Otakar Borůvka
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PART II.

BY

O. BORŮVKA.

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STUDIES ON MULTIPLICATIVE SYSTEMS.

PART II.

BY

O. BORŮVKA.

This paper is a sequel to the publication *Studies on multiplicative systems. Part I.* (Publ. Fac. Sci. univ. Masaryk, n° 245, 1937). It deals with m. systems without kernel, i. e. m. systems $\mathfrak{M}$ characterised in the way that for every element $a \in \mathfrak{M}$ there exists a positive integer α, called the index of a, such that a is product of α but not more than α elements of $\mathfrak{M}$.

16. M. systems homomorphically representable on the infinite cyclic m. system $\mathfrak{Z}$. Let A be a non-vacuous set. Any system of mutually exclusive subsets of A, covering A, is termed *decomposition of A*.

Let $\mathfrak{M}$ be a m. system. Let $(0 \neq)\, A \subset \mathfrak{M}$ and the sequence

$$\{A_1, A_2, \dots\} \tag{12}$$

be a decomposition of A. Let W_α, for $\alpha = 1, 2, \dots$, be defined by the formula

$$W_\alpha \quad \Sigma A_{\alpha_1} A_{\alpha_2} \dots A_{\alpha_\beta},$$

$\alpha_1, \alpha_2, \dots, \alpha_\beta$ being an arrangement of β equal or different positive integers such that $\alpha_1 + \alpha_2 + \dots + \alpha_\beta = \alpha$ and the summation being related to all arrangements of this kind for $\beta = 1, 2, \dots, \alpha$. If a factor of some term in the sum is vacuous, this term is to be replaced by the vacuous set. It is clear that W_α is a subset of $\mathfrak{M}$. We call it *aggregate α in the decomposition (12)*. For instance, we get

$$W_1 = A_1, \quad W_2 = A_2 \vee A_1^2, \qquad W_3 = A_3 \vee A_1 A_2 \vee A_2 A_1 \vee A_1^3, \dots$$

For $\alpha, \beta = 1, 2, \dots$, there evidently holds

$$W_\alpha W_\beta \subset W_{\alpha+\beta}. \tag{13}$$

The decomposition (12) of A is called *generating* if: 1° $A_1 \neq 0$ 2° any two aggregates α, β in the decomposition are mutually exclusive for $\alpha \neq \beta$.

1. Let $\mathfrak{M} = M_1 \vee M_2 \vee \dots$ be a m. system without kernel and be homomorphically representable on $\mathfrak{Z}$. Let $\mathfrak{F}(F_1, F_2, \dots)$ be a homomorphism of $\mathfrak{M}$ on $\mathfrak{Z}$. Let $A_\alpha = M_1 \cap F_\alpha$, for $\alpha = 1, 2, \dots$, so that the sequence $\{A_1, A_2, \dots\}$ is a decomposition of the excentrum of $\mathfrak{M}$. Then the aggregate α in this decomposition is precisely the set F_α.

Proof. From the relation

$$\mathfrak{M} = F_1 \vee F_2 \vee \dots$$

we get $$\mathfrak{M}^2 = \sum_{\alpha, \beta = 1, 2, \dots} F_\alpha F_\beta$$

and therefore $$\mathfrak{M}^2 = F_{22} \vee F_{23} \vee \dots,$$

$F_{2\alpha}$, for $\alpha = 2, 3, \dots$, being given by the formula

$$F_{2\alpha} = F_1 F_{\alpha-1} \vee F_2 F_{\alpha-2} \vee \dots \vee F_{\alpha-1} F_1.$$

According to **13**·*1* there holds $F_{2\alpha} \subset F_\alpha$ and therefore $A_\alpha \vee F_{2\alpha} \subset F_\alpha$, $\alpha = 2, 3, \dots$, Thus, from the relations

$$\begin{aligned} \mathfrak{M} = M_1 \vee \mathfrak{M}^2 &= A_1 \vee (A_2 \vee F_{22}) \vee (A_3 \vee F_{23}) \vee \dots \\ &= F_1 \vee \quad F_2 \quad \vee \quad F_3 \quad \vee \dots \end{aligned}$$

follows

$$F_\alpha = A_\alpha \vee F_1 F_{\alpha-1} \vee F_2 F_{\alpha-2} \vee \dots \vee F_{\alpha-1} F_1 \, (\alpha = 1, 2, \dots; F_1 = A_1).$$

Now let W_α be the aggregate α in the mentioned decomposition of the excentrum of $\mathfrak{M}$, $\alpha = 1, 2, \dots$ Obviously the equality $W_1 = F_1$ holds. Let us therefore suppose $W_1 = F_1$, $W_2 = F_2, \dots, W_{\alpha-1} = F_{\alpha-1}$, for some $\alpha > 1$. We get

$$F_\alpha = A_\alpha \vee W_1 W_{\alpha-1} \vee W_2 W_{\alpha-2} \vee \dots \vee W_{\alpha-1} W_1,$$

so that, according to the definition of W_α and by (13), results $F_\alpha \subset W_\alpha$. Consider an arbitrary element $a \in \mathfrak{M}$ contained in W_α. Then $a \in A_{\alpha_1} A_{\alpha_2} \dots A_{\alpha_\beta}$, $\alpha_1, \alpha_2, \dots, \alpha_\beta$ being a suitable arrangement of $(1 \leqq)\ \beta\ (\leqq \alpha)$ equal or different positive integers such that $\alpha_1 + \alpha_2 + \dots + \alpha_\beta = \alpha$. If $\beta = 1$ we get $\alpha_1 = \alpha$ and therefore $a \in A_\alpha \subset F_\alpha$. If $\beta > 1$ we get $1 \leqq \alpha_1 \leqq \alpha - 1$ and $\alpha_2, \dots, \alpha_\beta$ is an arrangement of (at most $\alpha - \alpha_1$) equal or different positive integers such that $\alpha_2 + \dots + \alpha_\beta = \alpha - \alpha_1$. Hence $a \in W_{\alpha_1} W_{\alpha - \alpha_1} \subset F_\alpha$, by induction. It results that $W_\alpha = F_\alpha$ for $\alpha = 1, 2, \dots$

·2. *Let $\mathfrak{M}$ be a m. system without kernel. $\mathfrak{M}$ is homomorphically representable on $\mathfrak{Z}$ if and only if there exists a sequence of sets which is a generating decomposition of the excentrum of $\mathfrak{M}$.*

Proof. a) Let $\mathfrak{M}$ be homomorphically representable on $\mathfrak{Z}$. Using the notations as in ·*1* the sequence $\{A_1, A_2, \dots\}$ is a generating decomposition of the excentrum M_1. In fact, the sets A_α, $\alpha = 1, 2, \dots$, are evidently mutually exclusive and cover M_1; as $A_1 = F_1$ we get $A_1 \neq 0$; by ·*1* are any two aggregates α, β in this decomposition mutually exclusive for $\alpha \neq \beta$.

b) Suppose that there exists a sequence of sets $\{A_1, A_2, \dots\}$ which is a generating decomposition of the excentrum M_1. Let W_α, $\alpha = 1, 2, \dots$, be the aggregate α in this decomposition. According to the supposition, the sets $W_1, W_2, \dots$ are mutually exclusive. Because (13) holds, we get $W_1{}^\alpha \subset W_\alpha$ for $\alpha = 1, 2, \dots$; hence $W_\alpha \neq 0$ since $W_1 = A_1 \neq 0$. Further, there holds $\mathfrak{M} = W_1 \vee W_2 \vee \ldots$ Thus $\mathfrak{F}(W_1, W_2, \ldots)$ is a homomorphism of $\mathfrak{M}$ on $\mathfrak{Z}$ (13 ·*1*). — We remark the following relation: $A_\alpha = M_1 \cap W_\alpha$, $\alpha = 1, 2, \ldots$

By the theorem in nº 15 and by 6·*2*, ·*2* we get the following result:

Let $\mathfrak{M}$ be a m. system without kernel. Let α be an integer > 1. Consider a sequence of sets which is a decomposition of the set of all elements of $\mathfrak{M}$ whose indices are $\alpha, \alpha+1, \ldots, 2\alpha-1$, the first set of the sequence being non-vacuous. Then there are two aggregates β, γ in this decomposition, $\beta \neq \gamma$, which have a common element.

·*3. Let $\mathfrak{M}$ be a m. system without kernel and be homomorphically representable on $\mathfrak{Z}$. There exist a (1, 1) correspondence between the homomorphic representations of $\mathfrak{M}$ on $\mathfrak{Z}$ and the sequences of sets which are generating decompositions of the excentrum of $\mathfrak{M}$.*

Proof. Consider the following correspondence between the homomorphic representations of $\mathfrak{M}$ on $\mathfrak{Z}$ and the sequences of sets which are generating decompositions of the excentrum M_1 of $\mathfrak{M}$: With any homomorphism $\mathfrak{F}(F_1, F_2, \ldots)$ the sequence of sets $A_\alpha = M_1 \cap F_\alpha$, $\alpha = 1, 2, \ldots$, which really is a generating decomposition of M_1 (·*2* a)), is associated. By ·*2* b) we see that for any sequence of sets which is a generating decomposition of M_1 there exists a homomorphism of $\mathfrak{M}$ on $\mathfrak{Z}$ to which the sequence in the mentioned correspondence belongs. By ·*1*, with any two different homomorphic representation of $\mathfrak{M}$ on $\mathfrak{Z}$ two different sequences are associated.

17. Uniquely decomposable m. systems. Let $\mathfrak{M} = M_1 \vee M_2 \vee \ldots$ be a m. system without kernel. $\mathfrak{M}$ is termed *uniquely decomposable* if every sequence of sets $\{A_1, A_2, \ldots\}$, A_1 being non-vacuous, which is a decomposition of the excentrum of $\mathfrak{M}$, is generating. According to 16 ·*2* ·*3*, if $\mathfrak{M}$ is uniquely decomposable it is homomorphically representable on $\mathfrak{Z}$ and there exists a (*1, 1*) correspondence between the homomorphic representations of $\mathfrak{M}$ on $\mathfrak{Z}$ and the sequences of sets $\{A_1, A_2, \ldots\}$, $A_1 \neq 0$, which are decompositions of the excentrum of $\mathfrak{M}$.

·*1. $\mathfrak{M}$ is uniquely decomposable if and only if all decompositions of every element of $\mathfrak{M}$ into prime-factors differ only by the order of factors.*

Proof. a) Let $\mathfrak{M}$ be uniquely decomposable. Firstly, it is easy to see that $\mathfrak{M}$ is homogeneous. In fact, otherwise there exists an element $a \in \mathfrak{M}$ of index ≥ 3 admitting two different decompositions into prime-factors

$$a = p_1 p_2 \cdots p_\alpha = p'_1 p'_2 \cdots p'_\beta,$$

with $1 < \alpha < \beta$. Then for every decomposition $\{A_1, A_2, \ldots\}$ of the excentrum of $\mathfrak{M}$ such that all prime-factors p as well as p' are in A_1, we get $a \in A_1^\alpha \cap A_1^\beta$; it follows that a is a common element to the aggregate α as well as to the aggregate β in this decomposition. Further, all decompositions of every element $a \in \mathfrak{M}$ into prime-factors differ only by the order of factors. Indeed, let

$$a = p_1 p_2 \ldots p_\alpha = p'_1 p'_2 \ldots p'_\alpha \qquad (\alpha > 2)$$

be two decompositions of a into prime-factors. If, for instance, the prime-factor p'_α is none of the factors $p_1, p_2, \ldots, p_\alpha$, then for every decomposition $\{A_1, A_2, \ldots\}$ of the excentrum of $\mathfrak{M}$ such that $p_1, p_2, \ldots p_\alpha$, p'_1, $p'_2, \ldots, p'_{\alpha-1} \in A_1, p'_\alpha \in A_2$, the element a is contained in the aggregate α in this decomposition as well as in the aggregate $\alpha + 1$.

b) Suppose that all decompositions of every element of $\mathfrak{M}$ into prime-factors differ only by the order of factors. Let $\{A_1, A_2, \ldots\}$ be a decomposition of the excentrum of $\mathfrak{M}$ such that $A_1 \neq 0$. Let W_α be the aggregate α in this decomposition, $\alpha = 1, 2, \ldots$ Let $a \in \mathfrak{M}$, $a \in W_\alpha \cap W_\beta$. Then $a \in A_{\alpha_1} A_{\alpha_2} \ldots A_{\alpha_\gamma} \cap A_{\beta_1} A_{\beta_2} \ldots A_{\beta_\delta}$, the integers α, β satisfying the relations $\alpha_1 + \alpha_2 + \ldots + \alpha_\gamma = \alpha$, $\beta_1 + \beta_2 + \ldots + \beta_\delta = \beta$. Therefore there exist prime-factors $p_{\alpha_1} \in A_{\alpha_1}, p_{\alpha_2} \in A_{\alpha_2}, \ldots, p_{\beta_\delta} \in A_{\beta_\delta}$ such that $a = p_{\alpha_1} p_{\alpha_2} \ldots p_{\alpha_\gamma} = p_{\beta_1} p_{\beta_2} \ldots p_{\beta_\delta}$. Because the sets $A_1, A_2, \ldots$ are mutually exclusive, from the relation $p_{\alpha_\nu} = p_{\beta_\mu}$ follows $A_{\alpha_\nu} = A_{\beta_\mu}$. As the decompositions of a differ only by the order of factors, we get $\alpha = \beta$.

