

Jarník's Notes of the Lecture Course Allgemeine Idealtheorie by B. L. van der Waerden (Göttingen 1927/1928)

Van der Waerden's Monograph Moderne Algebra

In: Jindřich Bečvář (author); Martina Bečvářová (author): Jarník's Notes of the Lecture Course Allgemeine Idealtheorie by B. L. van der Waerden (Göttingen 1927/1928). (English). Praha: Matfyzpress, 2020. pp. 207–220.

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VAN DER WAERDEN'S MONOGRAPH

MODERNE ALGEBRA

Van der Waerden's celebrated two-volume monograph *Moderne Algebra* was published by Springer in Berlin in the years 1930 and 1931, and it appeared in the book series *Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete* (Band XXXIII, Band XXXIV). This series was founded by Richard Courant (1888–1972) in 1921. In addition to Courant, it was edited by Wilhelm Blaschke (1885–1962), Max Born (1882–1970), and Carl Runge (1856–1927).

Both volumes of van der Waerden's monograph *Moderne Algebra* bear the subtitle *Unter Benutzung von Vorlesungen von E. Artin und E. Noether*, in which van der Waerden gave credit for the inspiration leading to his book.

In the preface to the first volume (*Vorwort*), he stated again that the lectures delivered by Artin in 1926 served as a starting point, but after subsequent rewriting and expansion of the text, being influenced by other sources and his own research, he finally departed and diversified from Artin's original conception in a significant way.

Das vorliegende Buch hat sich aus einer Ausarbeitung einer Vorlesung von E. Artin (Hamburg, Sommer 1926) entwickelt; es ist aber so vielen Umarbeitungen und Erweiterungen unterzogen und es sind so viele andere Vorlesungen und neuere Untersuchungen darin verarbeitet worden . . . , daß man die Artinsche Vorlesung nur schwer darin wird wiederfinden können. (MA-I, 1930, p. v)

Van der Waerden described the goals of the book in the first paragraph of the introduction (*Einleitung*) as follows:

Ziel des Buches. Die „abstrakte“, „formale“ oder „axiomatische“ Richtung, der die Algebra ihren erneuten Aufschwung in der jüngsten Zeit verdankt, hat vor allem in der Körpertheorie, der Idealtheorie, der Gruppentheorie und der Theorie der hyperkomplexen Zahlen zu einer Reihe von neuartigen Begriffsbildungen, zur Einsicht in neue Zusammenhänge und zu weitreichenden Resultaten geführt. In diese ganze Begriffswelt den Leser einzuführen, soll das Hauptziel dieses Buches sein. (MA-I, 1930, p. 1)

He called reader's attention to the fact that the introductory parts are rather brief (but without gaps) – *knapp (aber lückenlos)* – since a more detailed treatment of algebraic topics is available in the following, relatively recent

books dealing with group theory, field theory, and classical as well as linear algebra:

- A. Speiser: *Die Theorie der Gruppen von endlicher Ordnung*, 2nd edition, Springer, Berlin, 1927,
- H. Hasse: *Höhere Algebra I., II.*, Sammlung Göschen 931, 932, W. de Gruyter, Berlin, Leipzig, 1926, 1927,
- O. Haupt: *Einführung in die Algebra I., II.*, Akademische Verlagsgesellschaft, Leipzig, 1929,
- O. Perron: *Algebra I., II.*, W. de Gruyter, Berlin, Leipzig, 1927,
- M. Bôcher: *Introduction to Higher Algebra*, Macmillan, New York, 1907; German translation (H. Beck): *Einführung in die höhere Algebra*, Teubner, Leipzig, Berlin, 1910,
- L.E. Dickson: *Modern Algebraic Theories*, Sanborn, Chicago, New York, Boston, 1926; German translation (E. Bodewig): *Höhere Algebra*, Teubner, Leipzig, Berlin, 1929.¹

The sources for van der Waerden's monograph were also mentioned in the introduction:

