ and let u be the closure for P such that $(i) \times \mathcal{Q}_2$ are open subspaces of $\langle P, u\rangle$, $\mathcal{Q}_1$ is a subspace of $\langle P, u\rangle$ and $x \in u((i) \times X)$, where $X \subset |\mathcal{Q}_2|$, $x \in |\mathcal{Q}_1|$, if and only if x belongs to the closure of X in $\mathcal{R}$. It is easily seen that $\langle P, u\rangle$ is separated but not strongly separated (consider the points $\langle i, x\rangle$, $i = 1, 0$, $x \in |\mathcal{Q}_2|$). A strongly separated space which is not regular: Let $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{R}$ have the meaning from the preceding example and let $\mathcal{R}$ be strongly separated, and consider the subspace $|\mathcal{Q}_1| \cup ((0) \times |\mathcal{Q}_2|)$ of $\langle P, u\rangle$.

4. If a subspace $\mathcal{Q}$ of a separated space $\mathcal{P}$ is compact, then $|\mathcal{Q}|$ is closed (compare with 17 ex. 5). [Hint: if a net $\mathcal{N}$ ranging in $|\mathcal{Q}|$ converges to a point $|\mathcal{P}| - |\mathcal{Q}|$ then $\mathcal{N}$

has no accumulation point in $\mathscr{Q}$.] Hence a continuous mapping of a compact space into a separated space carries closed sets into closed sets. Such a mapping need not carry open sets into open sets (e.g. consider a continuous function on a bounded closed interval of reals).

5. Let $\mathscr{T}$ be a separated topological field which is not discrete (e.g. take R, Q or C as $\mathscr{T}$). The set $F = \mathbf{E}\{\langle x, y\rangle \mid \langle x, y\rangle \in \mathscr{T} \times \mathscr{T}, y = x^{-1}\}$ is closed in $\mathscr{T} \times \mathscr{T}$ (if $\{\langle x_a, x_a^{-1}\rangle\}$ converges to $\langle x, y\rangle$, then $y = x^{-1}$ because $x_a \,.\, x_a^{-1} = 1$ must converge to $x \,.\, y$, which shows that $x \,.\, y = 1$). The projection $|\mathscr{T}| - (0)$ of F into $\mathscr{T}$ is not closed. Hence projections of a product space onto the coordinate spaces need not carry closed sets into closed sets. The set $f = F \cup (\langle 0, 0\rangle)$ is closed in $\mathscr{T} \times \mathscr{T}$ but the mapping $f : \mathscr{T} \to \mathscr{T}$ is not continuous.

6. If m is the density character of a separated space P, then card $\mathscr{P} \leqq$ exp exp m (and hence the total character is at most exp exp exp m; this estimate is attained – an example may be obtained by modifying the closure structure of βX).

7. (a) Each βX is separated. [If $\mathscr{X}$ and $\mathscr{Y}$ are distinct ultrafilters then $X \cap Y = \emptyset$ for some X in $\mathscr{X}$ and Y in $\mathscr{Y}$.]

(b) The cardinal of any infinite closed subset F of βX is at least exp exp $\aleph_0$. [The cardinal of βX with X infinite is exp exp card X. If F is an infinite closed subset of βX such that $F \cap X$ is infinite, then $\overline{F \cap X} \subset F$ is a homeomorph of $\beta(F \cap X)$ and hence the cardinal of F is at least exp exp $\aleph_0$. If $F \cap (\beta X - X)$ is infinite then we can choose a sequence $\{\xi_n\}$ in $F - X$ and $X_n \in \xi_n$ such that $\{X_n\}$ is a disjoint family. Let $Z = \mathbf{E}\{X_n\}$. We shall construct a one-to-one mapping f of βZ onto the set $F_1 = \overline{\mathbf{E}\{\xi_n\}} \subset F$. If η is a free ultrafilter on Z then the sets $\bigcup\{Y_n \mid X_n \in N\}$, $Y_n \in \xi_n$, $N \in \eta$, form an ultrafilter $f\eta$ on X and $f\eta \in F_1$. It is easily seen that $f : \beta Z \to F$ is bijective.]

(c) There exists a family $\{\mathscr{P}_a \mid a \in A\}$ of countably compact subspaces of βN such that N $\subset \mathscr{P}_a$, $\{|\mathscr{P}_a| - \mathrm{N}\}$ is disjoint, card $A =$ exp exp $\aleph_0$. [It is sufficient to show that if the cardinal of a subset X of βN $-$ N is less than card A then there exists a countably compact subspace $\mathscr{P}$ such that N $\subset |\mathscr{P}|$, $|\mathscr{P}| \cap X = \emptyset$, card $|\mathscr{P}| \leqq$ exp $\aleph_0$. Since each infinite set has at least exp exp $\aleph_0$ cluster points, it has a cluster point in βN $- X$. Let φ be a single-valued relation which assigns to each infinite set a cluster point in βN $- X$; denote by $S(Y)$ the collection of all countable subsets of Y and define by induction $Y_0 = \mathrm{N}$, $Y_\alpha = S(\bigcup\{X_\beta \mid \beta < \alpha\}$, $X_\alpha = \varphi[Y_\alpha]$. It follows that $\bigcup\{Y_\alpha \mid \alpha < \omega_1\} = X_{\omega_1}$ is countably compact, card $X_{\omega_1} =$ exp $\aleph_0$ and $Y_{\omega_1} \cap X = \emptyset$.]

(d) The product of two countably compact spaces need not be countably compact. [Let $\mathscr{P}$ and $\mathscr{Q}$ be two countably compact subspaces of βN, N $= |\mathscr{P}| \cap |\mathscr{Q}|$. The set $N_1 = \{\langle x, x\rangle \mid x \in \mathrm{N}\}$ is closed in $\mathscr{P} \times \mathscr{Q}$; of course N_1 is discrete.]

8. Some examples. There exists a continuous mapping f of a dense subspace $\mathscr{R}$ of a regular separated space $\mathscr{Q}$ into a regular separated space $\mathscr{P}$ such that f has a continuous extension on each subspace $\mathscr{R} \cup (x)$ but not on $\mathscr{Q}$. (E.g., take an infinite separated space $\mathscr{S}$ with exactly one cluster point, say x, and an infinite set A, and fix an α in A. Next, let $\mathscr{P} = \Sigma\{\mathscr{S} \mid a \in (A - (\alpha))\}$ and let $\mathscr{Q}$ be the space defined as follows: the underlying set of $\mathscr{Q}$ is $\Sigma\{|\mathscr{S}| \mid a \in A\}$, $\mathscr{P}$ is an open subspace of $\mathscr{Q}$, $(\alpha) \times \mathscr{S}$ is a closed subspace of $\mathscr{Q}$, $(\alpha) \times (|\mathscr{S}| - (x))$ is an open subset of $\mathscr{Q}$, and $\langle\alpha, x\rangle$ belongs to the closure of a set X if and only if either $\langle\alpha, x\rangle$ belongs to the closure of $((\alpha) \times |\mathscr{S}|) \cap X$ in $(\alpha) \times \mathscr{S}$ or $X \cap (A \times (x))$ is infinite. Finally, fix a β in $A - (\alpha)$ and consider the mapping f of the subspace $\mathscr{R}$ of $\mathscr{Q}$, $|\mathscr{R}| = |\mathscr{Q}| - A \times (x)$, into $\mathscr{P}$ such that f is identity on $|\mathscr{R}| - (\alpha \times |\mathscr{S}|)$, and $f\langle\alpha, y\rangle = \langle\beta, y\rangle$ for $y \in |\mathscr{S}| - (x)$.)