A simple example of a commutative uniquely decomposable m. system is the m. system whose elements are the integers $2, 3, \ldots$ and the multiplication is defined in the usual way. For this m. system the set M_α, $\alpha = 1, 2, \ldots$, is clearly the set of integers > 1, which are products of precisely α prime numbers. In particular, the excentrum of this m. system consists of all prime numbers. Because two decompositions of every positive integer into prime numbers differ only by the order of factors, the m. system in question is uniquely decomposable.

An example of a uniquely decomposable non-commutative m. system is the m. system defined in the following way; The elements are positive integers with the exception of those whose symbol in the decimal system contains the figure 0. The multiplication is defined as follows: For $\alpha = \alpha_1 \alpha_2 \ldots \alpha_\gamma$, $\beta = \beta_1 \beta_2 \ldots \beta_\delta$, $\gamma, \delta \geqq 1$, where α_μ (β_ν) denote the figures of α (β) in the decimal system, $\alpha\beta$ is given by the formula: $\alpha\beta = \alpha_1 \alpha_2 \ldots \alpha_\gamma \beta_1 \beta_2 \ldots \beta_\delta$. Consequently, we get, for instance $1 . 2 = 12$, $14 . 23391 = 1423391$. It is clear that this multiplication is associative. The set M_α, $\alpha = 1, 2, \ldots$, for the m. system in question is the set of its elements, whose symbol in the decimal system contains α figures. In particular, the excentrum is the set $\{1, 2, \ldots 9\}$. As every number is completely determined by its figures and their order, the above m. system is uniquely decomposable. It is clear that it is non-commutative.

18. Countable m. systems without kernel. A m. system is called *countable* if the set of its elements is countable. Let $\mathfrak{M}$ be a countable m. system. Then the set of elements of $\mathfrak{M}$ can be ranged in a sequence, that is to say, it can be put into a $(1, 1)$ correspondence with the set of all positive integers. We say that $\mathfrak{M}$ *is ranged* if a ranging of it in a sequence has been chosen. Let $\mathfrak{M}$ be ranged. We denote by a_α the element of $\mathfrak{M}$, which corresponds to the positive integer α. Then $\mathfrak{M} = \{a_1, a_2, \ldots\}$. About two elements $a_\alpha, a_\beta \in \mathfrak{M}$ we say that a_α *precedes* (*follows*) a_β if $\alpha < \beta$ $(\alpha > \beta)$. $\mathfrak{M}$ is isomorphic to the m. system, whose elements are positive integers and the product $\alpha\beta$ of any ordered pair ~~of positive~~ integers α, β is given by the formula $a_\alpha a_\beta = a_{\alpha\beta}$. Any m. subsystem of ~~a countable~~ m. system is at most countable; that is to say, finite or countable.

·1. *Let $\mathfrak{M} = M_1 \vee M_2 \vee \ldots$ be a m. system without kernel. $\mathfrak{M}$ is countable if and only if its excentrum ~~is at~~ most countable.*

Proof. It is evident that the excentrum M_1 is at most countable if $\mathfrak{M}$ is countable. Let us therefore suppose, that ~~the~~ excentrum M_1 is at most countable. Obviously it is sufficient to prove ~~that the~~ set M_α, for $\alpha = 2, 3, \ldots$, is at most countable. Consider to this effect an $\alpha > 2$ and an $a \in M_\alpha$. There exists at least one ordered group $p_1, \ldots, p_\alpha$ of prime-factors such that $a = p_1 \ldots p_\alpha$. Let us associate with every element $a \in M_\alpha$ one group of this kind. Then with any two different elements of M_α two different groups are associated. The elements of M_α are therefore put into a $(1, 1)$ correspondence with the elements of a certain subset in the set formed by all ordered groups of α elements of M_1. Because M_1 is at most countable, this set of groups and thus every one of its subsets is at most countable. Consequently, M_α is at most countable.

If the excentrum M_1 is finite so is M_α, for $\alpha = 1, 2, \ldots$

19. Let $\mathfrak{M}$ be a countable m. system. A ranging $\mathfrak{M} = \{a_1\, a_2 \ldots\}$ is termed *increasing* if it has the following property: The product of any two elements follows each of them. Then we get by induction (related to α) that the product of any $\alpha + 1$ elements of $\mathfrak{M}$ follows the element a_α. $\mathfrak{M}$ is called *increasing* if there exists an increasing ranging of $\mathfrak{M}$.

For instance, the m. system $\mathfrak{Z}^\nu = \{z^\nu, z^{\nu+1} \ldots\}$ is obviously increasing and the above written ranging is increasing for every positive integer ν. This m. system is isomorphic to the m. system whose elements are all positive integers and the multiplication is given by the formula: $\alpha\beta = \alpha + \beta + \nu - 1$, for $\alpha, \beta = 1, 2, \ldots$ On the contrary, the m. system consisting of all positive integers with the usual multiplication is obviously not increasing. Every m. subsystem of an increasing m. system is increasing.

·1. *Let $\mathfrak{M}$ be a countable m. system. $\mathfrak{M}$ is increasing if and only if it is without kernel and every one of its elements admits only a finite number of different decompositions into prime-factors.*

Proof. a) Let $\mathfrak{M}$ be increasing. Let $\mathfrak{M} = \{a_1, a_2, \ldots\}$ be an increasing ranging of $\mathfrak{M}$. Let $a_\alpha \in \mathfrak{M}$. As the product of any $\alpha + 1$ elements of $\mathfrak{M}$ follows the element a_α, a_α is a product of at most α elements of $\mathfrak{M}$. Hence $\mathfrak{M}$ is without kernel. If a decomposition of a_α into prime-factors be given, every prime-factor precedes a_α because the ranging is increasing. But only $\alpha - 1$ elements precede a_α. Consequently, there exists only a finite number of prime-factors which appear in the decompositions of a_α into prime-factors. Hence a_α admits only a finite number of different decompositions into prime-factors.

b) Let $\mathfrak{M}$ be without kernel and let every element of $\mathfrak{M}$ admit only a finite number of decompositions into prime-factors. Let $a \in \mathfrak{M}$. Every element $b \in \mathfrak{M}$ such that $a = xb$ or $a = by$ or $a = xby$, for suitable $x, y \in \mathfrak{M}$, shall be called *part of a*. Every element of $\mathfrak{M}$ has only a finite number of parts. Indeed, otherwise there exists a sequence of mutually different parts $\{b_1, b_2, \ldots\}$ of some element $a \in \mathfrak{M}$. Let α be the index of a. The index of every part b_β of a is $< \alpha$. Consequently, the number of prime-factors which appear in the decompositions of all parts $b_1, b_2, \ldots$ into prime-factors is not finite. For any part b_β of a there exists at least one decomposition of a into prime-factors in which all prime-factors occuring in an arbitrary decomposition of b_β into prime-factors appear. Consequently, the number of prime-factors appearing in the decompositions of a into prime-factors is not finite. a admits therefore an infinite number of different decompositions into prime-factors-which is contradictory to the supposition. Let $\mathfrak{M} = \{a'_1, a'_2, \ldots\}$ be a ranging of $\mathfrak{M}$. Let A_1 be the set formed by the element a'_1 and all its parts. For every integer $\alpha > 1$ let A_α be the set formed by the element a'_β and by all its parts which do not appear in $A_1 \vee \ldots \vee A_{\alpha-1}$, a'_β being the first element in the sequence $\{a'_1, a'_2, \ldots\}$ not contained in $A_1 \vee \ldots \vee A_{\alpha-1}$. It is clear that the sets $A_1, A_2, \ldots$ are non-vacuous, mutually exclusive and (according to the above consideration) finite. There holds $\mathfrak{M} = A_1 \vee A_2 \vee \ldots$ Let $\{a_1, a_2, \ldots\}$ be a ranging of $\mathfrak{M}$ defined in the following way: 1° Every element of $A_{\alpha+1}$ follows every element of A_α, $\alpha = 1, 2, \ldots$, 2° the elements of A_α are ordered in an arbitrarily chosen way upon the only condition that the indices of them increase. This ranging is increasing. In fact, let us choose $a_\mu, a_\nu \in \mathfrak{M}$. Then $a_\mu \in A_\alpha$, $a_\nu \in A_\beta$, $a_\mu a_\nu \in A_\gamma$ for suitable α, β, γ. As a_μ (a_ν) forms a part of $a_\mu a_\nu$, we get $a_\mu, a_\nu \in A_1 \vee \ldots \vee A_\gamma$. Hence $\gamma \geq \alpha, \beta$. If $\gamma > \alpha$, the element $a_\mu a_\nu$ follows a_μ because every element of A_γ follows every element of A_α; if $\gamma = \alpha$, the element $a_\mu a_\nu$ follows a_μ because the index of $a_\mu a_\nu$ is greater than the index of a_μ. Similarly, the element $a_\mu a_\nu$ follows a_ν.

For instance, every countable m. system without kernel which has a finite excentrum is increasing. An example of a non-increasing countable m. system without kernel is the m. oversystem $\mathfrak{M} = \{a_1, z, a_2, z^2, a_3, z^3, \ldots\}$ of the infinite cyclic m. system $\mathfrak{Z} = \{z, z^2, z^3, \ldots\}$ at the following projection: $f(a_\alpha) = z$, $f(z^\alpha) = z^\alpha$ for $\alpha = 1, 2, \ldots$ By $6 \cdot 3$, the elements $a_1, a_2, \ldots$ are prime-factors of $\mathfrak{M}$; according to the definition of the multiplication in $\mathfrak{M}$, for any positive integer β and for $\alpha = 1, 2, \ldots$ there holds $a_\alpha z^\beta = z^{\beta+1}$. Hence $z^{\beta+1}$ admits an infinite number of different decompositions into prime-factors.

III. Structure of m. systems without kernel.

20. Decompositions of sets. Let M be a non-vacuous set. We have already defined (nº **16**) what is meant by a *decomposition of M*. As long as the contrary is not stated we suppose that the sets which are elements of a decomposition, are non-vacuous; we shall denote them by small Latin letters. If a decomposition D of M consists of the sets $\{a\}$, where $a \in M$, we write $D \sim M$.

Let D_1 be a decomposition of M. Let D_2 be a decomposition of M such that every element of D_1 is the sum of some subsets of M which are elements of D_2. D_2 is termed *lower decomposition of M with regard* (w. r.) *to D_1* and we say that it *is* (*lies*) *under D_1*. Analogously, D_1 is termed *upper decomposition of M w. r. to D_2* and we say that it *is* (*lies*) *over D_2*.

Let $a \in D_1$ and let A be the set of elements of D_2 whose sum is a. Then A is a decomposition of a and the system of elements of the A's, associated in this way with the elements of D_1, forms the set D_2. We get therefore D_2 from D_1 by replacing every element a of D_1 by a suitable decomposition of a; we get D_1 from D_2 by forming a suitable decomposition of D_2 and adding the subsets of M which are elements of D_2 and are contained in the same element of the decomposition. Inversely, if every one of the elements of D_1 be replaced by some of its own decomposition, we get a lower decomposition of M w. r. to D_1; we get an upper decomposition of M w. r. to D_2 by forming an arbitrary decomposition of D_2 and adding the subsets of M which are elements of D_2 and are contained in the same element of the decomposition.

1. *Let D_1, D_2 be decompositions of M. D_2 is under D_1 if and only if for $a \in D_1$, $b \in D_2$, $a \cap b \neq 0$ there holds $b \subset a$.*

Proof. a) Let D_2 be under D_1. Let $a \in D_1$, $b \in D_2$, $a \cap b \neq 0$. Then a is the sum of some sets which are elements of D_2. One of these is the set b because $a \cap b \neq 0$ and the sets of the system D_2 are mutually exclusive.

b) If the above property holds, every element a of D_1 is the sum of those elements of D_2 which have with a a common element of M.

21. Let M be non-vacuous set. Let D_1 be a decomposition of M. Let D_3 be a lower decomposition of M w. r. to D_1. Let D_2 be a decomposition of M such that it lies under D_1 and over D_3. We say that *D_2 is* (*lies*) *between D_1 and D_3*.

As D_3 (D_2) lies under D_1, the set A_3 (A_2) of the elements of D_3 (D_2) whose sum is a, for any $a \in D_1$, forms a decomposition of a; as D_2 is over D_3, every element of A_2 is the sum of some elements of A_3. Consequently, we get D_2 from D_1 and D_3 in the following way: For every $a \in D_1$ we form a suitable decomposition of A_3 and add the subsets of M which are elements of A_3 and are contained in the same element of the decomposition; these sums are the elements of D_2. Inversely, we get a decomposition of M between D_1 and D_3 by forming, for every $a \in D_1$, a decomposition of the set A_3 and by adding the subsets of M which are elements of A_3 and are contained in the same element of the decomposition.

22. Let M be a non-vacuous set. Let (D) be a non-vacuous system of decompositions of M. A decomposition $\mathring{D}$ of M which is over every decomposition belonging to (D) is termed *upper decomposition of M w. r. to (D)* and we say that *$\mathring{D}$ is (lies) over (D).*

The set of the upper decompositions of M w. r. to (D) is non-vacuous. In fact, the decomposition of M formed by the single element M is over every decomposition belonging to (D) and lies therefore over (D). This decomposition is the *largest decomposition $\mathring{D}_{max}$ of M over (D).* It is clear that every decomposition of M over (D) lies under $\mathring{D}_{max}$.

