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BAIRE ONE, NULL FUNCTIONS

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ABSTRACT. It is proved that a Baire one function which is zero almost everywhere can be written as the product of two derivatives. Moreover, if the function is non-negative, then the factors can be selected to be non-negative. In both cases the factors can be chosen to have arbitrarily small L^p norm for $1 \leq p < \infty$.

1. INTRODUCTION. It has been known for some time now that the class of derivatives does not behave well with respect to multiplication. In fact since the turn of the century a number of authors have obtained results which indicate when the product of two derivatives is again a derivative. A relatively complete summary of such results is contained in Fleissner [2].

More recently the focus of attention has turned towards the question of determining the class of functions which can be expressed as the product of two or more derivatives. This work contains results along these lines. Specifically it is first shown that if φ is a bounded, Baire one function that is zero almost everywhere, then φ can be written as the product of two bounded derivatives. Next it is shown that even if φ is not bounded, it is still the product of two derivatives of arbitrarily small L^p norms, $1 \leq p < \infty$. In any case, if φ is nonnegative, then the factors can be selected to be nonnegative.

2. PRELIMINARIES. Let $R = (-\infty, \infty)$. The only measure used is Lebesgue measure in R and each integral should be interpreted as the corresponding Lebesgue integral. For each $S \subset R$, $|S|$ denotes its outer measure, and χ_S its characteristic function. All functions will be real valued functions of a real variable. If S is an open set or an interval, then $\Delta(S)$ is the family of all

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finitely differentiable functions on S where differentiability is one-sided in the case of an endpoint of S that belongs to S. Further set $\Delta^{+}(S) = \{F \in \Delta(S) : F' \geq 0 \text{ on } S\}$, $\mathcal{D}(S) = \{F' : F \in \Delta(S)\}$ and $\mathcal{D}^{+}(S) = \{f \in \mathcal{D}(S) : f \geq 0 \text{ on } S\}$. We write Δ, Δ^{+} etc. for $\Delta(R)$, $\Delta^{+}(R)$ etc. Further let $\mathcal{T} = \{f \in \mathcal{D} : 0 \leq f < 2 \text{ on } R\}$.

The term "D-closed" refers to the Denjoy topology or the density topology on R as it is often called. (See for example [5]). Note that each set of measure zero is D-closed. Let C_{ap} denote the system of all functions approximately continuous on R (that is, continuous relative to the Denjoy topology). It is well known that each element of C_{ap} is a Baire one function.

Let $\mathfrak{A}$ be the collection of all sets $S \subset R$ such that S is both an F_σ set and a G_δ set. Such sets are often called ambiguous sets. As is well known, a function f is of Baire class one if and only if for each $c \in R$ $\{x : f(x) > c\}$ and $\{x : f(x) < c\}$ are F_σ sets. Consequently $S \in \mathfrak{A}$ if and only if χ_S is of Baire class one. It follows from Proposition 3 and Theorem 2 of [1] that $s \in \mathfrak{A}$ if and only if there are $F,G,H \in \Delta$ such that $\chi_S = F'G + H'$. The objective of the next section is to prove that there are $f,g \in \mathcal{T}$ such that $\chi_S = fg$ if and only if $S \in \mathfrak{A}$ and S is D-closed. This section is concluded with some facts that will be needed in the rest of the paper. The first has an easy proof which is therefore omitted. The second is a special case of the first.

2.1. THEOREM. Let V be open, A closed. Let $f \in \mathcal{D}(V \setminus A)$, $\alpha \in C_{ap}$. Let f and α be bounded and let $\alpha = 0$ on $V \cap A$. Set $f^* = 0$ on $V \cap A$, $f^* = \alpha f$ on $V \setminus A$. Then $f^* \in \mathcal{D}(V)$.

2.2. THEOREM. If f and α are bounded functions with $f \in \mathcal{D}$ and $\alpha \in C_{ap}$, then $\alpha f \in \mathcal{D}$.

The next result is a part of Theorem 3.2 in [5].