(b) If a space $\mathscr{P}$ is not regular, then there exists a continuous mapping f of a dense subspace $\mathscr{R}$ of a space $\mathscr{Q}$ into $\mathscr{P}$, which has a continuous domain-extension to each subspace $|\mathscr{R}| \cup (x)$ of $\mathscr{Q}$ but not to $\mathscr{Q}$. (E g., if $\mathscr{P} = \langle P, u\rangle$ is not regular, then there exists an x in P such that, denoting by $\mathscr{U}$ the neighborhood system at x, $uU - V \neq \emptyset$ for some $V \in \mathscr{U}$ and each $U \in \mathscr{U}$. Exhibit a topological closure v for P finer than u on $P - (x)$ such that the sets uU, $U \in \mathscr{U}$, form a local base at x, and consider the identity mapping of the subspace V of $\langle P, v\rangle$ into $\langle P, u\rangle$.)

9. In a regular topological space the collection $\mathscr{X}$ of all sets of the form $G \cap F$, G open, F closed, is relatively feebly locally determined (see 21 A.12). [Hint: Assuming that X relatively feebly locally belongs to $\mathscr{X}$, prove that $X \in \mathscr{X}$. It is sufficient to show that $\overline{X} - X$ is closed because $X = \overline{X} \cap (P - (\overline{X} - X))$. Clearly each point of $P - \overline{X}$ is an interior point of $P - (\overline{X} - X)$. Fix $x \in X$. Since X feebly locally belongs to $\mathscr{X}$ at x we can choose a neighborhood U of x such that $U \cap X = G \cap F$ where G is open and F is closed. Choose an open neighborhood V of x with $\overline{V} \subset U \cap G$. We have $V \cap \overline{X} \subset \overline{V \cap X} \subset \overline{U \cap X} \cap G \subset F \cap G = X \cap U \subset X$. Thus x is an interior point of $P - (\overline{X} - X)$.]

10. A subset X of a regular topological space is relatively locally closed if and only if $\overline{X} - X$ is closed (i.e. $X = F \cap G$ where F is closed and G is open, by 15 ex. 1).

11. Extension of mappings into complete metric spaces. (a) Let f be a continuous mapping of a dense subspace $\langle R, w\rangle$ of a separated topological space $\langle Q, v\rangle$ into a complete pseudometric space $\langle P, d\rangle$ (definition 22 ex. 17). There exists a continuous domain-extension g of f to a $\mathbf{G}_\delta$-subspace $\langle S, u\rangle$ of $\langle Q, v\rangle$ (i.e. such that S is a $\mathbf{G}_\delta$ in $\langle Q, v\rangle$). Proof: For each n in $\mathbb{N}$ let $\mathscr{U}_n$ be the collection of all open subsets U of $\langle Q, v\rangle$ such that the diameter of $f[U \cap R]$ is at most $(n + 1)^{-1}$. Put $U_n = \bigcup\mathscr{U}_n$, $S = \bigcap\{U_n\}$. For each x in $S - R$ let $\mathscr{U}_x$ be the neighborhood system at x, and $\mathscr{V}_x$ be the f-transform of the filter $R \cap [\mathscr{U}_x]$. Clearly $\mathscr{V}_x$ contains arbitrarily small sets and hence, $\langle P, d\rangle$ being complete, each $\mathscr{V}_x$ converges to a point gx. Since $\langle P, d\rangle$ is regular, $g : \langle S, u\rangle \to \langle P, d\rangle$ is continuous by 27 B.17 (b).

(b) State and prove (a) for uniformly continuous mappings of a dense subspace of a uniform space into a complete pseudometric space (use 27 B.18 (b)).

(c) If a dense subspace $\mathscr{Q}$ of a separated topological space $\mathscr{P}$ admits a metrization by a complete metric, then $|\mathscr{Q}|$ is a $\mathbf{G}_\delta$ in $\langle P, u\rangle$. [Let f be the identity mapping of the subspace $\mathcal{Q}$ of $\mathcal{P}$ onto $\mathcal{Q}$. By (a) f has a continuous domain-extension to a $\mathbf{G}_\delta$-subspace $\mathcal{S}$ of $\mathcal{P}$. By ex. 2 f has no continuous proper domain extension, and hence $\mathcal{S} = \mathcal{Q}$.]

(d) Suppose that a space $\mathcal{P}$ admits a metrization by a complete metric. A subspace $\mathcal{Q}$ of $\mathcal{P}$ admits a metrization by a complete metric if and only if $|\mathcal{Q}|$ is a $\mathbf{G}_\delta$ in $\mathcal{P}$. ["If" was proved in 22 ex. 7. Conversely, if $\mathcal{Q}$ admits a metrization by a complete metric then $|\mathcal{Q}|$ is a $\mathbf{G}_\delta$ in the subspace $|\mathcal{Q}|$ of $\mathcal{P}$ (by (c)) and hence in $\mathcal{P}$ because each closed subset of a pseudometrizable space is a $\mathbf{G}_\delta$.]

(e) Each homeomorphism f of a subspace $\mathcal{R}_1$ of a complete metric space $\mathcal{R}$ onto a subspace $\mathcal{P}_1$ of a complete metric space $\mathcal{P}$ has an extension to a homeomorphism g of a $\mathbf{G}_\delta$-subspace $\mathcal{R}_2$ of $\mathcal{R}$ onto a $\mathbf{G}_\delta$-subspace $\mathcal{P}_2$ of $\mathcal{P}$. Proof: Take a continuous extension f_1 of f to a mapping of a $\mathbf{G}_\delta$-subspace $\mathcal{R}'$ of $\mathcal{R}$ into $\mathcal{P}$ such that $\mathcal{R}_1$ is dense in $\mathcal{R}'$ (by (a)), and then a continuous extension of f^{-1} to a mapping f_2 of a $\mathbf{G}_\delta$-subspace $\mathcal{P}_2$ of $\mathcal{P}$ into $\mathcal{R}'$ such that $\mathcal{P}_1$ is dense in $\mathcal{P}_2$ (this is possible by (a) because by virtue of (c) the space $\mathcal{R}$ admits a metrization by a complete metric). The mappings $\mathsf{J} : \mathcal{P}_2 \to \mathcal{P}$ and $f_1 \circ f_2 : \mathcal{P}_2 \to \mathcal{P}$ coincide on a dense subspace of $\mathcal{P}_2$, namely on $\mathcal{P}_1$, and hence they are identical because $\mathcal{P}$ is separated. Thus $f_1 \circ f_2 : \mathcal{P}_2 \to \mathcal{P}_2$ is the identity mapping of $\mathcal{P}_2$ onto $\mathcal{P}_2$, and hence f_2 is a homeomorphism of $\mathcal{P}_2$ onto the subspace $\mathcal{R}_2$ whose underlying set is $f_2[|\mathcal{P}_2|]$. The set $|\mathcal{P}_2|$ is a $\mathbf{G}_\delta$ in $\mathcal{P}$ and hence $\mathcal{P}_2$ admits a metrization by a complete metric (by (d)), and thus $\mathcal{R}_2$ admits such a metrization; this implies that $|\mathcal{R}_2|$ is a $\mathbf{G}_\delta$ in $\mathcal{R}'$ (by (c)) and hence a $\mathbf{G}_\delta$ in $\mathcal{R}$.