·1. There exists a unique smallest decomposition $\mathring{D}_{min}$ of M over (D); i. e. such a decomposition of M over (D) that every decomposition of M over (D) lies over $\mathring{D}_{min}$.

Proof. Consider a decomposition $D_0 \in (D)$. Let $a_0, b_0 \in D_0$. Any ordered finite set of elements of D_0

$$\{a_1, \ldots, a_\alpha\}$$

shall be called *chain in (D) for a_0, b_0* if $a_1 = a_0$, $a_\alpha = b_0$ and if there exists for any two adjoining elements of the set an element of a suitable decomposition contained in (D) having with both a common element of M. The relation applying to two elements $a_0, b_0 \in D_0$ and defined in the way that there exists a chain in (D) for a_0, b_0 is clearly reflexive, symetric and transitive. Consequently, there exists such a decomposition D_{00} of the set D_0 that for any two elements of D_0 which are contained in the same element of D_{00} there exists a chain in (D), whereas for any two elements of D_0 which are not contained in the same element of D_{00} no such chain exists. The system of all subsets of M such that every subset is the sum of all elements of D_0 which are contained in the same element of D_{00} is a decomposition $\mathring{D}_{min}$ of M.

a) D_{min} lies over (D). In fact, let D be a decomposition in (D). We have to show that D is under $\mathring{D}_{min}$. Let $a \in D$, $\mathring{a}_{min} \in \mathring{D}_{min}$, $a \cap a_{min} \neq 0$. According to **20**·*1* it is sufficient to prove that $a \subset a_{min}$. As $\mathring{a}_{min}$ is the sum of some subsets of M which are elements of D_0 and since $a \cap \mathring{a}_{min} \neq 0$, there exists an $a_0 \in D_0$ such that $a_0 \subset \mathring{a}_{min}$, $a_0 \cap a \neq 0$. Let m be an element of M contained in a. Then there exists such a $b_0 \in D_0$ that $m \in b_0$. Evidently $\{a_0, b_0\}$ is a chain in (D) for a_0, b_0. Consenquently a_0, b_0 are in the same element of D_{00} and thus $b_0 \subset \mathring{a}_{min}$. Hence $m \in \mathring{a}_{min}$ and therefore $a \subset \mathring{a}_{min}$.

b) $\mathring{D}_{min}$ is a smallest decomposition of M over (D). In fact, let D be a decomposition of M over (D). We have to prove that $\mathring{D}_{min}$ is under $\mathring{D}$. Let $\mathring{a} \in \mathring{D}$, $a_{min} \in \mathring{D}_{min}$, $a \cap \mathring{a}_{min} \neq 0$. By **20·1** it is sufficient to show that $\mathring{a}_{min} \subset a$. By definition, $\mathring{a}$ is the sum of some elements of D_0 and similarly is a_{min}. As $\mathring{a} \cap \mathring{a}_{min} \neq 0$ there exists an element $a_0 \in D_0$ such that $a_0 \subset \mathring{a} \cap \mathring{a}_{min}$. Let $b_0 \in D_0$, $b_0 \subset \mathring{a}_{min}$. Then there exists a chain in (D) for a_0, b_0

$$\{a_1, \ldots, a_\alpha\} \qquad (a_1 \quad a_0, a_\alpha = b_0).$$

We clearly get $a_1 \subset \mathring{a}$. Let us therefore suppose that there holds, for some $(1 <) \beta (< \alpha - 1)$, $a_1, \ldots, a_\beta \subset \mathring{a}$. According to the definition of a chain in (D), there exists an element a of a suitable decomposition $D \in (D)$ which possesses common elements with a_β, $a_{\beta+1}$. Consequently $a \cap \mathring{a} \neq 0$. As D lies under $\mathring{D}$ there holds $a \subset \mathring{a}$. Hence $a_{\beta+1} \cap \mathring{a} \neq 0$ and since D_0 lies under $\mathring{D}$ we get $a_{\beta+1} \subset \mathring{a}$. Hence $a_\alpha \subset \mathring{a}$ and thus $\mathring{a}_{min} \subset a$.

c) $\mathring{D}_{min}$ is unique. In fact, let $\mathring{D}$ be also a smallest decomposition of M over (D). We have to show that every element $a_{min} \in \mathring{D}_{min}$ is an element of $\mathring{D}$. Consider an element $a \in D$ such that $a \cap \mathring{a}_{min} \neq 0$. Because $\mathring{D}_{min}$ is a smallest decomposition of M over (D) it lies under $\mathring{D}$ and therefore $\mathring{a}_{min} \subset \mathring{a}$; because $\mathring{D}$ is smallest it lies under $\mathring{D}_{min}$ and therefore $\mathring{a} \subset \mathring{a}_{min}$. Hence $\mathring{a}_{min} = \mathring{a}$.

23. Multiplicative nets. Let M be a non-vacuous set.

The *warp* α *based upon* (b. u.) M is the set formed by all ordered groups $(a_1, \ldots, a_\alpha)$ of α equal or different elements $a_1, \ldots, a_\alpha \in M$, $\alpha = 1, 2, \ldots$ Notation: $O_\alpha(M)$ or shorter O_α. The *warp b. u.* M is the set $O(M) = O_1(M) \vee O_2(M) \vee \ldots$; a shorter notation: O. The *length of an element* $(a_1, \ldots, a_\alpha) \in O$ is the number α. The *prolongation of an element* $(a_1, \ldots, a_\alpha)$ $\in O$ *by an element* $(b_1, \ldots, b_\beta) \in O$ is the element $(a_1, \ldots, a_\alpha, b_1, \ldots, b_\beta) \in O$. *A knot in* O, shorter *knot*, is any non-vacuous subset of O. If the set of lengths of the elements contained in a knot be bounded, elements of a greatest length exist in the knot; in this case the mentioned greatest length is the *length of the knot.* A knot is termed *homogeneous* if every one of its elements is of the same length α; in this case the knot is a subset of O_α and α is the length of the knot. Inversely, every non-vacuous subset of O_α is a homogeneous knot in O of length α. The set of all elements of a length $\alpha\ (\geq 1)$ contained in a knot is the *homogeneous component* α *of the knot.* Such a component is therefore the vacuous set or a homogeneous knot of length α. Should a knot have a length then the length of every one of its non-vacuous homogeneous components equals at most the length of the knot. The *prolongation of a knot* a *by knot* b is the knot formed by the different prolongations of every element of a by every element of b. Notation: $\widehat{ab}$. If a, b, c, be knots in O there holds, of course, $(\widehat{ab})\widehat{\ }c = a\widehat{\ }(\widehat{bc})$; the operation $\widehat{\ }$ is therefore associative. It is clear that it is also distributive. If $a\,(b)$ is of a length $\alpha\ (\beta)$ then $\widehat{ab}$ is of the length $\alpha + \beta$. If A, B denote non-vacuous sets of knots in O, the symbol $\widehat{AB}$ will denote the set of knots formed by the prolongations of every element of A by every element of B.

A *net b. u. M* is a decomposition of the warp $O(M)$ having the following properties:

1° For every element $(a) \in O_1(M)$ the set $\{(a)\}$ is an element of the decomposition;

2° every element of the decomposition possesses a length;

3° for every ordered pair a, b of elements of the decomposition there exists in the decomposition an element containing $\widehat{ab}$.

Usual notation: $\mathfrak{M}(M)$ or shorter $\mathfrak{M}$. Sometimes we say „net" instead of „net b. u. M" and „knot of a net" instead of „element of a net". A net $\mathfrak{M}$ is termed *homogeneous* if every one of its knots is homogeneous. — The simplest net b. u. M is such that every one of its knots is a set formed by a single element of $O(M)$ so that this net is equivalent to $O(M)$. The net in question shall be called the *smallest net b. u. M*. Notation: $\mathfrak{O}(M)$ or shorter $\mathfrak{O}$. Clearly $\mathfrak{O}(M)$ is a homogeneous net.

·*1. Let $\mathfrak{M}$ be a net b. u. M. Let $a_1, \ldots, a_\alpha, a \in \mathfrak{M}$, $\widehat{a_1 \ldots a_\alpha} \cap a \neq 0$; $\alpha \geq 2$. Then $\widehat{a_1 \ldots a_\alpha} \subset a$.*

Proof. In fact, firstly let us show that there exists a knot $a' \in \mathfrak{M}$ such that $\widehat{a_1 \ldots a_\alpha} \subset a'$. If $\alpha = 2$ such an a' exists because of the property 3° of a net. Let therefore be $\alpha > 2$ and let us suppose that there exists, for some $(2 \leq) \beta (\leq \alpha - 1)$, a knot $b' \in \mathfrak{M}$ such that $\widehat{a_1 \ldots a_\beta} \subset b'$. Then $\widehat{a_1 \ldots a_\beta a_{\beta+1}} \subset \widehat{b' a_{\beta+1}} \subset b$ for a suitable $b \in \mathfrak{M}$. Consequently there exists an $a' \in \mathfrak{M}$, $\widehat{a_1 \ldots a_\alpha} \subset a'$. — From the hypothesis follows $a' \cap a \neq 0$ and thus $a' = a$ because the knots of $\mathfrak{M}$ are mutually exclusive.

24. Let $\mathfrak{M}$ be a net b. u. M. By the property 3° of a net there exists in $\mathfrak{M}$, for every ordered pair of knots $a, b \in \mathfrak{M}$, precisely one knot containing $\widehat{ab}$. The correspondence, in which with every ordered pair of knots $a, b \in \mathfrak{M}$ precisely the mentioned knot is associated, defines in $\mathfrak{M}$ a multiplication ab. We call it the *multiplication in $\mathfrak{M}$*.

This multiplication is associative. In fact, let $a, b, c \in \mathfrak{M}$. By definition, $ab\,[(ab)c]$ is the knot of $\mathfrak{M}$ containing $\widehat{ab}\ \widehat{(ab)c}$. There holds therefore $(ab)c \supset \widehat{(ab)c} \supset \widehat{\widehat{(ab)}c} = \widehat{abc}$. A similar reasoning shows that $a(bc) \supset \widehat{abc}$. Hence $(ab)c = a(bc)$ because the knots of $\mathfrak{M}$ are mutually exclusive.

The net $\mathfrak{M}$ with the multiplication in $\mathfrak{M}$ is therefore a m. system; we call it *multiplicative* (m.) *net b. u. M*. Notation: $\mathfrak{M}(M)$ or shorter $\mathfrak{M}$. The m. net $\mathfrak{M}$ is *homogeneous* if the net $\mathfrak{M}$ is homogeneous. — The simplest m. net b. u. M is the *smallest m. net b. u. M*, $\mathfrak{O}(M)$. This m. net is isomorphic to the m. system whose elements are the elements of $O(M)$ and the multiplication is defined by the formula $ab = \widehat{ab}$, $\widehat{ab}$ being the prolongation of the element a by b. The m. net $\mathfrak{O}(M)$ is homogeneous.

25. Let $\mathfrak{M}$ be a m. net b. u. M.

·*1. The length of every knot ab of $\mathfrak{M}$ equals at least the sum of lengths of the factors.*

The proof follows easily from the definition of the multiplication in $\mathfrak{M}$.

·2. $\mathfrak{M}$ is a m. system without kernel.

Proof. If $\mathfrak{M}$ possesses a kernel, there holds $\mathfrak{M}^{\alpha} = \mathfrak{M}^{\alpha+\beta}$ for a suitable positive integer α and for every positive integer β. Let $a \in \mathfrak{M}^{\alpha}$ so that $a \in \mathfrak{M}^{\alpha+\beta}$. According to the property 2° of a net a has a determined length; on the other hand a is a product of $\alpha + \beta$ knots and therefore (by ·*1*) its length equals at least $\alpha + \beta$, for every positive integer β, which is contradictory. Similarly we find that there does not exist an $a \in \prod_{\alpha=1}^{\infty} \mathfrak{M}^{\alpha}$.

·3. The index of every knot in $\mathfrak{M}$ equals its length.

Proof. Let $a \in \mathfrak{M}$ and let α denote the index of a. Then a is product of α suitable knots of $\mathfrak{M}$ but not more than α knots of $\mathfrak{M}$. Let β be the length of a. According to ·*1* we get $\beta \geqq \alpha$. a contains an element $(a_1, \ldots, a_\beta) \in O$ of length β and is therefore the product of the knots $\{(a_1)\}, \ldots, \{(a_\beta)\} \in \mathfrak{M}$; hence $\beta \leq \alpha$.

·4. If $\mathfrak{M}$ is homogeneous then it is a homogeneous m. system.

Proof. According to ·*3* the prime-factors of the m. system $\mathfrak{M}$ are the knots of length *1*. By the definition of the multiplication in $\mathfrak{M}$, the product of α knots $\{(a_1)\}, \ldots, \{(a_\alpha)\} \in \mathfrak{M}$ of length *1* is the knot of $\mathfrak{M}$ which contains the element $(a_1, \ldots, a_\alpha) \in O$. This element is of length α. If $\mathfrak{M}$ is homogeneous then the knot in question is homogeneous and therefore every one of its elements is of the same length α. Thus the length of the knot is α and therefore (by ·*3*) its index is α. Consequently the product of any α prime-factors possesses index α.