2.3. THEOREM. If $|S| = 0$ and if φ is a Baire one function on R, then there is an $\alpha \in C_{ap}$ such that $\alpha = \varphi$ on S.

Our next assertion follows from a theorem on p. 257 of [3] by letting $\alpha = 1$.

2.4. THEOREM. Let A be a G_δ set and B, an F_σ set. If $A \subset B$, then there is an $M \in \mathfrak{A}$ such that $A \subset M \subset B$.

We now restate Lemma 12 on p. 29 of [6].

2.5. THEOREM. Let A and B be disjoing, D-closed, G_δ sets. Then there is an $\alpha \in C_{ap}$ such that $0 \leq \alpha \leq 1$ on R, $\alpha = 1$ on A, and $\alpha = 0$ on B.

Finally we establish a consequence of the above which will prove useful in the rest of the work.

2.6. THEOREM. Let A and B be as in 2.5. Then there are $\alpha,\beta \in C_{ap}$ such that $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $\alpha\beta = 0$ on R, $\alpha = 1$ on A and $\beta = 1$ on B.

PROOF. By 2.5 there is a $\gamma \in C_{ap}$ such that $\gamma = 1$ on A and $\gamma = 0$ on B. Let $A_1 = \{x : \gamma(x) \geq \frac{1}{2}\}$, $B_1 = \{x : \gamma(x) \leq \frac{1}{2}\}$. Then A and B_1 are disjoint, D-closed, G_δ sets. Applying 2.5 again we get an $\alpha \in C_{ap}$ such that $0 \leq \alpha \leq 1$ on R, $\alpha = 1$ on A and $\alpha = 0$ on B_1. Similarly there is a $\beta \in C_{ap}$ such that $0 \leq \beta \leq 1$ on R, $\beta = 0$ on A_1 and $\beta = 1$ on B. Clearly $\alpha\beta = 0$ on R so that α and β satisfy our requirements.

3. D-CLOSED AMBIGUOUS SETS. As was mentioned above the objective of this section is to show that those sets whose characteristic functions are the products of two derivatives from $\mathcal{T}$ are precisely the D-closed ambiguous sets. We begin with a lemma which clearly establishes one direction of this characterization. It is actually an easy consequence of Theorem 5.5 of [4] but we include a proof here for the sake of completeness.

3.1. LEMMA. Let m be a natural number, $a,b \in R$, $a < b$; set $J = [a,b]$. Let $f_1,\dots,f_m \in \mathcal{B}^+(J)$, $f_1 \cdots f_m = \chi_S$ on J, $\limsup_{x \searrow 0} |S \cap (a,x)|/(x-a) > 0$. Then $a \in S$.

PROOF. There is an $\epsilon \in (0,\infty)$ and numbers $x_n \in (a,b)$ such that $x_n \to a$ and $|S \cap (a,x_n)| > \epsilon(x_n - a)$ for $n = 1,2,\dots$. Choose an n and set $L = [a,x_n]$. By Hölder's inequality we have $\epsilon \leq \frac{1}{|L|}\int_L (f_1 \cdots f_m)^{1/m} \leq \Pi_{j=1}^m \left(\frac{1}{|L|}\int_L f_j\right)^{1/m}$. This easily implies that $\Pi_{j=1}^m f_j(a) > 0$ so that $a \in S$.

For the remainder of this section let S denote a fixed subset of R. As a notational convenience for each interval $J \subset R$ we will let $\mathcal{P}(J)$ be the set of all pairs (f,g), where $f,g \in \mathcal{B}^+(J)$, $f < 2$, $g < 2$ on J, $f = g = 1$ on $J \cap S$ and $fg = 0$ on $J \setminus S$. Let $\mathcal{J}$ be the system of all intervals J such that $\mathcal{P}(J) \neq \emptyset$.

We now prove four lemmas followed by the other direction of the desired characterization.