12. Two topologically equivalent uniformities coincide provide that they coincide on a dense subset.

13. If the density character of a regular topological space is m then the local character is at most exp m. The assumption of regularity is essential.

14. The box-product of a family of regular (separated) spaces is a regular (separated) space.

15. If $\mathcal{L}$ is a separated infinite-dimensional normed real module and $\{f_i\}$ is a finite family of linear functionals, then the set $\bigcap\{f_i^{-1}[0]\}$ is not bounded.

(Sections 28 – 30)

1. Exhibit a separated topological space on which every continuous function is constant. [E.g. let $\{K_n\}$ be a disjoint sequence of non-void finite subsets of $\mathsf{R} - \mathsf{Q}$ such that any non-void open subset of R intersects almost all K_n. Consider the closure u for Q such that $\mathsf{Q} - \mathsf{N}$ is an open subspace of $\langle \mathsf{Q}, u \rangle$ and each neighborhood of any $n \in \mathsf{N}$ is of the form $(n) \cup (U \cap \mathsf{Q} - \mathsf{N})$ with U a neighborhood of K_n in R; if f is a continuous function on $\langle \mathsf{Q}, u \rangle$ then $fq = \lim \{fn\} \mid n \in \mathsf{N}\}$ for each q in Q, and hence f is a constant function.]

2. Exact Borel sets. Let $\mathscr{P}$ be a closure space. With the notation of 22 ex. 10, if $\varrho_1\{X_a\} = \bigcup\{X_a\}$, $\varrho_2\{X_a\} = \bigcap\{X_a\}$, and $\mathscr{X}$ is the collection of all exact closed (exact open) sets, then the elements of $\mathscr{X}_a$ are termed exact $\mathbf{F}_a$ sets (exact $\mathbf{G}_a$ sets, respectively). Of course we use the term exact $\mathbf{F}_\sigma$ instead of exact $\mathbf{F}_1$, etc. Then exact $\mathbf{F}_0 \subset$ exact $\mathbf{G}_1$ and exact $\mathbf{G}_0 \subset$ exact $\mathbf{F}_1$, and hence exact $\mathbf{F}_{\omega_1}$ = exact $\mathbf{G}_{\omega_1}$ (by 22 ex. 10). The sets from exact $\mathbf{F}_{\omega_1}$ will be termed exact Borel sets; it should be remarked that these sets are often called Baire sets.

3. A family $\{f_a\}$ of functions on $\mathscr{P}$ is termed locally finite (point-finite) if the family $\{N(f_a)\}$ is locally finite (point-finite); here $N(f_a) = \mathbf{E}\{x \mid f_a x \neq 0\}$. If $f = \{f_a\}$ is point-finite, then

$$\{\langle x, y\rangle \rightarrow \Sigma\{|f_a x - f_a y|\}\}$$

is a pseudometric, denoted by d_f. If f is locally finite and all f_a are continuous, then d_f is a continuous pseudometric and each $N(f_a)$ is open in $\langle|\mathscr{P}|, d_f\rangle$.

(a) A cover $\mathscr{X}$ of $\mathscr{P}$ is uniformizable if and only if it is refined by a locally finite cover consisting of exact open sets.

A partition of the unity on $\mathscr{P}$ subordinated to a cover $\mathscr{X}$ of $\mathscr{P}$ is a locally finite family $f = \{f_a\}$ of non-negative continuous functions such that $\Sigma\{f_a x\} = 1$ for each x, and the family $\{N(f_a)\}$ refines $\mathscr{X}$.

(b) Some partition of unity is subordinated to $\mathscr{X}$ if and only if $\mathscr{X}$ is uniformizable. [Hint: given a uniformizable cover $\mathscr{X}$, take a locally finite family $\{f_a\}$ of non-negative functions such that $\{N(f_a)\}$ is a cover refining $\mathscr{X}$; put $g_a = f_a/\Sigma\{f_a\}$.]

4. Extension of unbounded functions. Let $\mathscr{Q}$ be a subspace of a closure space $\mathscr{P}$ such that each bounded continuous function on $\mathscr{Q}$ is the restriction of a bounded continuous function on $\mathscr{P}$ (i.e. such that the Čech proximity of $\mathscr{Q}$ is the relativization of the Čech proximity of $\mathscr{P}$). Then each continuous function on $\mathscr{Q}$ has a continuous extension to $\mathscr{P}$ provided that $|\mathscr{Q}|$ and any disjoint exact closed set are functionally separated. [Hint: Let φ be a homeomorphism of R onto $]\!]1, 1[\![$; given a continuous function f on $\mathscr{Q}$, consider a continuous extension g of $\varphi \circ f$ to $\mathscr{P}$ and $Z = \mathbf{E}\{x \mid |gx| \geqq 1\}$. Take a continuous function h, $0 \leqq h \leqq 1$, $h[|\mathscr{Q}|] \subset (1)$, $h[Z] \subset (0)$ and put $f^* = \varphi^{-1} \circ (g \,.\, h)$.]

5. Let $\mathscr{S} = \langle S, \sigma\rangle$ be a struct, $\varrho \subset \exp S \times \exp S$, and let A be a set. Suppose that $M\varrho N$ implies $M \subset N$, and consider the set Ψ of all covers X of $\mathscr{S}$ with $\mathbf{D}X = A$. If $X \in \Psi$ then we write $X = \{X_a\}$. For $X \in \Psi$ let Ψ_X denote the set of all $Y \in \Psi$ with $Y_a \varrho X_a$ or $Y_a = X_a$ for each a. Let $\leqq_X$ denote the order for Ψ_X defined as follows: $Y \leqq_X Z$ if and only if, for each a, $Z_a \neq X_a$ implies $Y_a = Z_a$.

(a) If $X \in \Psi$ is point-finite, then each monotone subset of Ψ_X has a lower bound; in particular, each element of Ψ_X is greater than a minimal element (see part III of the proof of Theorem 29 C.1).

(b) Suppose that $X \in \Psi$ is point-finite, and $Y \in \Psi_X$, $a \in A$ imply that there exists a $Z \leqq_X Y$ with $Z_a \varrho X_a$. Then there exists a Z in Ψ_X with $Z_a \varrho X_a$ for each a.