·5. Every m. system without kernel is isomorphic to a suitable m. net b. u. its excentrum.

Proof. Let $\overline{\mathfrak{M}} = M_1 \vee \overline{M}_2 \vee \ldots$ be a m. system without kernel. Let O be the warp b. u. M_1. Let $\mathfrak{J}$ stand for the correspondence defined in the following way: With every element $a \in \overline{\mathfrak{M}}$ the knot a in O formed by all elements $(p_1, \ldots, p_\alpha) \in O$ for which $p_1 \ldots p_\alpha = a$, is associated. The set of knots in O which are counterparts of the elements of $\overline{\mathfrak{M}}$ in $\mathfrak{J}$ is a net b. u. M_1; we denote it by $\mathfrak{M}$. In fact, it is easy to perceive that $\mathfrak{M}$ is a decomposition of the warp O possessing the properties 1° — 3° of a net. The correspondence $\mathfrak{J}$ is an isomorphism between the m. system $\overline{\mathfrak{M}}$ and $\mathfrak{M}$. Indeed, $\mathfrak{J}$ is clearly a (*1, 1*) correspondence. Further, from $a \longleftrightarrow a$, $\bar{b} \longleftrightarrow b$ ($a, \bar{b} \in \overline{\mathfrak{M}}$; $a, b \in \mathfrak{M}$) follows $a\bar{b} \longleftrightarrow ab$ because to the element ab corresponds in $\mathfrak{J}$ the element of $\mathfrak{M}$ containing the prolongation of every element belonging to a by every element belonging to b, i. e. the knot ab.

·6. Every homogeneous m. system is isomorphic to a suitable homogeneous m. net b. u. its excentrum.

Proof. Let notions and notations be the same as in the proof of ·*5*. Further, let $\overline{\mathfrak{M}}$ be homogeneous so that every element $a \in \overline{\mathfrak{M}}$ of an index α is product of precisely α prime-factors. Then the knot a which is associated with a in $\mathfrak{J}$ contains elements of O but only of length α. Hence a is homogeneous.

According to ·*2* ·*5* (·*4* ·*6*) the theory of m. systems without kernel (homogeneous m. systems) is equivalent to the theory of m. nets (homogeneous m. nets).

26. Consider the set M formed by the unique element z. Then $O(\{z\})$ is the set

$$\{(z),\ (z,z),\ (z,z,z),\ldots\}.$$

Consequently $\mathfrak{O}(\{z\})$ is given by

$$\{\{(z)\},\ \{(z,z)\},\ \{(z,z,z)\},\ldots\}$$

and this smallest m. net b. u. $\{z\}$ is clearly isomorphic to the infinite cyclic m. system $\mathfrak{Z}$. We call it the *infinite cyclic m. net* $\mathfrak{Z}$. We notice that the m. net in question is the unique m. net b. u. $\{z\}$. In fact, for every positive integer α there exists in $O(\{z\})$ a unique element of length α: $(\underbrace{z,\ldots,z}_{\alpha})$. Consequently, if some knot of a m. net b. u. $\{z\}$ contains two elements $(\underbrace{z,\ldots,z}_{\alpha}), (\underbrace{z,\ldots,z}_{\beta}) \in O(\{z\})$, $\alpha < \beta$, there does not exist in the m. net any knot of length α; the considered m. net is therefore not a m. system without kernel, which contradicts **25**·*2*.

27. Construction of the homogeneous m. nets b. u. a given set.

·*1. Let* $\mathfrak{M} = M_1 \vee M_2 \vee \ldots$ *be a homogeneous net b. u.* M. *The set* M_α *is a decomposition of* $O_\alpha(M)$ *and* $M_\alpha \frown M_\beta$ *is a decomposition of* $O_{\alpha+\beta}(M)$ *for* $\alpha, \beta = 1, 2, \ldots$

The proof follows from the supposition of homogenity of $\mathfrak{M}$ according to which every knot of M_α $[M_\alpha \frown M_\beta]$ is a subset of $O_\alpha(M)$ $[O_{\alpha+\beta}(M)]$.

·*2. Let the suppositions be the same as in* ·*1. The decomposition* $M_{\alpha+1}$ *of* $O_{\alpha+1}(M)$, *for* $\alpha = 1, 2, \ldots$, *is over the system of decompositions* $M_\nu \frown M_{\alpha+1-\nu}$ *of the set* $O_{\alpha+1}(M)$, $\nu = 1, \ldots, \alpha$.

Proof. Let $a_{\alpha+1} \in M_{\alpha+1}$, $a \in M_\nu \frown M_{\alpha+1-\nu}$ and let $a \cap a_{\alpha+1} \neq 0$. According to **20** ·*1* it is sufficient to show that $a \subset a_{\alpha+1}$. But by the definition of a there holds $a = a_\nu \frown a_{\alpha+1-\nu}$ for suitable $a_\nu \in M_\nu$, $a_{\alpha+1-\nu} \in M_{\alpha+1-\nu}$. Consequently $a_\nu \frown a_{\alpha+1-\nu} \cap a_{\alpha+1} \neq 0$ and we get, by **23**·*1*, $a \subset a_{\alpha+1}$.

·*3. Let* M *be a non-vacuous set. Let* $M_1 \sim O_1(M)$ *and let* $M_{\alpha+1}$ *be a decomposition of the set* $O_{\alpha+1}(M)$ *over the system of decompositions* $M_\nu \frown M_{\alpha+1-\nu}$ *of* $O_{\alpha+1}(M)$, *for* $\alpha = 1, 2, \ldots, \nu - 1, \ldots, \alpha$. *Then* $\mathfrak{M} = M_1 \vee M_2 \vee \ldots$ *is a homogeneous net b. u.* M.

Proof. It is clear that $\mathfrak{M}$ is a decomposition of $O(M)$ and that every one of its knots is homogeneous. We also perceive that the above decomposition possesses the properties 1° 2° of a net. We have therefore only to prove that it possesses the property 3°. Let $a, b \in \mathfrak{M}$ so that $a \in M_\alpha$, $b \in M_\beta$ for suitable α, β. Then $\widehat{ab} \in M_\alpha \frown M_\beta$ and according to the supposition, $M_\alpha M_\beta$ is a decomposition of $O_{\alpha+\beta}(M)$ under $M_{\alpha+\beta}$. Hence there exists in $M_{\alpha+\beta}$ a knot containing ab.

From $\cdot 1$ $\cdot 2$ $\cdot 3$ and **22**$\cdot 1$ we get the following construction of all homogeneous nets and consequently of all homogeneous m. nets b. u. a given non-vacuous set M:

Decompositions $M_1, M_2, M_3, \ldots$ of the sets $O_1(M)$, $O_2(M)$, $O_3(M), \ldots$ are to be formed in the following way: 1° $M_1 \sim O_1(M)$ 2° if $M_1, \ldots, M_\alpha$, for an $\alpha \geq 1$, have been formed, an arbitrary upper decomposition of the set $O_{\alpha+1}(M)$ w. r. to the smallest decomposition of $O_{\alpha+1}(M)$ over the system of decompositions $M_\nu \frown M_{\alpha+1-\nu}$, $\nu = 1, \ldots, \alpha$, is to be chosen for $M_{\alpha+1}$. Then the set $M_1 \vee M_2 \vee \ldots$ is a homogeneous net b. u. M.

$\cdot 4$. *If a homogeneous m. net possesses a single knot of index α (≥ 1) then it also possesses only a single knot of index $\alpha + \beta$, for every positive integer β.*

Proof. It is sufficient to consider the case $\beta = 1$. Let $\mathfrak{M} = M_1 \vee M_2 \vee \ldots$ be a homogeneous net b. u. M and let M_α, for a determined α, possesses only one knot. By $\cdot 1$, this knot is the set $O_\alpha(M)$. Let (D) denote the system of decompositions of $O_{\alpha+1}(M)$ $D_\nu = M_\nu \frown M_{\alpha+1-\nu}$, $\nu = 1, \ldots, \alpha$, and let $\mathring{D}_{min}$ stand for the smallest decomposition of $O_{\alpha+1}$ over (D). By $\cdot 2$ and **22**$\cdot 1$, $M_{\alpha+1}$ is a suitable decomposition of $O_{\alpha+1}$ over $\mathring{D}_{min}$. Therefore it is sufficient to prove that $\mathring{D}_{min}$ possesses only one knot. Let $\{(a)\} \frown O_\alpha$ be a knot of D_1 so that $\{(a)\} \in M_1$. As $\mathring{D}_{min}$ is over D_1 there exists an element $a_{min} \in D_{min}$, $\mathring{a}_{min} \supset \{(a)\} \frown O_\alpha$. By definition, $\mathring{a}_{min}$ is the sum of elements $\{(a')\} \frown O_\alpha$ of D_1 such that there exists a chain in (D) for $\{(a)\} \frown O_\alpha$, $\{(a')\} \frown O_\alpha$. Consequently, it suffices to prove that such a chain exists for every element $\{(a')\} \frown O_\alpha \in D_1$. Let us choose an element $(a_1, \ldots, a_\alpha) \in O_\alpha$. Then $\{(a, a_1, \ldots, a_\alpha)\} = \{(a)\} \frown \{(a_1, \ldots, a_\alpha)\} \subset \{(a)\} \frown O_\alpha$ and analogously $\{(a', a_1, \ldots, a_\alpha)\} \subset \{(a')\} \frown O_\alpha$. Further, $\{(a, a_1, \ldots, a_\alpha)\} = \{(a, a_1, \ldots, a_{\alpha-1})\} \frown \{(a_\alpha)\} \subset O_\alpha \frown \{(a_\alpha)\} \in D_\alpha$ and similarly $\{(a', a_1, \ldots, a_\alpha)\} \subset O_\alpha \frown \{(a_\alpha)\}$. Hence D_x contains the element $O_\alpha \frown \{(a_\alpha)\}$ which possesses a common element with $\{(a)\} \frown O_\alpha$ as well as with $\{(a')\} \frown O_\alpha$. Thus $\{(a)\} \frown O_\alpha$, $\{(a')\} \frown O_\alpha$ form a chain in (D) for these elements.

$\cdot 5$. *Let M be a non-vacuous set. There exists such a homogeneous m. net that its excentrum is equivalent to M and there exists but one single element of index $\geq \alpha + 1$ for a given positive integer α.*

Proof. Let $M_1 \sim O_1(M)$. In the case $\alpha > 2$ we define the sets $M_1, \ldots, M_\alpha$ in the following way: If $M_1, \ldots, M_\beta$, for a ($1 \leq$) β ($\leq \alpha - 1$), have been formed, we choose for $M_{\beta+1}$ an arbitrary upper decomposition of $O_{\beta+1}(M)$ w. r. to the system $(D_{\beta+1})$ of decompositions $M_\nu \frown M_{\beta+1-\nu}$ of $O_{\beta+1}$, $\nu = 1, \ldots, \beta$. For $\alpha \geq 1$, $\gamma > \alpha$ let $M_{\gamma+1}$ be the decomposition of $O_{\gamma+1}(M)$ formed by the single element $O_{\gamma+1}$. Then $M_{\gamma+1}$ is a decomposition of $O_{\gamma+1}$ over $(D_{\gamma+1})$, $(D_{\gamma+1})$ having an analogous meaning to $(D_{\beta+1}.)$ According to $\cdot 3$, $\mathfrak{M} = M_1 \vee M_2 \vee \ldots$ is a homogeneous net and evidently possesses the above mentioned properties.

28. Upper and lower m. nets with regard to a given m. net. Let $\mathfrak{M}$ be a net event. a m. net b. u. M. By definition, the net $\mathfrak{M}$ is a decomposition of the warp $O(M)$. Every net $\mathfrak{M}$ b. u. M which lies under this decomposition *is (lies) under the net* $\mathfrak{M}$ and is termed *lower net w. r. to* $\mathfrak{M}$; the m. net $\mathfrak{M}$ *is (lies) under the m. net* $\mathfrak{M}$ and is termed *lower m. net w. r. to* $\mathfrak{M}$. $\mathfrak{M}$ *is (lies) over* $\underset{\circ}{\mathfrak{M}}$ and is termed *upper net* event. *upper m. net w. r. to* $\underset{\circ}{\mathfrak{M}}$. If $\underset{\circ}{\mathfrak{M}}$ be under the m. net $\mathfrak{M}$, then the prime-factors of the m. nets $\mathfrak{M}$, $\mathfrak{M}$ are clearly the same. The simplest lower net w. r. to $\mathfrak{M}$ is $\mathfrak{O}(M)$; it shall be called the *smallest net under (w. r. to)* $\mathfrak{M}$ or the *support of* $\mathfrak{M}$. The set of knots in $O(M)$ which are the different non-vacuous homogeneous components of the knots forming $\mathfrak{M}$, is clearly a homogeneous net b. u. M and lies under $\mathfrak{M}$. This net event. m. net is the *largest homogeneous net* event. *m. net under* $\mathfrak{M}$.