- E. Artin: *Vorlesung über Algebra*, Hamburg, Sommersemestr 1926,
- E. Artin, W. Blaschke, O. Schreier, B.L. van der Waerden: *Seminar über Idealtheorie*, Hamburg, Wintersemestr 1926/1927,
- B.L. van der Waerden: *Vorlesung über allgemeine Idealtheorie*, Göttingen, Wintersemestr 1927/1928,²
- E. Noether: *Vorlesungen über Gruppentheorie und hyperkomplexe Zahlen*, Göttingen, Wintersemestr 1924/1925, Wintersemestr 1927/1928.³

Still in the introduction, van der Waerden remarked that the exposition of field theory in his book follows the landmark work

- E. Steinitz: *Algebraische Theorie der Körper*.⁴

Later in 1975, he provided a very detailed account of all sources and inspirations leading to the *Moderne Algebra* in the paper *On the Sources of*

¹ For more details on the algebra textbooks by Hasse, Haupt, Perron, Bôcher, and Dickson, see the chapter *Monographs and Textbooks in Algebra at the Turn of the 20th Century*.

² This is precisely the lecture course recorded by Vojtěch Jarník.

³ These lectures were compiled by Emmy Noether in her extensive work *Hyperkomplexe Größen und Darstellungstheorien*, *Mathematische Zeitschrift* 30(1929), pp. 641–692. For more information on Emmy Noether's mathematical achievements see [BS] and [Ko].

⁴ *Journal für die reine und angewandte Mathematik* 137(1910), pp. 167–309. Van der Waerden stated in a footnote that the work of Steinitz will soon appear in a book, which indeed happened (*Algebraische Theorie der Körper*, W. de Gruyter, Berlin, Leipzig, 1930, 150 pages). A supplement to this book entitled *Erläuterungen zu Ernst Steinitz Algebraische Theorie der Körper* (27 pages) was written by Reinhold Baer (1902–1979) and Helmut Hasse (1898–1979).

My Book Moderne Algebra [Wa3]. In particular, he underscored the inspiring mathematical environment in Göttingen and Hamburg, and listed the names of several mathematicians with whom he discussed algebraic topics and drew from their work.⁵ However, a direct stimulus for the book came from Emil Artin:

He had promised to write a book on algebra for the “Yellow Series” of Springer. We decided that I should take lecture notes and that we should write the book together. Courant, the editor of the series, agreed. Artin’s lectures were marvellous. I worked out my notes and showed Artin one chapter after another. He was perfectly satisfied and said, “Why don’t you write the whole book?” ([Wa3], p. 38)

Let us also mention that in both volumes, van der Waerden provided (right after the tables of contents) a scheme of dependencies between the material of all the seventeen chapters of both volumes: *Leitfaden. Übersicht über die Kapitel der beiden Bände und ihre logische Abhängigkeit.*

* * *

The text of both volumes of van der Waerden’s monograph evolved in subsequent editions, and the author described the most important changes in his paper [Wa3].

In later editions, when the importance of valuations became more and more obvious, a separate chapter was devoted to fields with valuations ... ([Wa3], p. 40)

We now briefly mention the changes performed by van der Waerden in the second and third edition of the first volume (1937, 1950), and in the second edition of the second volume (1940). The number of chapters in both volumes (10 + 7) remained unchanged. Their structure was to a large extent preserved, but there were alterations in their contents. Certain sections were omitted, new sections appeared, and the material was sometimes reorganized, expanded or simplified. One can get a rough overview of the changes by comparing the tables of contents of both editions.⁶ There were only minor changes in the scheme of dependencies between the chapters in the second edition, and the list of suggested reading in the second edition of the first volume included as an additional item the problem book

- H. Hasse: *Aufgabensammlung zur Höheren Algebra*, Sammlung Göschen 1082, W. de Gruyter, Berlin, 1934.

⁵ In chapter 6 of the book [So7], Alexander Soifer describes the history of the origin of van der Waerden’s book *Moderne Algebra*. On the basis of the study of preserved sources and memories of leading mathematicians, he analyses Artin’s and Emmy Noether’s role, influence and contribution to the creation of the book.

⁶ The table of contents in the second edition of the first volume describes the correspondence between the sections of both editions.