(c) Let $M \varrho N$ if and only if $M = N$ or $M = \emptyset$. It follows from (a) that any point-finite cover X of $\mathscr{S}$ has a "minimal" subcover Y, i.e. no proper subfamily of Y is a cover.

6. A semi-uniformizable topological space $\mathscr{P}$ is normal provided that it is the union of a locally finite family of closed normal subspaces.

7. A semi-uniformizable space with at most one cluster point is hereditarily paracompact.

8. (a) Let X be the set of all isolated points of T_{ω_1}, and Y be its complement in T_{ω_1}. The set X is locally $\mathbf{F}_\sigma$ in T_{ω_1}, but not $\mathbf{F}_\sigma$ (each closed subset of X is finite), and Y is locally $\mathbf{G}_\delta$ but not $\mathbf{G}_\delta$. Thus neither $\mathbf{F}_\sigma$-sets nor $\mathbf{G}_\delta$-sets need be locally determined in a normal space.

The set X is relatively locally $\mathbf{F}_\sigma$ in T_{ω_1+1} but not $\mathbf{F}_\sigma$, Y is relatively locally $\mathbf{G}_\delta$ but not $\mathbf{G}_\delta$; thus neither $\mathbf{F}_\sigma$-sets nor $\mathbf{G}_\delta$-sets need be relatively locally determined in a paracompact space (or even in a compact separated space).

(b) Any continuous pseudometric on T_{ω_1} has an extension to a continuous pseudometric for T_{ω_1+1}, and consequently the fine uniformity of T_{ω_1} is the relativization of the fine uniformity of T_{ω_1+1}. [Hint: First prove that if F_1 and F_2 are two disjoint closed sets in T_{ω_1} then at least one of the sets F_1 and F_2 is bounded. Then prove that any continuous function on T_{ω_1} is constant on some set $\leqq [\alpha]$ with $\alpha < \omega_1$.]

9. A space is pseudometrizable if and only if it is paracompact and there exists a sequence $\{\mathscr{X}_n\}$ of interior covers such that $\{\operatorname{st}(x, \mathscr{X}_n)\}$ is a local base at x for each x.

10. If $\{X_a\}$ and $\{Y_b\}$ are locally finite families in a normal space such that X_a and Y_b are functionally separated for each a and b, then $\bigcup\{X_a\}$ and $\bigcup\{Y_b\}$ are functionally separated.

11. A space $\mathscr{P}$ is uniformizable (or regular) if and only if each cover $(U, |\mathscr{P}| - (x))$ with U a neighborhood of x is uniformizable (semi-uniformizable, respectively).

(Sections 31 – 35)

1. For each set P the mapping $\{u \to \{uX \mid X \subset P\}\} : \mathbf{C}(P) \to \langle \exp P, \subset \rangle^{\exp P}$ is completely join-stable. For each u in $\boldsymbol{\tau}\mathbf{C}(P)$ let $\mathscr{O}(u)$ be the collection of all u-open sets; the mapping $\{u \to \mathscr{O}(u)\} : \boldsymbol{\tau}\mathbf{C}(P) \to \langle V, \supset \rangle$ is completely join-stable (but not meet-stable), where V is the universal class.

2. If u is the infimum of a non-void collection $\{u_a\}$ in $\mathbf{C}(P)$ and $X \subset P$, then $x \in uX$ if and only if for each finite cover $\{X_i\}$ of X there exists an X_i such that $x \in u_a X_i$ for each a.

3. If a closure u has a lower modification in $\boldsymbol{\tau}\mathbf{C}$, then $u \in \boldsymbol{\tau}\mathbf{C}$ (show that $u = \sup\{v \mid v \leqq u, v \in \boldsymbol{\tau}\mathbf{C}\}$). On the other hand if u has a lower modification in $\boldsymbol{\upsilon}\mathbf{C}$ then u need not be uniformizable (consider $\mathbf{C}(P)$ where $\operatorname{card} P = 2$).

4. The class of all pseudometrizable spaces is countably meet-stable in **C**. [Hint: If u_n is pseudometrized by d_n then $\inf\{u_n\}$ is pseudometrized by $\Sigma\{2^{-n}\min(d_n, 1)\}$.]

5. Let D be the class of all dense-in-themselves closure operations (a closure is called dense-in-itself if each point is a cluster point.)

(a) If u is finer than v and $u \in D$ then $v \in D$.

(b) If $\{u_a\}$ is a down directed (in particular, if $\mathbf{E}\{u_a\}$ is a non-void and monotone) family in D, then $\inf\{u_a\} \in D$. Consequently, each closure of D is coarser than a minimal element of D (which is said to be a fine dense-in-itself closure operation).

(c) If $K \subset \mathbf{C}$ is completely meet-stable then $K \cap D$ has property (b). In particular, there exist fine topological, uniformizable, regular, etc., dense-in-themselves closure operations.

(d) A closure u for P is a fine dense-in-itself closure operation if and only if $[\mathscr{U}_x] \cap \cap (P - (x))$ is an ultrafilter on $P - (x)$ for each x (here $\mathscr{U}_x$ is the neighborhood system at x in $\langle P, u\rangle$).

(e) A topological dense-in-itself closure u for P is a fine topological dense-in-itself closure operation if and only if the conditions $x \in uX$, X is dense-in-itself imply that $(x) \cup X$ is a neighborhood of x (or equivalently, each dense-in-itself set is open).

6. The class of all locally connected spaces is inductive-stable. [Either prove that a space inductively generated by a single mapping from a locally connected space is locally connected, and make use of the fact that the class of all locally connected closure operations is completely join-stable in **C**, or prove that any quotient of a locally connected space is locally connected and the sum of any family of locally connected spaces is locally connected.]

7. $K \subset \mathbf{C}$ is projective-stable if and only if, for each $\mathscr{Q}$ in **C**, there exists a $\kappa\mathscr{Q}$ in K, $|\kappa\mathscr{Q}| = |\mathscr{Q}|$ such that $f : \mathscr{Q} \to \mathscr{P}$ with $\mathscr{P}$ in K is continuous if and only if $f : \kappa\mathscr{Q} \to \mathscr{P}$ is continuous (of course, $\kappa\mathscr{Q}$ is the upper modification of $\mathscr{Q}$ in K). A similar result holds for inductive-stable classes.

8. If $\mathscr{Q}$ is a quotient of $\mathscr{P}$ then the density character of $\mathscr{Q}$ is at most that of $\mathscr{P}$, but the local character and the total character of $\mathscr{P}$ may be greater than the corresponding characters of $\mathscr{P}$ (consider the space obtained by identifying the points of a line in $\mathsf{R} \times \mathsf{R}$). If $\mathscr{P}$ is separated then $\mathscr{Q}$ need not be separated (consider a uniformizable space which is not normal, choose disjoint closed sets F_1 and F_2 which are not separated, and consider the space obtained by identifying the points of F_1 and of F_2. The points F_1 and F_2 are not separated).