·1. *Let* $\mathfrak{M}$ *be a net under* $\mathfrak{M}$. *Let* $a, \underset{\circ}{b} \in \mathfrak{M}$, $\underset{\circ}{a} \subset a \in \mathfrak{M}$, $\underset{\circ}{b} \subset b \in \mathfrak{M}$. *Then* $\underset{\circ}{a}\underset{\circ}{b} \subset ab$.

Proof. By definition, $\underset{\circ}{a}\underset{\circ}{b}$ is the element of $\mathfrak{M}$ which contains the knot $\widehat{\underset{\circ}{a}\underset{\circ}{b}}$. But $\widehat{ab} \subset \widehat{ab} \subset ab$ so that $\widehat{\underset{\circ}{a}\underset{\circ}{b}} \subset \underset{\circ}{a}b \cap ab \neq 0$. Consequently, by **20**·*1*, we get $\underset{\circ}{a}\underset{\circ}{b} \subset ab$.

·2. *Every homogeneous lower net w. r. to* $\mathfrak{M}$ *lies under the largest homogeneous net under* $\mathfrak{M}$.

Proof. Let $\mathfrak{M}$ ($\mathfrak{H}$) be a homogeneous (the largest homogeneous) net under $\mathfrak{M}$. We have to prove that $\mathfrak{M}$ lies under $\mathfrak{H}$. Let $\underset{\circ}{a} \in \underset{\circ}{\mathfrak{M}}$, $h \in \mathfrak{H}$, $\underset{\circ}{a} \cap h \neq 0$. It is sufficient to show that $\underset{\circ}{a} \subset h$. According to the definition, h is the set of all elements of $O(M)$ which are of the same length and lie in a knot $a \in \mathfrak{M}$. Hence $\underset{\circ}{a} \cap a \neq 0$ and thus, by **20**·*1*, there holds $\underset{\circ}{a} \subset a$, because $\underset{\circ}{\mathfrak{M}}$ is under $\mathfrak{M}$. Consequently, we get $\underset{\circ}{a} \subset h$ because $\underset{\circ}{a}$ is homogeneous.

29. Let $\mathfrak{M} = M_1 \vee M_2 \vee \ldots$ be a (m.) net b. u. M. Let $(0 \neq)$ $A \subset \mathfrak{M}$. Every (m.) lower net $\underset{\circ}{\mathfrak{M}}$ w. r. to $\mathfrak{M}$ such that the elements of A are at the same time elements of $\underset{\circ}{\mathfrak{M}}$ is a *lower* (m) *net w. r. to* $\mathfrak{M}$ *generated by* A. Since every lower m. net w. r. to $\mathfrak{M}$ possesses the same prime-factors as $\mathfrak{M}$ we may study the lower (m.) nets w. r. to $\mathfrak{M}$ generated by A supposing that $A \supset M_1$.

·1. *There exists a unique smallest net* $\underset{\circ}{\mathfrak{M}}_{min}$ *under (w. r. to)* $\mathfrak{M}$ *generated by* A; *i. e. such a lower net w. r. to* $\mathfrak{M}$ *generated by* A *that every lower net w. r. to* $\mathfrak{M}$ *generated by* A *lies over* $\underset{\circ}{\mathfrak{M}}_{min}$.

Proof. Let us associate with every group $a_1, a_2, \ldots, a_\alpha$ of α equal or different elements of A, $\alpha = 1, 2, \ldots$, the following knot in $O(M)$: $a_1 \frown a_2 \ldots \frown a_\alpha$. Let $\widehat{O}$ be the set of all these knots so that $\widehat{O} = \sum_{\alpha\;1}^{\infty} \underbrace{A \frown A \frown \ldots \frown A}_{\alpha}$.

For $(a_1, \ldots, a_\alpha) \in O(M)$, $a_1, \ldots, a_\alpha \in M$, we get $\{(a_1)\}, \ldots, \{(a_\alpha)\} \in M_1 \subset A$ and thus $\{(a_1)\} \frown \ldots \frown \{(a_\alpha)\} = \{(a_1, \ldots, a_\alpha)\} \in \widehat{O}$. Consequently $\widehat{O}$ covers $O(M)$. Let $\widehat{a}$, $\widehat{b} \in \widehat{O}$. A *chain for* $\widehat{a}$, $\widehat{b}$ is an ordered finite set of elements of $\widehat{O}$:

$$\{\widehat{a}_1, \ldots, \widehat{a}_\alpha\}$$

such that $\widehat{a}_1 = \widehat{a}$, $\widehat{a}_\alpha = \widehat{b}$ and any two adjoining elements possess a common element of $O\,(M)$. The relation for two elements a, $\widehat{b} \in \widehat{O}$ defined in the way that there exists a chain for $\widehat{a}$, $\widehat{b}$, is obviously reflexive, symetric and transitive. Consequently, there exists such a decomposition D of $\widehat{O}$ that there exists a chain for any two elements of $\widehat{O}$ lying in the same element of the decomposition, whereas no chain exists for any two elements of $\widehat{O}$ lying in different elements. The system of the subsets of $O\,(M)$ such that every subset is the sum of all elements of $\widehat{O}$ lying in the same element of D, is a decomposition $\mathfrak{M}_{min}$ of $O\,(M)$.

a) $\mathfrak{M}_{min}$ is a decomposition of the net $\mathfrak{M}$. In fact, let $\mathfrak{q}_{min} \in \mathfrak{M}_{min}$, $\mathfrak{a} \in \mathfrak{M}$, $\mathfrak{q}_{min} \cap \mathfrak{a} \neq 0$. It suffices to prove that $\mathfrak{q}_{min} \subset \mathfrak{a}$. By definition, $\mathfrak{q}_{min}$ is the sum of some elements of $\widehat{O}$. Since $\mathfrak{q}_{min} \cap \mathfrak{a} \neq 0$ there exists an element $\widehat{a} \in \widehat{O}$ such that $\mathfrak{a} \cap \widehat{a} \neq 0$. By the definition of $\widehat{a}$, we get $\widehat{a}$ $\widehat{a}_1 \frown \ldots \frown a_\alpha$, $a_1, \ldots, a_\alpha$ being suitable knots of $\mathfrak{M}$. According to **23·1** there holds $a \subset \mathfrak{a}$. Let $\widehat{b} \in \widehat{O}$, $\widehat{b} \subset \mathfrak{q}_{min}$. Then there exists a chain for $\widehat{a}$, $\widehat{b}$:

$$\{\widehat{a}_1, \ldots, \widehat{a}_\alpha\} \qquad\qquad (\widehat{a}_1 = \widehat{a},\ \widehat{a}_\alpha = \widehat{b}).$$

We clearly get $\widehat{a}_1 \subset \mathfrak{a}$. Let us therefore suppose that $\widehat{a}_1, \ldots, \widehat{a}_\beta \subset \mathfrak{a}$ holds for some $(1 \leq)$ β $(< \alpha - 1)$. According to the definition of a chain we get $\widehat{a}_\beta \cap \widehat{a}_{\beta+1} \neq 0$ and thus $\mathfrak{a} \cap \widehat{a}_{\beta+1} \neq 0$; hence $\widehat{a}_{\beta+1} \subset \mathfrak{a}$, by **23·1**. Consequently $a_\alpha \subset \mathfrak{a}$; $\mathfrak{q}_{min} \subset \mathfrak{a}$.

b) $\mathfrak{M}_{min}$ is a net b. u. M. We have only to show that $\mathfrak{M}_{min}$ possesses the properties 1°—3° of a net.

1° For every element $(a) \in O_1\,(M)$ the set $\{(a)\}$ is an element of $\mathfrak{M}_{min}$. Indeed, for every element $(a) \in O_1\,(M)$ the set $\{(a)\}$ is an element of $\widehat{O}$ and clearly there does not exist any chain for this element and any other element of $\widehat{O}$ different from it.

2° Every element of $\mathfrak{M}_{min}$ possesses a length, because it is (by a)) a subset of some element of $\mathfrak{M}$.

3° For every ordered pair of elements $\mathfrak{q}_{min}$, $\mathfrak{b}_{min}$ of $\mathfrak{M}_{min}$ there exists in $\mathfrak{M}_{min}$ an element containing $\mathfrak{q}_{min} \frown \mathfrak{b}_{min}$. Firstly, let us show that if $\widehat{a}$, $\widehat{a}' \subset \mathfrak{q}_{min}$, $\widehat{b}$, $\widehat{b}' \subset \mathfrak{b}_{min}$, $\widehat{a} \frown \widehat{b} \subset \mathfrak{c}_{min}$ then $\widehat{a}' \frown \widehat{b}' \subset \mathfrak{c}_{min}$. In fact, if the suppositions hold there exists a chain $\widehat{a}_1, \ldots, \widehat{a}_\alpha$ for $\widehat{a}'$, $\widehat{a}$ $(\widehat{a}_1 = \widehat{a}',\ \widehat{a}_\alpha = \widehat{a})$ and a chain $\widehat{b}_1, \ldots, \widehat{b}_\beta$ for $\widehat{b}'$, $\widehat{b}$ $(\widehat{b}_1 = \widehat{b}',\ \widehat{b}_\beta = \widehat{b})$. We may suppose β α; as if for instance $\beta < \alpha$, we prolong the chain for $\widehat{b}'$, $\widehat{b}$ by adding $\alpha - \beta$ knots equal to $\widehat{b}_\beta$. Now it suffices to show that

$$\{\widehat{a}_1 \frown \widehat{b}_1, \ldots, \widehat{a}_\alpha \frown \widehat{b}_\alpha\}$$

is a chain for $\widehat{a}_1 \frown \widehat{b}_1$, $\widehat{a}_\alpha \frown \widehat{b}_\alpha$; i. e. that for $1 < \gamma \leq \alpha - 1$ both knots $\widehat{a}_\gamma \frown \widehat{b}_\gamma$, $\widehat{a}_{\gamma+1} \frown \widehat{b}_{\gamma+1}$ possess a common element of $O\,(M)$. But both knots $\widehat{a}_\gamma$, $\widehat{a}_{\gamma+1}$ $[\widehat{b}_\gamma, \widehat{b}_{\gamma+1}]$ possess a common element $(a_1, \ldots, a_\mu)$ $[(b_1, \ldots, b_\nu)]$ of $O\,(M)$, and therefore the knots $\widehat{a}_\gamma \frown \widehat{b}_\gamma$, $\widehat{a}_{\gamma+1} \frown \widehat{b}_{\gamma+1}$ both contain the element $(a_1, \ldots, a_\mu, b_1, \ldots, b_\nu) \in O\,(M)$. — Now let $\widehat{a}$ $[\widehat{b}]$ represent $\mathfrak{q}_{min}$ $[\mathfrak{b}_{min}]$ so that $\mathfrak{q}_{min}$ $[\mathfrak{b}_{min}]$ is the sum of elements $\widehat{a}'$ $[\widehat{b}']$ such that there exists a chain for a, $\widehat{a}'$ $[b, b']$. Let $\widehat{a} \frown \widehat{b} \subset \mathfrak{c}_{min}$. Every element of

$\widehat{\mathfrak{q}_{min}\,\mathfrak{b}_{min}}$ is contained in $\widehat{a'\,b'}$, a', b' being suitable elements having the mentioned property. Hence $\widehat{a'\,b'} \subset \mathfrak{c}_{min}$, $\widehat{\mathfrak{q}_{min}\,\mathfrak{b}_{min}} \subset \mathfrak{c}_{min}$.

c) $\mathfrak{M}_{min}$ is a smallest net under $\mathfrak{M}$ generated by A. Indeed, let $\underset{\circ}{\mathfrak{M}}$ ~~be an arbitrary lower net w. r. to $\mathfrak{M}$ generated by A.~~ We have to show that $\mathfrak{M}$ lies over $\mathfrak{M}_{min}$. Let $\mathfrak{q} \in \underset{\circ}{\mathfrak{M}}$, $\mathfrak{q}_{min} \in \mathfrak{M}_{min}$, $\mathfrak{q} \cap \mathfrak{q}_{min} \neq 0$. ~~It suffices~~ to prove that $\mathfrak{q}_{min} \subset \mathfrak{q}$. Let $\widehat{a}$ represent the knot $\mathfrak{q}_{min}$ so that $\mathfrak{q}_{min}$ is the sum of elements $\widehat{a'}$ such that there exists a chain for $\widehat{a}, \widehat{a'}$. There holds $\widehat{a} = \widehat{a_1} \ldots \widehat{a_\alpha}$ for suitable knots $a_1, \ldots, a_\alpha \in A$. According to our assumption there holds $a_1, \ldots, a_\alpha \in \underset{\circ}{\mathfrak{M}}$ and therefore, by **23**·*1*, $\widehat{a} \subset \mathfrak{b}$, $\mathfrak{b}$ being a suitable element of $\mathfrak{M}$. From the existence of a chain for $\widehat{a}, \widehat{a'}$ easily follows $\widehat{a'} \subset \mathfrak{b}$ and hence $\mathfrak{q}_{min} \subset \mathfrak{b}$. From $\mathfrak{q} \cap \mathfrak{q}_{min} \neq 0$ results $\mathfrak{b} = \mathfrak{q}$ since the knots of $\mathfrak{M}$ are mutually exclusive.

d) $\mathfrak{M}_{min}$ is unique. The proof is analogous to the proof in **22**·*1* c).

·*2. If* $A = M_1$ *then* $\underset{\circ}{\mathfrak{M}}_{min}$ *is the support* $\mathfrak{O}(M)$ *of* $\mathfrak{M}$.