The first chapter of the first volume remained unchanged. In the second one, the material of Section 7 (*Untergruppen*) was expanded and divided in two sections (*Untergruppen* and *Das Rechnen mit Komplexen. Nebenklassen*). The third chapter included a short new section *Vektorräume und hyperkomplexe Systeme* (4 pages). The fourth chapter incorporated two sections *Die Resultante zweier Polynome* and *Die Resultante als symmetrische Funktion der Wurzeln* that were moved from the second volume, as well as a new section *Partialbruchzerlegung der rationalen Funktionen*.

The fifth chapter included a rewritten section *Lineare Gleichungen über einen Schiefkörper* (2 pages) that was originally in the second volume. On the other hand, Section 36 *Einfache transzendente Erweiterungen* was moved from the fifth to the eighth chapter.

There were no major changes in chapters six and seven. The eighth chapter *Geordnete und wohlgeordnete Mengen* (*Geordnete Mengen, Das Auswahlpostulat und der Wohlordnungssatz, Die transfiniten Induktion*, altogether 6 pages) was completely omitted.

The original ninth chapter became chapter eight, and it was rewritten; it incorporated the previously mentioned Section 36 from the fifth chapter, as well as a new section *Differentiation der algebraischen Funktionen* (5 pages).

The material of the original tenth chapter *Reelle Körper* was, mainly under the influence of Max Deuring (1907–1984), rewritten, substantially expanded, and split in two chapters: *Reelle Körper* and *Bewertete Körper*. This is why the number of chapters in the first volume was preserved. Information on new journal articles was appended (MA-I, 1937, pp. 245, 266).

In the preface to the second edition of the first volume, van der Waerden commented on the changes he had made:

... habe ich danach gestrebt, das Buch wieder ganz auf die Höhe der Zeit zu bringen. Dazu war es allererst notwendig, die Grundlagen der Bewertungstheorie ausführlicher und gründlicher zu behandeln.

... aus dem ersten Bande ein für Anfänger brauchbares Elementarbuch der Algebra mit Ausnahme der Determinantentheorie zu machen.

... habe ich versucht, bedenkliche mengentheoretische Schlußweisen in der Algebra möglichst zu vermeiden.

... wurden diejenigen Teile der Körpertheorie, die auf dem Auswahlpostulat und dem Wohlordnungssatz beruhten, in der neuen Auflage ganz weggelassen. (MA-I, 1937, pp. v–vi)

Throughout the whole text, the author performed small corrections and various improvements (*Viele kleinere Verbesserungen, meist didaktischer Natur*, ..., MA-I, 1937, p. vi).

We remark that in Germany, the second edition of the first volume was sold out as early as in 1944.

In the third edition of the first volume (1950), the first chapter was enlarged with three sections that comprised chapter eight in the first edition, and were suppressed in the second edition.

In the fifth chapter, the sections *Vollkommene und unvollkommene Körper*⁷ and *Normen und Spuren* were shortened, and the last section entitled *Die Ausführung der körpertheoretischen Operationen in endlichvielen Schritten* (4 pages) was omitted. In the seventh chapter, the section *Die Perioden der Kreisteilungsgleichung* was rewritten and incorporated into the previous section, and the section *Die metazyklischen Gleichungen von Primzahlgrad* was omitted. In the eighth chapter, the section *Der Transzendenzgrad* was substantially expanded and split in two sections – *Algebraische Abhängigkeit und Unabhängigkeit* and *Der Transzendenzgrad*.

The tenth chapter *Bewertete Körper* was subject to major changes and expansion. The section *Bewertung von algebraischen Erweiterungskörpern* was replaced by the following six sections: *Bewertung von algebraischen Erweiterungskörpern: Perfekter Fall*, *Bewertung von algebraischen Erweiterungskörpern: Allgemeinen Fall*, *Bewertungen von algebraischen Zahlkörpern*, *Bewertungen des rationalen Funktionenkörpers $\Delta(x)$* , *Bewertungen von algebraischen Funktionenkörpern*, *Die abstrakte Riemannsche Fläche*. The page extent from the second edition (pp. 245–266) rapidly increased (pp. 248–285).