9. The graph of a correspondence of $\mathscr{P}$ into $\mathscr{Q}$ is closed provided that the following conditions are fulfilled:

(a) $\mathbf{D}f$ is closed.

(b) f is upper semi-continuous.

(c) If $\mathscr{U}$ is the neighborhood system of $f[x]$ then $\bigcap\{\bar{U} \mid U \in \mathscr{U}\} = f[x]$ (in particular, $f[x]$ is closed).

If $\mathscr{Q}$ is regular then condition (c) may be replaced by the weaker condition that each $f[x]$ is closed.

10. Every space admits a single-valued determining relation. [Hint: If $\mathscr{C}$ is a determining relation, $\mathscr{N} \in \mathbf{D}\mathscr{C}$, then the set of all $\langle \mathscr{N}, x \rangle \in \mathscr{C}$ can be replaced by a set consisting of points $\langle \mathscr{N}_x, x \rangle$, $x \in \mathscr{C}[\mathscr{N}]$, such that $x \neq y \Rightarrow \mathscr{N}_x \neq \mathscr{N}_y$ (by a formal change of the underlying sets).]

11. Topological modification of sequential closure operations. (a) If u is a sequential closure operation for a set P and $u_\alpha X = u(\bigcup\{u_\beta X \mid \beta < \alpha\})$ for each $X \subset P$, then each u_α is a closure operation and $u_{\omega_1} = \sup\{u_\alpha \mid \alpha < \omega_1\}$ is the topological modification of u.

(b) Let $\mathscr{P}$ and $\mathscr{R}$ be closure spaces and let $\langle F, u \rangle$ be the set $\mathbf{F}(\mathscr{P}, \mathscr{R})$ endowed with the sequential modification of the closure of pointwise convergence, and let v be the topological modification of u. The elements of $v\mathbf{C}(\mathscr{P}, \mathscr{R})$ are called the Baire mappings of $\mathscr{P}$ into $\mathscr{R}$, and the family $\{u_\alpha \mathbf{C}(\mathscr{P}, \mathscr{R}) \mid \alpha < \omega_1\}$ is called the Baire classification of Baire mappings. The elements of $u_\alpha \mathbf{C}(P, \mathscr{R})$ are termed the mappings of the α-th Baire class. Prove: an $f \in \mathbf{F}(P, \mathsf{R})$ is a Baire function if and only if $f^{-1}[X]$ is a Baire set in $\mathscr{P}$ for each closed (open, Baire) set in R.

12. Sequentially compact spaces.

An **L**-closure u for a set P is a coarse **L**-closure if and only if each sequence has a convergent subsequence (i.e. if $\langle P, u \rangle$ is sequentially compact).

Each sequentially compact space is countably compact, and the converse is true for each space with a countable local character. The class of all sequentially compact spaces is closed under formation of countable products and closed subspaces. The product of a sequentially compact space with a countably compact space is countably compact. (Remember that the product of two countably compact spaces need not be countably compact.)

13. For each **S**-closure u the closure $\sigma\upsilon u$ is the upper modification of u in $\sigma\upsilon\mathbf{C}$. A uniformizable space $\mathscr{Q}$ is the uniformizable modification of an **S**-space if and only if a function f on $\mathscr{Q}$ is continuous whenever the following condition is fulfilled: if a sequence S converges to x in $\mathscr{Q}$, then the sequence $f \circ S$ converges to fx in R.

14. The class $\sigma\upsilon\mathbf{C}$ is completely meet-stable in **S**, and each accrete closure belongs to $\sigma\upsilon\mathbf{C}$.

15. The classes **S**, **L**, $\sigma\upsilon\mathbf{C}$, $\sigma\upsilon\mathbf{C} \cap \mathbf{L}$, $\mathbf{S} \cap \upsilon\, \mathbf{C}_1$ and $\mathbf{L} \cap \upsilon\mathbf{C}$ are closed under sums.

16. A space $\mathscr{P}$ belongs to $\sigma\upsilon\mathbf{C}$ if and only if $\mathscr{P}$ is a homeomorph of a subspace of $\mathscr{P}' \times \sigma\mathsf{R}^{\aleph}$ where $\mathscr{P}'$ is the accrete space such that $|\mathscr{P}| = |\mathscr{P}'|$ and $\aleph$ is an appropriate cardinal (use 35 E.3).

(Sections 36–40)

1. A closure operation u for a set P is induced by a uniformity which is a coarse semi-uniformity if and only if $P = \bigcup\{F_i\}$ where $\{F_i\}$ is a disjoint family of closed accrete subspaces and F_i are finite and open except one, but the complements of neigborhods of its points must be finite.

2. (a) Let f be a uniformly continuous mapping of a uniform space $\mathscr{P} = \langle P, \mathscr{U}\rangle$ into a uniform space $\mathscr{Q} = \langle Q, \mathscr{V}\rangle$. If $\mathscr{R} = \langle R, \mathscr{W}\rangle$ is a dense subspace of $\mathscr{P}$ such that $g = f : \mathscr{R} \to \mathscr{Q}$ is a projective generating mapping, then f is a projective generating mapping. [If $V \in \mathscr{V}$ is a regular closed set in $\mathscr{Q} \times \mathscr{Q}$ then $(f \times f)^{-1}[V]$ is the closure of $(g \times g)^{-1}[V]$. If $U \in \mathscr{U}$ and U' is a closed element of $\mathscr{U}$ contained in U then $(g \times g)^{-1}$ $[V] \subset U' \cap (R \times R)$ for some regular closed $V \in \mathscr{W}$ and hence $(f \times f)^{-1}[V]$ is contained in $U' \subset U$.]

(b) Let $\mathscr{R}$ be a dense subspace of a uniform space $\mathscr{Q}$. If $\mathscr{R}$ is pseudometrizable or proximally coarse then so is $\mathscr{Q}$.

3. Hewitt uniformities. A Hewitt uniformity is defined to be a uniformity $\mathscr{U}$ projectively generated by all continuous functions (hence, the proximally coarse uniformity topologically equivalent to $\mathscr{U}$ is a Čech uniformity).

(a) A separated uniform space $\mathscr{P} = \langle P, \mathscr{U}\rangle$ is a uniform homeomorph of a subspace of some $\mathsf{R}^{\aleph}$ if and only if $\mathscr{U}$ is a Hewitt uniformity.

(b) $\langle P, \mathscr{U}\rangle$ is projectively generated by a mapping into a product $\mathsf{R}^{\aleph}$ if and only if $\mathscr{U}$ is a Hewitt uniformity.

(c) The Hewitt uniformity of a closure space is defined to be the uniformly finest continuous Hewitt uniformity. A closure space $\mathscr{P}$ is said to be pseudocompact if the Čech uniformity of $\mathscr{P}$ and the Hewitt uniformity of $\mathscr{P}$ coincide, i.e. if each continuous function is bounded.