The proof follows easily from the construction of $\underset{\circ}{\mathfrak{M}}_{min}$.

·*3. If* $\mathfrak{M}$ *is the smallest net b. u.* M *then* $\underset{\circ}{\mathfrak{M}}_{min} = \mathfrak{M}$ for every $A \supset M_1$.

Proof. Every element $a \in \mathfrak{M}$ is formed by a unique element of $O(M)$. Every element $\mathfrak{q}_{min} \in \mathfrak{M}_{min}$ is a non-vacuous subset of a suitable $a \in \mathfrak{M}$ and therefore $\mathfrak{q}_{min} = a$.

30. Construction of the homogeneous lower m. nets with regard to a given m. net.

·*1. Let* $\mathfrak{M} = M_1 \vee M_2 \vee \ldots$ *be a homogeneous net b. u.* M. *Let* $\underset{\circ}{\mathfrak{M}} = \underset{\circ}{M}_1 \vee \underset{\circ}{M}_2 \vee \ldots$ *be a homogeneous lower net w. r. to* $\mathfrak{M}$. $\underset{\circ}{M}_{\alpha+1}$ *is a decomposition of* $O_{\alpha+1}(M)$ *and lies between* $M_{\alpha+1}$ *and the smallest decomposition of* $O_{\alpha+1}(M)$ *over the system of decomposition* $\underset{\circ}{M}_\nu \underset{\circ}{M}_{\alpha+1-\nu}$ *of* $O_{\alpha+1}(M)$; $\alpha = 1, 2, \ldots$, $\nu = 1, \ldots, \alpha$.

Proof. According to **27**·*1*, $\underset{\circ}{M}_{\alpha+1}$ is a decomposition of $O_{\alpha+1}$ and by **27**·*2* and **22**·*1* it is an upper decomposition w. r. to the smallest decomposition of $O_{\alpha+1}$ over the system of decomposition $\underset{\circ}{M}_\nu \underset{\circ}{M}_{\alpha+1-\nu}$ of the set $O_{\alpha+1}$, $\nu = 1, \ldots, \alpha$. $M_{\alpha+1}$ is also a decomposition of $O_{\alpha+1}$. It therefore only remains to prove that $\underset{\circ}{M}_{\alpha+1}$ lies under $M_{\alpha+1}$; for which purpose it is sufficient to show that for $\mathfrak{q} \in \underset{\circ}{M}_{\alpha+1}$, $a \in M_{\alpha+1}$, $\mathfrak{q} \cap a \neq 0$ there holds $\mathfrak{q} \subset a$. But $\mathfrak{q} \in \underset{\circ}{\mathfrak{M}}$, $a \in \mathfrak{M}$ and by hypothesis, $\underset{\circ}{\mathfrak{M}}$ lies under $\mathfrak{M}$. Consequently $\mathfrak{q} \subset a$.

·*2. Let* $\mathfrak{M} = M_1 \vee M_2 \vee \ldots$ *be a homogeneous net b. u.* M. *Let* $\underset{\circ}{M}_1 \sim O_1(M)$ *and let* $\underset{\circ}{M}_{\alpha+1}$ *be an arbitrary decomposition of the set* $O_{\alpha+1}(M)$ *between* $M_{\alpha+1}$ *and the smallest decomposition of* $O_{\alpha+1}(M)$ *over the system of decompositions* $\underset{\circ}{M}_\nu \underset{\circ}{M}_{\alpha+1-\nu}$ *of* $O_{\alpha+1}(M)$; $\alpha = 1, 2, \ldots$, $\nu = 1, \ldots, \alpha$. *Then* $\underset{\circ}{\mathfrak{M}} = \underset{\circ}{M}_1 \vee \underset{\circ}{M}_2 \vee \ldots$ *is a homogeneous lower net of* $\mathfrak{M}$.

Proof. According to **27**·*3*, $\underset{\circ}{\mathfrak{M}}$ is a homogeneous net b. u. M. For $\mathfrak{q} \in \underset{\circ}{\mathfrak{M}}$, $a \in \mathfrak{M}$, $\mathfrak{q} \cap a \neq 0$ there holds $\mathfrak{q} \in \underset{\circ}{M}_\alpha$, $a \in M_\beta$, α, β being suitable positive integers. From $\mathfrak{q} \cap a \neq 0$ follows $\beta = \alpha$. As $\underset{\circ}{M}_\alpha$ lies under M_α, we get $\mathfrak{q} \subset a$. Consequently $\underset{\circ}{\mathfrak{M}}$ is a lower net w. r. to $\mathfrak{M}$.

From **28**·*2* and ·*1* ·*2* the following construction of all homogeneous lower m. nets of a given m. net $\mathfrak{M}$ b. u. a given non-vacuous set M results:

Take the largest homogeneous lower net $\mathfrak{H} = H_1 \vee H_2 \vee \ldots$ under $\mathfrak{M}$ and form the decompositions $\mathring{M}_1, \mathring{M}_2, \mathring{M}_3, \ldots$ of the sets $O_1(M), O_2(M), O_3(M), \ldots$ in the following way; 1° $\mathring{M}_1 \sim O_1(M)$ 2° if $\mathring{M}_1, \ldots, \mathring{M}_\alpha$ have been formed for some $\alpha \geq 1$, a decomposition of $O_{\alpha+1}(M)$ between $H_{\alpha+1}$ and the smallest decomposition of $O_{\alpha+1}(M)$ over the system of decompositions $\mathring{M}_\nu \mathring{M}_{\alpha+1-\nu}$, $\nu = 1, \ldots, \alpha$, of $O_{\alpha+1}(M)$ is to be chosen for $\mathring{M}_{\alpha+1}$. The set $\mathring{M}_1 \vee \mathring{M}_2 \vee \ldots$ is a homogeneous lower net w. r. to $\mathfrak{M}$.

31. Homomorphic representations of m. systems without kernel. Let $\mathfrak{M}$, $\overline{\mathfrak{M}}$ be m. systems.

Every correspondence $\mathfrak{F}$ between the elements of $\mathfrak{M}$ and the elements of $\overline{\mathfrak{M}}$ which has the following properties is called *homomorphic representation of* $\mathfrak{M}$ *in* $\overline{\mathfrak{M}}$:

1° Every element $a \in \mathfrak{M}$ is associated with a single element $\bar{a} \in \overline{\mathfrak{M}}$; we write $a \to \bar{a}\ (\mathfrak{F})$, shorter $a \to \bar{a}$, or, if desired, $\bar{a} = \mathfrak{f}\, a$.

2° For $a, b \in \mathfrak{M}$, $a \to \bar{a}$, $b \to \bar{b}$ there holds $ab \to \bar{a}\bar{b}$, i. e. $\mathfrak{f}\, a\, \mathfrak{f}\, b = \mathfrak{f}\, ab$. $\bar{a}$ is the *counterpart of* a, a is an *antecedent of* $\bar{a}$ in $\mathfrak{F}$. The set $\bar{A} \subset \overline{\mathfrak{M}}$ of the counterparts in $\mathfrak{F}$ of a set A of elements belonging to $\mathfrak{M}$ is the *counterpart of the set* A in $\mathfrak{F}$ and A is an *antecedent of* $\bar{A}$ in $\mathfrak{F}$; we write $A \to \bar{A}\ (\mathfrak{F})$, shorter $A \to \bar{A}$ or, if wanted, $\bar{A} = \mathfrak{f}\, A$. For the sake of brevity we sometimes say f. i. "$\bar{a}$ is the counterpart ($\mathfrak{F}$) of a" instead of "$\bar{a}$ is the counterpart of a in $\mathfrak{F}$". — If $\mathfrak{F}$ possesses the properties 1° 2° as well as the further property:

3° Every element of $\overline{\mathfrak{M}}$ is the counterpart of at least one element of $\mathfrak{M}$, we call $\mathfrak{F}$ homomorphic representation of $\mathfrak{M}$ *on* $\overline{\mathfrak{M}}$ or *homomorphism* of $\mathfrak{M}$ on $\overline{\mathfrak{M}}$. We say that $\mathfrak{M}$ *is homomorphically representable in (on)* $\overline{\mathfrak{M}}$ if there exists a homomorphic representation of $\mathfrak{M}$ in (on) $\overline{\mathfrak{M}}$.

·*1. $\mathfrak{M}$ is homomorphically representable in (on) $\overline{\mathfrak{M}}$ if and only if there exists for every element $\bar{a} \in \overline{\mathfrak{M}}$ a set $F_{\bar{a}} \subset \mathfrak{M}$ such that 1° the system of the sets $F_{\bar{a}}$ is a decomposition of $\mathfrak{M}$ 2° $F_{\bar{a}} F_{\bar{b}} \subset F_{\bar{a}\bar{b}}$ for $\bar{a}, \bar{b} \in \overline{\mathfrak{M}}$ (3° $F_{\bar{a}} \neq 0$ for every $\bar{a} \in \overline{\mathfrak{M}}$).*

In the case of a homomorphic representation *in* $\overline{\mathfrak{M}}$ some sets $F_{\bar{a}}$ may be, of course, vacuous.

Proof. Let $\mathfrak{M}$ be homomorphically representable in (on) $\overline{\mathfrak{M}}$ so that there exists a homomorphic representation $\mathfrak{F}$ of $\mathfrak{M}$ in (on) $\overline{\mathfrak{M}}$. For $\bar{a} \in \overline{\mathfrak{M}}$ let $F_{\bar{a}}$ denote the set of antecedents of the element $\bar{a}$ in $\mathfrak{F}$. Evidently the system of the sets $F_{\bar{a}}$ possesses the above properties. Inversely, if there exists a system of sets having the above mentioned properties, the correspondence $\mathfrak{F}$ between the elements of $\mathfrak{M}$ and the elements of $\overline{\mathfrak{M}}$ defined in manner that every element of $F_{\bar{a}}$ is associated with $\bar{a}$, is a homomorphic representation of $\mathfrak{M}$ in (on) $\overline{\mathfrak{M}}$.

·*2. Let $\mathfrak{M}$ be homomorphically representable ($\mathfrak{F}$) in $\overline{\mathfrak{M}}$. Then $\mathfrak{f}\, \mathfrak{M}$ is a m. subsystem in $\overline{\mathfrak{M}}$.*

Proof. Let $\bar{a}, \bar{b} \in \mathfrak{f}\,\mathfrak{M}$ so that there exists $a, b \in \mathfrak{M}$, $\bar{a} = \mathfrak{f}\,a, \bar{b} = \mathfrak{f}\,b$. As $\overline{ab} = \mathfrak{f}\,ab$ we get $\overline{ab} \in \mathfrak{f}\,\mathfrak{M}$.

$\cdot 3$. *Let $\mathfrak{M}$ be homomorphically representable ($\mathfrak{F}$) in $\overline{\mathfrak{M}}$. Let $\overline{\mathfrak{M}}$ be without kernel. Then $\mathfrak{f}\,\mathfrak{M}$ is without kernel.*

The proof follows from $\cdot 2$ and $8 \cdot 2$.

$\cdot 4$. *Let $\mathfrak{M}$ be homomorphically representable ($\mathfrak{F}$) in $\overline{\mathfrak{M}}$. Let α be an arbitrary positive integer. Then $\mathfrak{f}\,(\mathfrak{M})^\alpha = (\mathfrak{f}\,\mathfrak{M})^\alpha$.*

Proof. Let $\bar{a} \in \mathfrak{f}\,(\mathfrak{M})^\alpha$ so that $\bar{a}$ is the counterpart of the product of α elements $a_1, \ldots, a_\alpha \in \mathfrak{M}$. For the counterparts $\bar{a}_1, \ldots, a_\alpha \in \mathfrak{f}\,\mathfrak{M}$ of these elements there holds $\bar{a}_1 \ldots \bar{a}_\alpha = \mathfrak{f}\,a_1 \ldots a_\alpha = \bar{a}$. Hence $\bar{a} \in (\mathfrak{f}\,\mathfrak{M})^\alpha$. Inversely, let $\bar{a} \in (\mathfrak{f}\,\mathfrak{M})^\alpha$ so that $\bar{a}$ is the product of α suitable elements $\bar{a}_1, \ldots, \bar{a}_\alpha \in \mathfrak{f}\,\mathfrak{M}$. These elements being the counterparts of some elements $a_1, \ldots, a_\alpha \in \mathfrak{M}$, we get $a_1 \ldots a_\alpha \in \mathfrak{M}$ and $\mathfrak{f}\,a_1 \ldots a_\alpha = \bar{a}_1 \ldots \bar{a}_\alpha = \bar{a}$. Consequently $\bar{a} \in \mathfrak{f}\,(\mathfrak{M})^\alpha$.