Concerning the most important changes, van der Waerden wrote the following in the preface:

Schon in der zweiten Auflage wurde die Bewertungstheorie stark ausgebaut. Sie hat inzwischen in der Zahlentheorie und in der algebraischen Geometrie ihre Wichtigkeit immer mehr erwiesen. Daher habe ich das Kapitel Bewertungstheorie sehr viel ausführlicher und deutlicher gemacht.

Vielfachem Wunsche entsprechend, habe ich die Abschnitte über Wohlordnung und transfinite Induktion, die in der zweiten Auflage weggefallen waren, wieder aufgenommen und darauf fußend die Steinitzsche Körpertheorie wieder in voller Allgemeinheit gebracht. (MA-I, 1950, p. v)

As we already mentioned, van der Waerden took into account stimuli from his colleagues and friends, which provided inspiration for various corrections and improvements of the text. The most distinctive comments, which led to changes in the third edition of the first volume, came from Oscar Zariski (1899–1986) and Wouter Peremans (1921–1999).

The list of recommended reading in the third edition of the first volume no longer included Bôcher's book. There were no changes in the scheme of dependencies with respect to the second edition.

In the second volume (Chapters 11 to 17), the numbering of chapters and sections continues uninterrupted from the first volume. In the first edition,

⁷ The passage entitled *Wurzelkörper* was left out.

the two volumes contain Sections 1–70 and 71–128, respectively. In the second edition, the section numbers are 1–76 and 77–133, respectively.⁸

The eleventh chapter of the second edition no longer contained two sections on resultants (they were moved to the fourth chapter of the second edition of the first volume), and there were no major changes in chapters twelve and fourteen.

In chapter thirteen, the section *Parameterdarstellung algebraischer Mannigfaltigkeiten* was transformed into the section *Die Nullstellen eines Primideales*. The section *Spezialfälle und Anwendungen des Noetherschen Satzes* was rewritten and appended to the previous section *Der Noethersche Satz*. The final section *Der Grad einer Mannigfaltigkeit und die Schnittpunkte mit linearen Räumen* was omitted.

In the fifteenth chapter, two terminological curiosities appear in the titles of sections. In the title of Section 107, the original term *Körper* (according to the first edition, it need not be commutative) was replaced by the new term *Schiefkörper*. In the title of Section 108, the term *Hauptidealringe* was changed to *Euklidische Ringe* – the terminological relationship to MA-I is clear (see Section 16 of the first edition, and Section 18 of the second edition).

Chapter sixteen witnessed another terminological shift in the titles of sections: the term *Automorphismenring* from the first edition was altered to *Endomorphismenring* in the second edition.⁹ The last section of this chapter was substantially rewritten, slightly expanded, and the original title *Produkte von hyperkomplexen Systemen. Erweiterung des Grundkörpers* changed to *Das Verhalten der halbeinfachen hyperkomplexen Systeme bei Erweiterung des Grundkörpers*.

The last chapter was enlarged with three sections: *Halbgruppen von linearen Transformationen und ihr Verhalten bei Erweiterung des Grundkörpers*, *Die Brauerschen Algebrenklassen. Charakterisierung der Zerfällungskörper, Verschränkte Produkte; Faktorensysteme*. Naturally, the number of pages increased (pp. 177–212 in the first edition, pp. 167–219 in the second edition).

In the preface to the second edition of the second volume (1940), van der Waerden characterized the most significant changes as follows:

... die letzten beiden Kapitel, die von hyperkomplexen Zahlen und ihren Darstellungen handeln, entsprechend der stürmischen Entwicklung der neueren Algebrentheorie, erheblich erweitert und umgestaltet worden. Dafür konnte bei

⁸ However, in contrast to the second edition of the first volume, the table of contents of the second edition of the second volume no longer provides the correspondence between the sections of the first and second editions.