(d) Every countably compact space is pseudocompact (if f is continuous and $|fx_n| > n$ then $\{x_n\}$ is locally finite), a normal pseudocompact space is countably compact (if $\{x_n\}$ is a locally finite sequence of points of a normal space then some subsequence $\{x'_n\}$ is discrete, and hence $\bigcup\{(\overline{x'_n})\}$ is a closed subspace F; if f is n on $(\overline{x'_n})$ then f is an unbounded continuous function on F which has a continuous extension).

(e) A space $\mathscr{P}$ is pseudocompact if there exists a dense subspace $\mathscr{R}$ of $\mathscr{P}$ such that no infinite family of non-void subsets of $\mathscr{R}$ is locally finite in $\mathscr{P}$.

(f) There exists an infinite separated uniformizable pseudocompact space such that no infinite subspace is countably compact. Let $R \subset \beta\mathsf{N}$, $R \supset \mathsf{N}$, card $R =$ $= \exp \aleph_0$, and let each infinite subset of N have a cluster point in R. Let $<$ be a minimal well-order for $R - \mathsf{N}$. Let X be the set of all $x \in R$ such that x does not belong to the closure of the set X_x of all $y \in R$, $y < x$ $(y \neq x)$. The subspace $P =$ $= \mathsf{N} \cup X$ of $\beta\mathsf{N}$ is pseudocompact; this will be proved by showing that each infinite subset of N has a cluster point in X. Let Y be an infinite subset of N and let x be the smallest element of R such that x belongs to the closure $\overline{Y}$ of Y in $\beta\mathsf{N}$. The set $\overline{Y}$ is open in $\beta\mathsf{N}$, hence a neighborhood of x in $\beta\mathsf{N}$ and $\overline{Y} \cap X_x = \emptyset$. Hence $x \notin \overline{X}_x$. The space P has the following property: no countable subset of $X = P - \mathsf{N}$ has $\exp \aleph_0$ cluster points. In fact, if Z is countable subset of X, then $Z \subset X_x$ for some x in R because $\langle R, < \rangle$ contains no countable cofinal subset, and consequently card $\overline{Z}^P \leqq$ $\leqq$ card $(X_x \cap X) < \exp \aleph_0$; the first inequality follows from the fact that $X_x \cap X$ is closed in P, and the second from the fact that $<$ is minimal. That no infinite subspace of P is countably compact now follows from 17 ex. 5 (each infinite set contains an infinite set without a cluster point).

4. A pseudometrizable uniformity is not a proximally fine semi-uniformity if there exists a Cauchy sequence $\{x_n\}$ such that the family $\{\overline{(x_n)}\}$ is disjoint, in particular if the induced closure is not quasi-discrete. [Hint: Let $\langle P, \mathscr{U} \rangle$ be a uniform space and let $\{x_n\}$ be a sequence in P such that $\{\overline{(x_n)}\}$ is disjoint, and consider the set $X =$ $= \{x_{2k} \to x_{2k+1}\}$. Then the set of all $U - X - X^{-1}$, $U \in \mathscr{U}$ is a base for a semi-uniformity $\mathscr{V}$ proximally equivalent to $\mathscr{U}$; clearly $\mathscr{V} \neq \mathscr{U}$.]

A uniformly quasi-discrete uniformity is a proximally fine semi-uniformity. The converse does not hold.

5. Monotone uniformities. Let $\mathscr{P} = \langle P, \mathscr{U} \rangle$ be a semi-uniform space. $\mathscr{P}$ is defined to be monotone if $\mathscr{U}$ has a monotone base (i.e. a base $\mathscr{V}$ such that $\langle \mathscr{V}, \subset \rangle$ is monotone); e.g. every semi-pseudometric space is monotone. The uniform character of $\mathscr{P}$ is the smallest cardinal of a base for $\mathscr{P}$.

(a) Theorem. Every monotone uniform space is proximally fine. This follows from the following

(b) Theorem. If f is a proximally continuous mapping of a monotone uniform space $\mathscr{P}$ into a uniform space $\mathscr{Q}$, then f is uniformly continuous.

The proof of (b) is similar to that of 25 A.14; instead of sequences use monotone nets (as the ordered domain take a base for the uniform structure of $\mathscr{P}$ minimally well-ordered by $\supset$; exhibit a cofinal set such that the ranges of restricted nets are distant in $\mathscr{Q}$).

6. Locally fine uniform spaces. A semi-uniform space $\mathscr{P} = \langle P, \mathscr{U} \rangle$ is said to be locally fine (a current term open to serious criticism) if $\{\mathsf{J} : U[x] \to \mathscr{P} \mid x \in P\}$ inductively generates $\mathscr{P}$ for each U in $\mathscr{U}$; here $U[x]$ is considered as a semi-uniform subspace of $\mathscr{P}$.

(a) The class of all locally fine spaces is inductive-stable in $\mathbf{U}$. Denote by λ the corresponding lower modification.

(b) If $\mathscr{U}$ is a uniformity then so is $\lambda\mathscr{U}$. (It is obvious that $\lambda\mathscr{U}$ is locally fine in the "uniform sense", i.e. that the families in question are inductive generating in the uniform sense.)

(c) Given $\mathscr{U}$ one obtains $\lambda\mathscr{U}$ by transfinite iteration of the following operation: the set of all

$$\bigcup\{(U[x] \times U[x]) \cap U_x \mid x \in P\},\quad U_x \in \mathscr{U},\quad U \in \mathscr{U},$$

is a base for a semi-uniformity which is inductively generated by the family $\{\mathsf{J} : U[x] \to$ $\to P \mid x \in P\}$.

(d) Any fine uniform space is locally fine. Any subspace of a locally fine space is a locally fine space; in particular, every subfine space (i.e. a subspace of a fine space) is locally fine. No example of a locally fine uniform space which is not subfine is known.

(e) A presheaf $\mathscr{S}$ of sets over a uniform space is said to be projective in the uniform sense if the usual conditions are fulfilled with restriction to uniform spaces. A space $\mathscr{P}$ is locally fine if and only if, for each uniform space $\mathscr{Q}$, the presheaf of all uniformly continuous mappings of $\mathscr{P}$ onto $\mathscr{Q}$ is projective in the uniform sense.

(f) The following is an interesting property of some uniform spaces $\mathscr{P}$: $f : \mathscr{P} \to \mathscr{Q}$ is uniformly continuous provided that f is locally uniformly continuous, i.e. each point has a neighborhood U such that the restriction of f on U is uniformly continuous, or in other words, for each space $\mathscr{Q}$ the presheaf of all uniformly continuous mappings of $\mathscr{P}$ into $\mathscr{Q}$ is projective at $|\mathscr{P}|$.

(g) Every uniformly continuous function on a subspace $\mathscr{P}$ of a locally fine uniform space $\mathscr{Q}$ has a uniformly continuous extension to $\mathscr{Q}$. [Hint: Given f on $\mathscr{P}$ choose a uniform cover $\mathscr{U}$ of $\mathscr{Q}$ such that the diameter of $f[U \cap |\mathscr{P}|]$ is at most 1 for each U in $\mathscr{U}$, and apply appropriately the extension theorem for bounded functions to obtain "uniformly locally" a uniformly continuous extension.]