$\cdot 5$. *Let $\mathfrak{M}$ be homomorphically representable in $\overline{\mathfrak{M}}$. Let $\overline{\mathfrak{M}}$ be without kernel. Then $\mathfrak{M}$ is without kernel.*

Proof. Let $\mathfrak{F}$ be a homomorphism of $\mathfrak{M}$ in $\overline{\mathfrak{M}}$. If $\mathfrak{M}$ possesses a kernel, we get $\mathfrak{M}^\alpha = \mathfrak{M}^{\alpha+1}$ for a suitable positive integer α. Then $\mathfrak{f}\,(\mathfrak{M})^\alpha = \mathfrak{f}\,(\mathfrak{M})^{\alpha+1}$ and therefore $(\mathfrak{f}\,\mathfrak{M})^\alpha = (\mathfrak{f}\,\mathfrak{M})^{\alpha+1}$, by $\cdot 4$. Consequently $\mathfrak{f}\,\mathfrak{M}$ possesses a kernel, which contradicts $\cdot 3$. If there exists an $a \in \mathfrak{M}$ such that $a \in \mathfrak{M}^\alpha$ for every positive integer α, it follows $\mathfrak{f}\,a \in \mathfrak{f}\,(\mathfrak{M})^\alpha = (\mathfrak{f}\,\mathfrak{M})^\alpha \subset \overline{\mathfrak{M}}^\alpha$ for every positive integer α and therefore $\overline{\mathfrak{M}}$ is not without kernel.

$\cdot 6$. *Let the suppositions be the same as in $\cdot 5$. Let $a \in \mathfrak{M}$ and let α be the index of the element a in $\mathfrak{M}$. Then the index of $\mathfrak{f}\,a$ in $\mathfrak{f}\,\mathfrak{M}$ is $\geq \alpha$.*

Proof. According to the definition of α we get $a \in \mathfrak{M}^\alpha$ and by $\cdot 4$ there holds $\mathfrak{f}\,a \in (\mathfrak{f}\,\mathfrak{M})^\alpha$. Consequently the index of the element $\mathfrak{f}\,a$ in $\mathfrak{f}\,\mathfrak{M}$ equals at least α.

From this theorem particularly follows that every prime-factor of $\mathfrak{f}\,\mathfrak{M}$ is the counterpart of some prime-factor of $\mathfrak{M}$.

$\cdot 7$. *Let $\mathfrak{M}$ be homomorphically representable ($\mathfrak{F}$) on $\overline{\mathfrak{M}}$. Let $\overline{\mathfrak{M}}$ be without kernel. Let $\bar{a} \in \overline{\mathfrak{M}}$ and let α be the index of $\bar{a}$. There exists an antecedent of $\bar{a}$ in $\mathfrak{F}$ the index of which equals α.*

Proof. By the suppositions and by $\cdot 4$ there holds $\bar{a} \in \overline{\mathfrak{M}}^\alpha \wedge \overline{\mathfrak{M}}^{\alpha+1} = \mathfrak{f}\,(\mathfrak{M})^\alpha \wedge \mathfrak{f}\,(\mathfrak{M})^{\alpha+1}$ so that $\bar{a}$ is the counterpart of a suitable element $a \in \mathfrak{M}^\alpha \wedge \mathfrak{M}^{\alpha+1}$.

32. Complete antecedents of a given m. net. Let $\overline{\mathfrak{M}} = \overline{M}_1 \vee \overline{M}_2 \vee \ldots$ be a m. net b. u. $\overline{M}$. Any m. net $\mathfrak{M} = M_1 \vee M_2 \vee \ldots$ b. u. a set M, homomorphically representable ($\mathfrak{F}$) *on* $\overline{\mathfrak{M}}$ together with the homomorphism $\mathfrak{F}$ is termed *complete* (c.) *antecedent of* $\overline{\mathfrak{M}}$; notation ($\mathfrak{M}$, $\mathfrak{F}$). $\mathfrak{M}$ is the *antecedent* and $\mathfrak{F}$ the *homomorphism belonging to* ($\mathfrak{M}$, $\mathfrak{F}$). ($\mathfrak{M}$, $\mathfrak{F}$) is *homogeneous* if $\mathfrak{M}$ is homogeneous. If $\mathfrak{M}$ is the smallest m.

net $\mathfrak{O}(M)$ b. u. M, $(\mathfrak{O}, \mathfrak{F})$ is a *smallest c. antecedent of* $\overline{\mathfrak{M}}$. For the sake of brevity we sometimes say "c. antecedent" instead of "c. antecedent of $\overline{\mathfrak{M}}$".

Let $(\mathfrak{M}, \mathfrak{F})$ be a c. antecedent of $\overline{\mathfrak{M}}$.

$\cdot 1$. *Let $\underset{\circ}{\mathfrak{M}}$ be a lower m. net w. r. to $\mathfrak{M}$. Let $\underset{\circ}{\mathfrak{F}}$ be the correspondence between the elements of $\underset{\circ}{\mathfrak{M}}$ and those of $\overline{\mathfrak{M}}$ defined in the following way: With every element $\underset{\circ}{a} \in \mathfrak{M}$ the element $\bar{a} \in \overline{\mathfrak{M}}$, where $\underset{\circ}{a} \subset a \in \mathfrak{M}$, is associated. Then $\underset{\circ}{\mathfrak{F}}$ is a homomorphism of $\underset{\circ}{\mathfrak{M}}$ on $\overline{\mathfrak{M}}$.*

Proof. Every element of $\underset{\circ}{\mathfrak{M}}$ is contained in a certain element of $\mathfrak{M}$ and is therefore associated $(\underset{\circ}{\mathfrak{F}})$ with an element of $\overline{\mathfrak{M}}$. Every element of $\overline{\mathfrak{M}}$ is the counterpart $(\mathfrak{F})$ at least of one element of $\mathfrak{M}$ and is therefore counterpart $(\underset{\circ}{\mathfrak{F}})$ at least of one element of $\underset{\circ}{\mathfrak{M}}$. Let $\underset{\circ}{a}, \underset{\circ}{b} \in \underset{\circ}{\mathfrak{M}}$, $\underset{\circ}{a} \to \bar{a}$, $\underset{\circ}{b} \to \bar{b}\,(\underset{\circ}{\mathfrak{F}})$. For suitable $a, b \in \mathfrak{M}$ there holds: $\underset{\circ}{a} \subset a$, $b_0 \subset b$ and $a \to \bar{a}$, $b \to \bar{b}$ $(\mathfrak{F})$. Consequently $ab \to \bar{a}\bar{b}$ $(\mathfrak{F})$. According to 28$\cdot 1$ we get $\underset{\circ}{a}\underset{\circ}{b} \subset ab \to \bar{a}\bar{b}$ $(\mathfrak{F})$ and therefore $\underset{\circ}{a}\underset{\circ}{b} \to \bar{a}\bar{b}$ $(\underset{\circ}{\mathfrak{F}})$.

We say that the homomorphism $\underset{\circ}{\mathfrak{F}}$ of $\underset{\circ}{\mathfrak{M}}$ on $\overline{\mathfrak{M}}$ is *generated by* $\mathfrak{F}$. The c. antecedent $(\underset{\circ}{\mathfrak{M}}, \underset{\circ}{\mathfrak{F}})$ of $\overline{\mathfrak{M}}$ is termed *lower c. antecedent w. r. to* $(\mathfrak{M}, \mathfrak{F})$ and we say that it *is (lies) under* $(\mathfrak{M}, \mathfrak{F})$.

$\cdot 2$. *Let $\mathring{\mathfrak{M}}$ be an upper m. net w. r. to $\mathfrak{M}$ such that every one of its elements contains only elements of $\mathfrak{M}$ associated $(\mathfrak{F})$ with the same element of $\overline{\mathfrak{M}}$. Let $\mathring{\mathfrak{F}}$ be the correspondence between the elements of $\mathring{\mathfrak{M}}$ and those of $\overline{\mathfrak{M}}$ defined in the following way: With every element $\mathring{a} \in \mathring{\mathfrak{M}}$ the element $\bar{a} \in \overline{\mathfrak{M}}$, where $\mathring{a} \supset a \in \mathfrak{M}$, is associated. Then $\mathring{\mathfrak{F}}$ is a homomorphism of $\mathring{\mathfrak{M}}$ on $\overline{\mathfrak{M}}$.*

The proof is analogous to the proof of $\cdot 1$.

We say that the homomorphism $\mathring{\mathfrak{F}}$ of $\mathring{\mathfrak{M}}$ on $\overline{\mathfrak{M}}$ is *generated by* $\mathfrak{F}$. The c. antecedent $(\mathring{\mathfrak{M}}, \mathring{\mathfrak{F}})$ of $\overline{\mathfrak{M}}$ is termed *upper c. antecedent w. r. to* $(\mathfrak{M}, \mathfrak{F})$ and we say that it *is (lies) over* $(\mathfrak{M}, \mathfrak{F})$.

$\cdot 3$. *$(\underset{\circ}{\mathfrak{M}}, \underset{\circ}{\mathfrak{F}})$ lies under $(\mathfrak{M}, \mathfrak{F})$ if and only if $(\mathfrak{M}, \mathfrak{F})$ is over $(\underset{\circ}{\mathfrak{M}}, \underset{\circ}{\mathfrak{F}})$.*

Proof. Suppose that $(\underset{\circ}{\mathfrak{M}}, \underset{\circ}{\mathfrak{F}})$ lies under $(\mathfrak{M}, \mathfrak{F})$. Then every element $a \in \mathfrak{M}$, $a \to \bar{a}\,(\mathfrak{F})$, is the sum of some elements $\underset{\circ}{a} \in \underset{\circ}{\mathfrak{M}}$ for which $\underset{\circ}{a} \to \bar{a}\,(\underset{\circ}{\mathfrak{F}})$. $\mathfrak{M}$ is therefore an upper m. net w. r. to $\underset{\circ}{\mathfrak{M}}$ such that every one of its elements contains only elements of $\underset{\circ}{\mathfrak{M}}$ associated $(\underset{\circ}{\mathfrak{F}})$ with the same element of $\overline{\mathfrak{M}}$ and further, $\mathfrak{F}$ is generated by $\underset{\circ}{\mathfrak{F}}$. Inversely, if $(\mathfrak{M}, \mathfrak{F})$ is over $(\underset{\circ}{\mathfrak{M}}, \underset{\circ}{\mathfrak{F}})$, then every element $a \in \mathfrak{M}$, $a \to \bar{a}\,(\mathfrak{F})$ is the sum of some elements $\underset{\circ}{a} \in \underset{\circ}{\mathfrak{M}}$ for which $\underset{\circ}{a} \to \bar{a}\,(\underset{\circ}{\mathfrak{F}})$; $\underset{\circ}{\mathfrak{M}}$ is therefore a lower m. net w. r. to $\mathfrak{M}$ and $\underset{\circ}{\mathfrak{F}}$ is generated by $\mathfrak{F}$.

33. Let $(\mathfrak{M}, \mathfrak{F})$ be a c. antecedent of $\overline{\mathfrak{M}}$. By definition, every lower m. net $\underset{\circ}{\mathfrak{M}}$ w. r. to $\mathfrak{M}$ determines univocally a c. antecedent of $\overline{\mathfrak{M}}$ under $(\overline{\mathfrak{M}}, \mathfrak{F})$. We get the *smallest c. antecedent of $\overline{\mathfrak{M}}$ under (w. r. to)* $(\mathfrak{M}, \mathfrak{F})$ if we choose for $\underset{\circ}{\mathfrak{M}}$ the m. support of $\mathfrak{M}$; notation $(\underset{\circ}{\mathfrak{M}}_{min}, \underset{\circ}{\mathfrak{F}}_{min})$.

In order to define the largest c. antecedent of $\overline{\mathfrak{M}}$ over $(\mathfrak{M}, \mathfrak{F})$ we are going to prove the following theorem:

$\cdot 1$. *Let $\mathring{\mathfrak{M}}_{max}$ be the set of knots in the warp $O(M)$ defined in the* *following way: Every prime-factor of* $\mathfrak{M}$ *is an element of* $\mathring{\mathfrak{M}}_{max}$. *Any other element of* $\mathring{\mathfrak{M}}_{max}$ *is the sum of all elements of* $\mathfrak{M}$ *which are associated* ($\mathfrak{F}$) *with the same element of* $\overline{\mathfrak{M}}$. *Then* $\mathring{\mathfrak{M}}_{max}$ *is an upper m. net w. r. to* $\mathfrak{M}$.

Proof. It evidently suffices to prove that $\mathring{\mathfrak{M}}_{max}$ possesses the properties 1^0 — 3^0 of a net. $\mathring{\mathfrak{M}}_{max}$ evidently possesses the property 1^0. 2^0 Every element of $\mathring{\mathfrak{M}}_{max}$ is of a certain length. In fact, let $\mathring{a} \in \mathring{\mathfrak{M}}_{max}$. Let $a \in \mathfrak{M}$, $a \subset \mathring{a}$. If a is a prime-factor of $\mathfrak{M}$, we get $a = \mathring{a}$ so that $\mathring{a}$ is of a length equal *1*. Let us therefore suppose that the length α of a is ≥ 2. Let $a \to \bar{a} \in \mathfrak{M}$ ($\mathfrak{F}$). Let $\bar{\alpha}$ be the length of $\bar{a}$. By 31·*6* we get $\alpha \leq \alpha$. $\mathring{a}$ being the sum of all elements a such that $a \to \bar{a}$ ($\mathfrak{F}$), the last inequality proves the proposition. 3^0 For every ordered pair of elements $\mathring{a}, \mathring{b} \in \mathring{\mathfrak{M}}_{max}$ there exists in $\mathring{\mathfrak{M}}_{max}$ an element containing $a\mathring{b}$. In fact, according to the definition of $\mathring{\mathfrak{M}}_{max}$ we get $\mathring{a} = \Sigma a$, $\mathring{b} = \Sigma b$, the first (second) sum being related to some elements a $[b] \in \mathfrak{M}$; the counterpart ($\mathfrak{F}$) of any $a[b]$ occuring in the first (second) sum is the same element $\bar{a}$ $[\bar{b}]$ of $\overline{\mathfrak{M}}$. We get $\widehat{\mathring{a}\mathring{b}} = \Sigma\Sigma\, \widehat{ab}$. For any a, b appearing in this sum there holds $\widehat{ab} \subset ab \to \bar{a}\bar{b}$ ($\mathfrak{F}$). Let $\mathring{c}$ be the element of $\mathring{\mathfrak{M}}_{max}$ containing ab. Then $\widehat{\mathring{a}\mathring{b}} = \Sigma\Sigma\, \widehat{ab} \subset \mathring{c}$.