⁹ Unter „Automorphismen“ von \mathfrak{M} verstehen wir in diesem Paragraphen alle Operatorhomomorphismen von \mathfrak{M} in sich, also nicht nur die 1-Isomorphismen von \mathfrak{M} auf sich. (MA-II, 1931, p. 165). Unter einem Endomorphismus von \mathfrak{M} verstehen wir ... einen Operatorhomomorphismus von \mathfrak{M} in sich. (MA-II, 1940, p. 153).

den Polynomidealen einiges weggelassen werden, was mehr in die algebraische Geometrie als in die Algebra gehört. (MA-II, 1940, p. v)

We point out that the fourth edition of the first volume and the third edition of the second volume of van der Waerden's monograph appeared in 1955 under the shortened title *Algebra*. This change was suggested by the algebraist and number theorist Heinrich Karl Theodor Brandt (1886–1954) in 1952, who in his review of the third edition of the first volume wrote:

*Was den Titel anbetrifft, so würde ich es begrüßen, wenn in der 4. Auflage der schlichtere, aber kräftigere Titel „Algebra“ gewählt würde. Ein Buch, das so viel an bester Mathematik bietet, wie sie war, ist und sein wird, sollte nicht durch den Titel den Verdacht erwecken, als ob es nur einer Modeströmung folgte, die gestern noch unbekannt bekannt war und vielleicht morgen vergessen sein wird.*¹⁰

It is remarkable that van der Waerden kept working on the text of his monograph for over three decades. The most important changes were always mentioned in the prefaces to the individual editions. Let us briefly mention the editions from the years 1966/1967.

In the seventh edition of the first volume published in 1966, the sections on ordered sets, the axiom of choice and transfinite induction were removed from the first chapter and transformed into a new ninth chapter entitled *Ordnung und Wohlordnung von Mengen* (6 pages).¹¹ Besides this, the volume was enlarged with a new chapter *Vektorräume und Tensorräume* (22 pages). Some ideas from Emil Artin's celebrated book¹² were utilized in the part dealing with the Galois theory. The chapter *Bewertete Körper* was moved to the second volume. Thus, the seventh edition of the first volume has 11 chapters.

Among other things in the preface, van der Waerden noted:

Als die erste Auflage geschrieben wurde, war sie als Einführung in die neuere abstrakte Algebra gedacht. Teile der klassischen Algebra, insbesondere die Determinantentheorie, wurden als bekannt vorausgesetzt. Heute aber wird das Buch vielfach von Studenten als erste Einführung in die Algebra benutzt. Daher wurde es notwendig, ein Kapitel über „Vektorräume und Tensorräume“ einzufügen, in dem die Grundbegriffe der linearen Algebra, insbesondere der Determinantenbegriff erörtert werden. (MA-I, 1966, p. vi)

The structure of the fifth edition of the second volume from 1967, which followed immediately after the seventh edition of the first volume, differed from that of the previous editions. A comparison with the first edition from 1931 documents the development of algebra and research in algebra since the end of

¹⁰ Jahresbericht der Deutschen Mathematiker-Vereinigung 55(1952), p. 48.

¹¹ Zorn's lemma was derived as an immediate consequence of the axiom of choice, and a similar procedure, following Hellmuth Kneser (1898–1973), was used to prove the well-ordering theorem.

¹² E. Artin: *Galois Theory*, Lectures delivered at the University of Notre Dame, The University of Notre Dame Press, 1942, 2nd revised edition 1944, 1946.

the 1920s till the 1960s, as well as van der Waerden's great effort devoted to his monograph in the years 1930 to 1967. He remarked in the preface:

Der Band besteht jetzt aus drei voneinander unabhängigen Gruppen von je drei Kapiteln:

Kapitel 12–14: Lineare Algebra, Algebren, Darstellungstheorie.

Kapitel 15–17: Idealtheorie.

Kapitel 18–20: Bewertete Körper, Algebraische Funktionen, Topologische Algebra.

If we examine the evolution of the second volume of van der Waerden's monograph, we have to admire his effort to keep the text up-to-date from the viewpoint of both mathematics as well as its exposition.¹³

* * *

Van der Waerden's monograph *Moderne Algebra* was very successful, and represented the culmination of a certain era in the development of algebra. It placed emphasis especially on the theory of abstract groups, rings, ideals, hypercomplex numbers and fields. The algebra of structures was conceived axiomatically, and was built on top of set-theoretic foundations. It was subject to evolution, being abreast of the times for more than three decades. Let us recall what Saunders Mac Lane (1909–2005) wrote in his short paper *Van der Waerden's Modern Algebra* [Ma] in 1997:

This beautiful and eloquent text served to transform the graduate teaching of algebra, not only in Germany, but elsewhere in Europe and the United States. ...