7. The projective limit of a presheaf $\mathscr{S}$ over a set $\langle A, \leqq \rangle$ may be empty even if all connecting mappings are surjective (this is elementary) and $\langle A, \leqq \rangle$ is left-directed. E.g. let A be the set of all countable ordinals, $\leqq$ be the inverse of the usual order for ordinals, S_a be the set of all order embeddings f of the set of all $\beta < a$ into Q such that $\mathbf{E}f$ is right-bounded, and f_{ab} assign to each $f \in S_a$ the restriction of f to S_b. Clearly $\varprojlim \mathscr{S} = \emptyset$ (there exists no order-embedding of countable ordinals into Q) and all f_{ab} are surjective.

(Section 41)

1. Pseudometric spaces. (a) A pseudometric space is complete if and only if the uniform space is complete.

(b) Let $\mathscr{Q}$ be a completion of a uniform space $\mathscr{P}$. Each uniformly continuous pseudometric d on $\mathscr{P}$ is the restriction of the unique uniformly continuous pseudometric d^* on $\mathscr{Q}$; in addition, d induces the uniform structure of $\mathscr{P}$ if and only if d^* induces the uniform structure of $\mathscr{Q}$.

(c) The existence of completions may be proved as follows: If each pseudometrizable space has a completion then any uniform space has a completion; indeed, any uniform space can be embedded in a product $\Pi\{\mathscr{P}_a\}$ of pseudometrizable spaces and if $\mathscr{P}^*_a$ is a completion of $\mathscr{P}_a$ then $\Pi\{\mathscr{P}^*_a\}$ is a completion of $\Pi\{\mathscr{P}_a\}$; the closure of the range of the embedding is the required completion. Next, each metrizable space $\langle P, d \rangle$ can be embedded into $\mathbf{U}^*(\langle P, d \rangle, \mathsf{R})$ (by the relation $\{x \to \{y \to d\langle x, y \rangle\} : : \langle P, d \rangle \to \mathsf{R}\}$). If $\mathscr{Q}$ is P endowed with the uniformly accrete uniformity then any pseudometric space $\langle P, d \rangle$ can be embedded into $\mathscr{Q} \times \mathbf{U}^*(\langle P, d \rangle, \mathsf{R})$.

2. A completion of $\mathscr{P}$ is totally bounded if and only if $\mathscr{P}$ is totally bounded.

3. Completions of topological rings and modules. (a) Any separated topological ring is a dense topological subring of a complete topological ring which is said to be its completion. If $\mathscr{X}_1$ and $\mathscr{X}_2$ are Cauchy filters in $\mathscr{R} = \langle R, +, \cdot, u \rangle$, then $y = [\mathscr{X}_1] . [\mathscr{X}_2]$ is a base for a Cauchy filter; in fact, given a neighborhood U of 0 we can choose a neighborhood V of 0 and $X'_i \in \mathscr{X}_i$ such that $X'_1 . V + V . X'_2 \subset U$; if $X_i \in \mathscr{X}_i$, $X_i \subset X'_i$ and $X_i - X_i \subset V$, then $X_1 . X_2 - X_1 . X_2 \subset X'_1 . V + V . X'_2 \subset U$.

(b) Any separated topological linear space or algebra has a completion. [It must be shown that the external multiplication can be extended continuously; i.e. if $\mathscr{X}$ is a Cauchy filter and r is a scalar, then $r . \mathscr{X}$ is a Cauchy filter.]

4. Two complete topologically equivalent uniformities need not coincide. E.g., R is complete and $\langle \mathsf{R}, \mathscr{U} \rangle$ is complete if $\mathscr{U}$ is the Hewitt uniformity for R; clearly the uniformities are different.

5. Hypercomplete uniform spaces. (a) The Hausdorff hyperspace of a semi-uniform space $\mathscr{P} = \langle P, \mathscr{U} \rangle$ is the semi-uniform space $H(\mathscr{P}) = \langle \exp' P, \mathscr{U}^* \rangle$ where $\mathscr{U}^*$ has the collection of all $U^* = \mathbf{E}\{\langle X, Y \rangle \mid X \subset U[Y], Y \subset U[X]\}$, $U \in \mathscr{U}$, for a base. If $\mathscr{P}$ is a uniform space then $H(\mathscr{P})$ is a uniform space. The mapping $f = \{x \to (x)\} : \mathscr{P} \to H(\mathscr{P})$ is an embedding: if $\mathscr{P}$ is a separated uniform space then $\mathbf{E}f$ is closed in $H(\mathscr{P})$. Consequently, if $\mathscr{P}$ is separated and $H(\mathscr{P})$ is complete, then $\mathscr{P}$ is complete. If $\mathscr{P}$ is complete then $H(\mathscr{P})$ need not be complete; it can be proved that $H(\mathscr{P})$ is complete if and only if $\mathscr{P}$ is paracompact and the locally fine modification of the uniform structure of $\mathscr{P}$ is fine. If $H(\mathscr{P})$ is complete then $\mathscr{P}$ is said to be hypercomplete. Any complete metrizable space is hypercomplete (prove!).

(b) Let $\mathscr{P}$ be a semi-uniform space and let $E(\mathscr{P})$ be $\exp'(P \times P)$ endowed with the semi-uniformity which has the collection of all $\mathbf{E}\{\langle X, Y \rangle \mid U \circ X \supset Y, U \circ Y \supset X\}$, $U \in \mathscr{U}$, for a base. The mapping $f : \{x \to \langle z, x \rangle\} : \mathscr{P} \to E(\mathscr{P})$ is an embedding, and $\mathscr{P}$ is a uniform space if and only if $E(\mathscr{P})$ is a uniform space. $\mathscr{P}$ is hypercomplete if and only if $E(\mathscr{P})$ is complete.

6. Counter-examples to compactness. (a) Find a compact space $\mathscr{P}$ such that some interior cover contains no finite interior cover. [Hint: Let N be an open discrete infinite subset of a separated compact space $\langle P, u \rangle$, $R = P \cup (x)$ with $x \notin P$, and let v be a closure for R such that N is an open discrete subspace of $\langle R, v \rangle$, $R - N$ is a compact subspace of $\langle R, v \rangle$ with only one accumulation point, namely x, $R - N$ is a neighborhood of x, U is a neighborhood of a $y \in P - N$ if and only if $y \in U$ and $U \cup (P - N)$ is a neighborhood of y in $\langle P, u \rangle$. $\langle R, v \rangle$ is compact and there exists an interior cover which contains no interior cover of cardinal less than card $(P - N)$.]

(b) Find a space $\langle S, w \rangle$ which is not compact but each infinite subset has a complete accumulation point. [Hint: Let card $N = \aleph_0$ in (a), choose $P_1 \subset P - N$ such that each subset of N has an accumulation point in P_1 but $N \cup P_1$ is not compact, and let $\langle S, w \rangle$ be the subspace of $\langle R, v \rangle$ with $S = N \cup P_1 \cup (x)$.]