By definition, $\mathring{\mathfrak{M}}_{max}$ is an upper m. net w. r. to $\mathfrak{M}$ such that every one of its elements contains only elements of $\mathfrak{M}$ associated ($\mathfrak{F}$) with the same element of $\overline{\mathfrak{M}}$. We may therefore choose for $\mathring{\mathfrak{M}}$ the m. net $\mathring{\mathfrak{M}}_{max}$ in order to get an upper c. antecedent ($\mathring{\mathfrak{M}}_{max}$, $\mathring{\mathfrak{F}}_{max}$) w. r. to ($\mathfrak{M}$, $\mathfrak{F}$). The latter is the *largest c. antecedent of* $\overline{\mathfrak{M}}$ *over* (*w. r. to*) ($\mathfrak{M}$, $\mathfrak{F}$). It is clear that any upper c. antecedent of $\overline{\mathfrak{M}}$ w. r. to ($\mathfrak{M}$, $\mathfrak{F}$) lies under ($\mathring{\mathfrak{M}}_{max}$, $\mathring{\mathfrak{F}}_{max}$).

Remark. Let $\bar{a} \in \overline{\mathfrak{M}}$ be of index $\alpha > 2$. It might be shown that there exists in $\mathring{\mathfrak{M}}_{max}$ at most one single element which is not a prime-factor and is associated ($\mathring{\mathfrak{F}}_{max}$) with $\bar{a}$; its index equals precisely α. Accordingly, the set of the antecedents ($\mathring{\mathfrak{F}}_{max}$) of a given element $\bar{a} \in \overline{\mathfrak{M}}$ is composed by a set of prime-factors of $\mathring{\mathfrak{M}}_{max}$ and by a unique further element a whose index equals α, α being the index of $\bar{a}$. The set A of the mentioned prime-factors may be, of course, vacuous. If we remove from $\mathring{\mathfrak{M}}_{max}$ all prime- factors appearing in the sets A associated with the elements $\bar{a} \in \overline{\mathfrak{M}}$ which are not prime, we get a m. system $\mathfrak{N}$ without kernel. It is easy to see that every element of $\mathfrak{N}$ possesses in $\mathfrak{N}$ the same index as in $\mathring{\mathfrak{M}}_{max}$. The homomorphism of $\mathfrak{N}$ on $\overline{\mathfrak{M}}$ established by $\mathring{\mathfrak{F}}_{max}$ is *equiindicial*, which means that the index of the counterpart of every element $n \in \mathfrak{N}$ equals the index of n.

·*2. Let* ($\underset{\circ}{\mathfrak{M}}$, $\underset{\circ}{\mathfrak{F}}$) *be a c. antecedent of* $\overline{\mathfrak{M}}$ *under* ($\mathfrak{M}$, $\mathfrak{F}$). *The largest c. antecedent of* $\mathfrak{M}$ *over* ($\underset{\circ}{\mathfrak{M}}$, $\underset{\circ}{\mathfrak{F}}$) *is the same as the largest c. antecedent of* $\overline{\mathfrak{M}}$ *over* ($\mathfrak{M}$, $\mathfrak{F}$).

Proof. Let ($\underset{\circ}{\mathfrak{M}}_{max}$, $\underset{\circ}{\mathfrak{F}}_{max}$) [($\mathring{\mathfrak{M}}_{max}$, $\mathring{\mathfrak{F}}_{max}$)] be the largest c. antecedent of $\overline{\mathfrak{M}}$ w. r. to ($\underset{\circ}{\mathfrak{M}}$, $\underset{\circ}{\mathfrak{F}}$) [($\mathfrak{M}$, $\mathfrak{F}$)]. Let $\underset{\circ}{a}_{max} \in \underset{\circ}{\mathfrak{M}}_{max}$, $\underset{\circ}{a}_{max} \to \bar{a} \in \overline{\mathfrak{M}}$ ($\underset{\circ}{\mathfrak{F}}_{max}$). According to the definition of $\underset{\circ}{\mathfrak{M}}_{max}$, $\underset{\circ}{a}_{max}$ is the sum of some elements $\underset{\circ}{a} \in \mathfrak{M}$ for which $\underset{\circ}{a} \to \bar{a}$ ($\underset{\circ}{\mathfrak{F}}$). Each of these $\underset{\circ}{a}$ is contained in some $a \in \mathfrak{M}$

corresponding ($\mathfrak{F}$) to the same element $\bar{a}$, because ($\mathfrak{M}$, $\mathfrak{F}$) is over ($\underset{\circ}{\mathfrak{M}}$, $\underset{\circ}{\mathfrak{F}}$), by 32·3. We get therefore $\underset{\circ}{a}_{max} \subset \Sigma a$ for some suitable $a \in \mathfrak{M}$, $a \to \bar{a}\,(\mathfrak{F})$. By the definition of ($\mathring{\mathfrak{M}}_{max}$, $\mathring{\mathfrak{F}}_{max}$) there exists an $a_{max} \in \mathring{\mathfrak{M}}_{max}$ containing all these a and $\mathring{a}_{max} \to \bar{a}\,(\mathring{\mathfrak{F}}_{max})$. We get therefore $\underset{\circ}{\mathring{q}}_{max} \subset \mathring{a}_{max} \to \bar{a}\,(\underset{\circ}{\mathring{\mathfrak{F}}}_{max})$. It remains to prove that $a_{max} \subset \underset{\circ}{a}_{max}$. According to the definition of $\mathring{\mathfrak{M}}_{max}$, $\mathring{a}_{max}$ is the sum of some elements $a \in \mathfrak{M}$ for which $a \to \bar{a}\,(\mathfrak{F})$. Each of these a is the sum of some elements $\underset{\circ}{q} \in \mathfrak{M}$ such that $\underset{\circ}{q} \to \bar{a}\,(\underset{\circ}{\mathfrak{F}})$. Consequently we get $a_{max} \subset \Sigma\, \underset{\circ}{q}$ for some suitable $\underset{\circ}{q} \in \underset{\circ}{\mathfrak{M}}$, $\underset{\circ}{q} \to a\,(\underset{\circ}{\mathfrak{F}})$. According to the definition of ($\underset{\circ}{\mathring{\mathfrak{M}}}_{max}$, $\underset{\circ}{\mathring{\mathfrak{F}}}_{max}$) each of these $\underset{\circ}{q}$ is contained in $\underset{\circ}{\mathring{a}}_{max}$. Thus $a_{max} \subset \underset{\circ}{\mathring{a}}_{max}$.

34. Determination of the c. antecedents of a given m. net. Let $\overline{\mathfrak{M}}$ be a m. net.

·1. *Any c. antecedent of $\overline{\mathfrak{M}}$ lies under the largest c. antecedent w. r. to a smallest c. antecedent of $\overline{\mathfrak{M}}$.*

Proof. Let ($\mathfrak{M}$, $\mathfrak{F}$) be a c. antecedent of $\overline{\mathfrak{M}}$. By 32·3, ($\mathfrak{M}$, $\mathfrak{F}$) lies under the largest c. antecedent ($\mathring{\mathfrak{M}}_{max}$, $\mathring{\mathfrak{F}}_{max}$) w. r. to ($\underset{\circ}{\mathfrak{M}}$, $\mathfrak{F}$). According to 33·2, ($\mathring{\mathfrak{M}}_{max}$, $\mathring{\mathfrak{F}}_{max}$) is the largest c. antecedent of $\overline{\mathfrak{M}}$ over the smallest c. antecedent ($\mathfrak{M}_{min}$, $\mathfrak{F}_{min}$) w. r. to ($\mathfrak{M}$, $\mathfrak{F}$). But the latter is a smallest c. antecedent of $\mathfrak{M}$.

We get therefore the following determination of c. antecedents of a given m. net:

All c. antecedents of a given m. net $\overline{\mathfrak{M}}$ are precisely the lower c. antecedents w. r. to the largest c. antecedents of $\overline{\mathfrak{M}}$ lying over the smallest c. antecedents of $\overline{\mathfrak{M}}$.

35. Construction of the smallest c. antecedents of a given m. net. Let $\overline{\mathfrak{M}} = \overline{M}_1 \vee \overline{M}_2 \vee \ldots$ be a m. net b. u. $\overline{M}$. Let ($\mathfrak{O}$, $\mathfrak{F}$) be a smallest c. antecedent of $\overline{\mathfrak{M}}$ so that $\mathfrak{O} = M_1 \vee M_2 \vee \ldots$ is the smallest m. net b. u. a non-vacuous set M. The homomorphism $\mathfrak{F}$ determines univocally a partial correspondence $\mathfrak{F}^*$ between the elements of M_1 and the elements of a subset $\mathfrak{f}^* M_1$ of $\overline{\mathfrak{M}}$; $\mathfrak{F}^*$ is defined in the way that with every element of M_1 the same element of $\overline{\mathfrak{M}}$ in $\mathfrak{F}^*$ as well as in $\mathfrak{F}$ is associated. Hence $\mathfrak{f}^* M_1 = \mathfrak{f} M_1$. According to 31·6 we get $\overline{M}_1 \subset \mathfrak{f}^* M_1$ so that the power of M_1 and therefore the power of M equals at least the power of $\overline{M}_1$. Let $\{(a_1, \ldots, a_\alpha)\} \in M_\alpha$, $\alpha \geq 2$, and $a_1 \to \bar{a}_1, \ldots, a_\alpha \to \bar{a}_\alpha\,(\mathfrak{F})$ and therefore ($\mathfrak{F}^*$). As $\{(a_1, \ldots, a_\alpha)\} = \{(a_1)\} \ldots \{(a_\alpha)\} \to \bar{a}_1 \ldots \bar{a}_\alpha\,(\mathfrak{F})$, the element $\{(a_1, \ldots, a_\alpha)\}$ is associated ($\mathfrak{F}$) with the product of the counterparts ($\mathfrak{F}^*$) of $\{(\bar{a}_1)\}, \ldots, \{(\bar{a}_\alpha)\}$.

Inversely, let us choose a set M of a power equal at least to the power of $\overline{M}_1$ and let us establish an arbitrary correspondence $\mathfrak{F}^*$ between the elements of M_1, where M_1 denotes the excentrum of the smallest m. net b. u. M: $\mathfrak{O}(M) = M_1 \vee M_2 \vee \ldots$, and the elements of a subset $\mathfrak{f}^* M_1 \supset \overline{M}_1 \subset \overline{\mathfrak{M}}$, such that with every element of M_1 one single element of $\mathfrak{f}^* M_1$ is associated. Let $\mathfrak{F}$ be the correspondence between the elements of $\mathfrak{O}(M)$ and those of $\overline{\mathfrak{M}}$ defined in the following way: Every element of M_1 is associated with the same element of $\overline{\mathfrak{M}}$ in $\mathfrak{F}$ as in $\mathfrak{F}^*$; every element $\{(a_1, \ldots, a_\alpha)\}$ e M_α, $\alpha \geqq 2$, is associated with the product of the counterparts ($\mathfrak{F}^*$) of the elements $\{(a_1)\}, \ldots, \{(a_\alpha)\}$ e M_1. It is clear that $\mathfrak{F}$ is a homomorphism of $\mathfrak{D}(M)$ on $\overline{\mathfrak{M}}$ so that $(\mathfrak{D}, \mathfrak{F})$ is a smallest c. antecedent of $\overline{\mathfrak{M}}$. In this way we get a construction of all smallest c. antecedents of $\overline{\mathfrak{M}}$.

36. Construction of the homogeneous c. antecedents of a given m. net. Let $\overline{\mathfrak{M}}$ be a m. net. The following construction of all homogeneous c. antecedents of $\overline{\mathfrak{M}}$ results from our provious considerations:

Choose a smallest c. antecedent $(\mathfrak{D}, \mathfrak{F}_{min})$ of $\overline{\mathfrak{M}}$ (**35**) and form the largest c. antecedent $(\mathfrak{M}_{max}, \mathfrak{F}_{max})$ over $(\mathfrak{D}, \mathfrak{F}_{min})$ (**33**); form an arbitrary homogeneous lower m. net $\mathfrak{M}$ w. r. to $\mathfrak{M}_{max}$ (**30**) and consider the homomorphism $\mathfrak{F}$ of $\mathfrak{M}$ on $\overline{\mathfrak{M}}$ generated by $\mathfrak{F}_{max}$ (**32**). Then $(\mathfrak{M}, \mathfrak{F})$ is a homogeneous c. antecedent of $\overline{\mathfrak{M}}$.