It was van der Waerden who understood the real thrust of abstract algebra and who presented it abstractly but without pedantry. ([Ma], p. 321)

Victor J. Katz and Karen Hunger Parshall, leading contemporary historians of mathematics and experts in the history of modern algebra, highly appreciate the role of van der Waerden's monograph in the development of modern algebra. In the preface to their book *Taming the Unknown* [KP] they wrote:

*The textbook *Moderne Algebra* (1930–1931) ... became the manifesto for this new definition of algebra that has persisted into the twenty-first century. ([KP], p. 10)*

* * *

Further evidence of the success of van der Waerden's monograph is provided by the following list of its editions and translations, which is surely incomplete. We have also included some information on published reviews.¹⁴

¹³ Remark that in the Table 1 on p. 904 of his article [Sl], K.-H. Schlote provides an overview of the variations in distribution of chapters between the first (1930–1931), third (1950–1955) and seventh/fifth (1966–1967) editions of van der Waerden's book.

¹⁴ We refer to reviewing journals using the easily understandable abbreviations JFM

German – Volume I.

- *Moderne Algebra. Unter Benutzung von Vorlesungen von E. Artin und E. Noether.* Erster Teil. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band XXXIII., Verlag von Julius Springer, Berlin, 1930, viii + 243 pages.

JFM 56.0138.01 (W. Specht)

O. Ore: Bulletin of the American Mathematical Society 38(1932),
pp. 327–329

O. Ore: The American Mathematical Monthly 38(1931), p. 226

H. Hahn, O. Taussky: Monatshefte für Mathematik und Physik 39(1932),
pp. A11–A12

D.: Nieuw Archief voor Wiskunde (2) 17(1931), pp. 101–102

U. Wegner: Zeitschrift für mathematischen und naturwissenschaftlichen
Unterricht 63(1932), pp. 249–250

Th. Skolem: Norsk Matematisk Tidsskrift 13(1931), pp. 58–59

J. von Neumann: Acta Scientiarum Mathematicarum (Szeged) 5(1932),
pp. 259–260

W.L. Ferrar: The Mathematical Gazette 15(1931), No. 214, pp. 437–438

B. Levi: Bollettino della Unione matematica Italiana 10(1931),
pp. 239–243¹⁵

- 2nd edition: 1937, x + 272 pages.

JFM 63.0082.06 (W. Specht), Zbl 0016.33902 (A.A. Albert)

O. Ore: Bulletin of the American Mathematical Society 44(1938), p. 320

N. Hofreiter: Monatshefte für Mathematik und Physik 45(1936),
pp. A25–A26

A.R.R.: The Mathematical Gazette 21(1937), No. 245, p. 299

W. Thimm: Jahresbericht der Deutschen Mathematiker-Vereinigung
49(1939), p. 81

- Reprint (2nd ed.): G.E. Stechert, New York, 1943, 272 pages.

MR 0009016

(Jahrbuch über die Fortschritte der Mathematik, 1868–1942), Zbl (Zentralblatt für Mathematik und ihre Grenzgebiete, 1931 – present), MR (Mathematical Reviews, 1940 – present).

¹⁵ Review of both volumes as well as of the book E. Steinitz: *Algebraische Theorie der Körper* from 1930.