(c) Find a space $\langle P, u\rangle$ such that each monotone centred collection has a cluster point, but some infinite subset N of $\langle P, u\rangle$ has no complete accumulation point. [Hint: Let $\langle P, u\rangle$ be a space such that P is the disjoint union $N \cup M \cup (x)$, where $N = \bigcup\{N_n \mid n \in \mathsf{N}\}$ with $N_n \subset N_{n+1}$, card $N_{n+1} >$ card N_n, N_n are compact subspaces of $\langle P, u\rangle$; $M \cup (x)$ is a compact space with only one accumulation point, namely x, each point of M has a neighborhood U such that $U \cap N$ is countable, and finally, each sequence $\{x_n\}$, $x_n \in N_{n+1} - N_n$, has a limit point in M.]

7. If $\bigcap\overline{\mathscr{U}} = X$ in a compact space, $\mathscr{U}$ is a filter, and each $U \in \mathscr{U}$ is a neighborhood of X, then $\mathscr{U}$ is the neighborhood system of X. If a compact space is regular and $\bigcap\overline{\mathscr{U}} = (x)$ where $\mathscr{U}$ is a collection of neighborhoods of (x), then $\mathscr{U}$ is a local sub-base at x.

8. Correspondences. (a) If f is an upper semi-continuous full correspondence of a compact space $\mathscr{P}$ onto a topological space $\mathscr{Q}$ and if the fibres $f[x]$, $x \in \mathscr{P}$, are compact, then $\mathscr{Q}$ is compact. [If $\mathscr{U}$ is an additive open cover of $\mathscr{Q}$, then the sets $V_U = \mathbf{E}\{x \mid f[x] \subset U\}$, $U \in \mathscr{U}$, are open and cover $\mathscr{P}$.]

(b) If $f : \mathscr{P} \to \mathscr{Q}$ is a correspondence such that gr f is closed in $\mathscr{P} \times \mathscr{Q}$ and $\mathscr{Q}$ is compact, then f is upper semi-continuous. [Hint: Assume that $\{x_a\}$ converges to x in $\mathscr{P}$, U is a neighborhood of $f[x]$ in $\mathscr{Q}$ and $X_a = f[x_a] - U \neq \emptyset$ for each a. Choose y_a in X_a and take a generalized subnet $\{y'_b\}$ of $\{y_a\}$ which converges to a point y. If $\{x'_b\}$ is the corresponding generalized subnet of $\{x_a\}$, then $\langle x'_b, y'_b\rangle \in$ gr f, $\{\langle x'_b, y'_b\rangle\}$ converges to $\langle x, y\rangle$ and $\langle x, y\rangle \notin$ gr f.]

(c) Let f be an upper semi-continuous full correspondence of a compact space $\mathscr{P}$ onto a separated topological space $\mathscr{Q}$ such that the fibres $f[x]$ are compact. Then f is inversely upper semi-continuous. [$\mathscr{Q}$ is regular and the fibres $f[x]$ are closed, hence gr f is closed, which implies that f^{-1} is upper semi-continuous.]

9. In any compact topological group the right uniformity, the left uniformity and the two-sided uniformity coincide. Hence, even if all group uniformities coincide then the group need not be commutative.

10. Let $\mathscr{F}$ be a collection of continuous functions on a compact space $\mathscr{P} = \Pi\{\mathscr{P}_a\}$ containing each function $f \circ \mathrm{pr}_a : \mathscr{P} \to \mathsf{R}$ with $f \in \mathbf{C}^*(\mathscr{P}_a, \mathsf{R})$. Then the smallest algebra containing $\mathscr{F}$ is dense in $\mathbf{C}^*(\mathscr{P}, \mathsf{R})$. (Apply the Stone-Weierstrass Theorem.)

11. βX is a Čech-Stone compactification of X endowed with the discrete closure structure. [Prove that each bounded function on X has a continuous extension to βX.]

12. If $\mathscr{P}$ is an infinite discrete space, then card $|\beta\mathscr{P}|$ = exp exp card $|\mathscr{P}|$ (Pospíšil). [Use the fact that there are exactly exp exp X ultrafilters on an infinite set X. An alternate proof: Consider any one-to-one mapping f of $\mathscr{P}$ onto a dense subset of $\mathscr{Q} = [\![0, 1]\!]^{\exp|\mathscr{P}|}$ (22 A.10); f is continuous and has a continuous extension to a mapping of $\beta\mathscr{P}$ onto $\mathscr{Q}$. Hence card $|\beta\mathscr{P}| \geqq$ card $|\mathscr{Q}|$ = exp exp card $|\mathscr{P}|$. On the other hand, if the density character of a separated space $\mathscr{P}$ is m, then card $\mathscr{P} \leqq$ exp exp m.]

13. A uniformizable space is normal if and only if for each two distinct maximal centered collections $\mathscr{F}_1$ and $\mathscr{F}_2$ of closed sets some $F_1 \in \mathscr{F}_1$ and $F_2 \in \mathscr{F}_2$ are separated.

14. Wallman compactification. Let $\mathscr{P}$ be a topological space and let X be the set of all $x \in |\mathscr{P}|$ and all maximal centered collections $\mathscr{F}$ of closed sets in $\mathscr{P}$ with $\bigcap \mathscr{F} = \emptyset$. For each closed F in $\mathscr{P}$ let F^* be the union of F and the set of all $\mathscr{F} \in X - |\mathscr{P}|$ such that $F \in \mathscr{F}$. The collection of all F^* is a closed base for a topological closure u for X. The space $\langle X, u\rangle$ is called the Wallman compactification of $\mathscr{P}$. Prove:

(a) $\langle X, u\rangle$ is compact.

(b) $\mathsf{J} : \mathscr{P} \to \langle X, u\rangle$ is an embedding and $|\mathscr{P}|$ is dense.

(c) $\langle X, u\rangle$ is separated if and only if $\mathscr{P}$ is separated and normal.

(d) The Wallman proximity of $\mathscr{P}$ is a relativization of the Wallman proximity of $\langle X, u\rangle$.

(e) The following conditions are equivalent: $\langle X, u\rangle$ is uniformizable, $\mathscr{P}$ is normal, $\langle X, u\rangle = \beta\mathscr{P}$.

15. If a regular space $\mathscr{P}$ contains a dense subspace $\mathscr{Q}$ such that each net in $\mathscr{Q}$ has an accumulation point in $\mathscr{P}$ ($\mathscr{Q}$ is said to be compact in $\mathscr{P}$), then $\mathscr{P}$ is compact.

16. If f is a non-negative real-valued relation on $I = [\![0, 1]\!]$, then $f : I \to \mathsf{R}$ is upper semi-continuous (in the sense of 18 D.1) if and only if the correspondence $\sum\{[\![0, fx]\!] \mid x \in I\} : I \to \mathsf{R}$ is upper semi-continuous.