- 3rd edition: 1950, viii + 292 pages.
 Zbl 0037.01903 (L. Rédei), MR 0037277 (I. Kaplansky)
 D. Zelinsky: Bulletin of the American Mathematical Society 57(1951), p. 206
 G.A. Dirac: The Mathematical Gazette 35(1951), No. 313, p. 203
 M. Draeger: Journal of Applied Mathematics and Mechanics (Zeitschrift für
 angewandte Mathematik und Mechanik) 31(1951), p. 155
 H. Brandt: Jahresbericht der Deutschen Mathematiker-Vereinigung
 55(1952), pp. 47–48
- 4th edition: *Algebra*, 1955, viii + 292 pages.
 Zbl 0067.00501 (H. Orsinger), MR 0069787
 E. Kamke: Jahresbericht der Deutschen Mathematiker-Vereinigung
 59(1957), p. 12
- 5th edition: 1960, vii + 292 pages.
 Zbl 0087.25903, MR 0122834
 H. Grunsky: Jahresbericht der Deutschen Mathematiker-Vereinigung
 63(1961), p. 46
- 6th edition: 1964, viii + 292 pages.
 MR 0177027
- 7st edition: Heidelberger Taschenbücher 12, Springer, Berlin, Heidelberg,
 New York, 1966, x + 271 pages.
 Zbl 0137.25403, MR 0263581
- 8th edition: 1971, xi + 272 pages.
 Zbl 0221.12001
- 9st edition: Springer, Berlin, Göttingen, Heidelberg, 1993, xii + 272 pages.
 Zbl 0781.12002 (P. Bundschuh)

German – Volume II.

- *Moderne Algebra. Unter Benutzung von Vorlesungen von E. Artin und E. Noether.* Zweiter Teil. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band XXXIV., Verlag von Julius Springer, Berlin, 1931, vii + 216 pages.

JFM 57.0153.03 (W. Specht), Zbl 0002.00804 (H. Grell)

O. Ore: Bulletin of the American Mathematical Society 38(1932),
pp. 327–329

O. Taussky: Monatshefte für Mathematik und Physik 40(1933), pp. A3–A4

D.: Nieuw Archief (2) 17(1932), pp. 187–188

U. Wegner: Zeitschrift für mathematischen und naturwissenschaftlichen
Unterricht 63(1932), pp. 249–250

Th. Skolem: Norsk Matematisk Tidsskrift 14(1932), pp. 101–102

J. von Neumann: Acta Scientiarum Mathematicarum (Szeged) 5(1932),
pp. 259–260

W.L. Ferrar: The Mathematical Gazette 16(1932), No. 218, p. 147

B. Levi: Bollettino della Unione matematica Italiana 10(1931), pp. 239–243

- 2nd edition: 1940, viii + 224 pages.

JFM 66.0034.01 (H.L. Schmid), Zbl 0022.29801 (M. Deuring),
MR 0002841

N. Hofreiter: Monatshefte für Mathematik und Physik 49(1941), p. A21

G. Köthe: Jahresbericht der Deutschen Mathematiker-Vereinigung 51(1941),
pp. 74–75

- Reprint (2nd ed.): G.E. Stechert, New York, 1943, 224 pages.

MR 0009016

- 3rd edition: *Algebra*, 1955, viii + 224 pages.

Zbl 0067.00502 (H. Orsinger), MR 0072829

E. Kamke: Jahresbericht der Deutschen Mathematiker-Vereinigung
59(1957), p. 12

- 4th edition: 1959, ix + 275 pages.

Zbl 0192.33001, MR 0122835 (L.J. Paige)

D. Zelinsky: Bulletin of the American Mathematical Society 67(1961),
pp. 83–86

G. Köthe: Jahresbericht der Deutschen Mathematiker-Vereinigung 63(1961),
pp. 23–24

- 5th edition: Heidelberg Taschenbücher 23, Springer, Berlin, Heidelberg, New York, 1967, x + 300 pages.

Zbl 0192.33002, MR 0233647

- 6th edition: *Mit einem Geleitwort von Jürgen Neukirch*, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1993, xii + 300 pages.

Zbl 0781.12003 (P. Bundschuh)

English – Volume I.

- *Modern Algebra I*. Translated from the 2nd revised German edition by Fred Blum. With revisions and additions by the author. Ungar, New York, 1949, xii + 264 pages.

Zbl 0033.10102, Zbl 0039.00902, MR 0029363

D. Zelinsky: Bulletin of the American Mathematical Society 57(1951), p. 206

- 2nd revised edition: 1953, xii + 264 pages, 6th print 1966.
- *Algebra I*. Translated from the 7th German edition by Fred Blum and John R. Schulenberger, Ungar, New York, 1970, xiv + 265 pages.

MR 0263582

- Reprint: Springer, New York, 1991, xiv + 265 pages.

Zbl 0724.12001 (K.-B. Gundlach), MR 1080172

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