3.2. LEMMA. Let $a_1,a_2,b_1,b_2 \in R$, $a_1 < a_2 < b_1 < b_2$. Let $(f_j,g_j) \in \mathcal{P}([a_j,b_j])$ $(j = 1,2)$. Then there is a pair $(f,g) \in \mathcal{P}([a_1,b_2])$ such that $f = f_1$, $g = g_1$ on $[a_1,a_2]$ and $f = f_2$, $g = g_2$ on $[b_1,b_2]$.

PROOF. If there is a $c \in [a_2,b_1] \cap S$, then $f_1(c) = \dots = g_2(c) = 1$ and we set $f = f_1$, $g = g_1$ on $[a_1,c]$, $f = f_2$, $g = g_2$ on $[c,b_2]$. Otherwise we have $f_1g_1 = f_2g_2 = 0$ on $[a_2,b_1]$.

Then we choose a $c \in (a_2,b_1)$ and construct functions f,g such that $f=f_1$, $g=g_1$ on $[a_1,a_2]$, $f(c)=g(c)=0$, $f=f_2$, $g=g_2$ on $[b_1,b_2]$ and that f and g are linear on each of the intervals $[a_2,c]$ and $[c,b_1]$. It is easy to see that $(f,g) \in \mathcal{P}(]a_1,b_2])$.

3.3. LEMMA. Let L be an open interval. Suppose that for each $x \in L$ there is an open interval I such that $x \in I \in \mathcal{J}$. Then $L \in \mathcal{J}$.

PROOF. Choose numbers $x_n \in L$ $(n=0,\pm 1,\pm 2,\ldots)$ such that

$$(1) \qquad x_{n-1} < x_n, \quad \inf_n x_n = \inf L, \quad \sup_n x_n = \sup L.$$

It follows easily from 3.2 that $[x_{n-1},x_{n+1}] \in \mathcal{J}$ for each n. Let $(f_n,g_n) \in \mathcal{P}([x_{n-1},x_{n+1}])$. Applying 3.2 once more we get a pair $(f,g) \in \mathcal{P}(L)$ such that $f(x_n)=f_n(x_n)$, $g(x_n)=g_n(x_n)$ for each n.

3.4. LEMMA. Let $a,b \in R$, $a<b$, $q \in (0,b-a)$. Set $J=[a,b]$. Let $(f_0,g_0) \in \mathcal{P}(J)$. Then there is a pair $(f,g) \in \mathcal{P}(J)$ such that $f=f_0$, $g=g_0$ on $[a,b]$, $q<\int_J f \leq |J|$ and $q<\int_J g \leq |J|$.

PROOF. Choose a $p \in (0,1)$ such $|J|p > q(2-p)$. Set $S_1 = (S \cap J) \cup \{a,b\}$, $S_0 = J \setminus S_1$. It follows from 3.1 that S_1 is a D-closed, G_δ set. If $|S_0|=0$, we define $A=B=\emptyset$. Otherwise $|S_0|/(2-p) < |S_0|$ and we choose disjoint closed sets $A,B \subset S_0$ such that $|A|=|B|$ and that $|A \cup B| > |S_0|/(2-p)$. It follows from 2.6 that there are $\alpha,\beta,\gamma \in C_{ap}$ with $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $0 \leq \gamma \leq 1$, $\alpha\beta = \alpha\gamma = \beta\gamma = 0$ on R, $\alpha = 1$ on A, $\beta = 1$ on B and $\gamma = 1$ on S_1. Define $f = 2p\,\alpha + \gamma f_0$, $g = 2p\,\beta + \gamma g_0$ on J. It is easy to see that $(f,g) \in \mathcal{P}(J)$ and that $f=f_0$, $g=g_0$ on $\{a,b\}$. Since $2|A|(2-p) \geq |S_0|$, we have $q < \frac{p}{2-p}(|S_0|+|S_1|) \leq |S_1| + 2p|A| \leq \int_J f \leq$
$\leq |S_1| + 2p|A| + 2(|S_0| - 2|A|) = |J| + |S_0| - 2|A|(2-p) \leq |J|$; similarly for g.

3.5. LEMMA. Let $a,b \in R$, $a<b$; set $L=(a,b)$. Let $L \in \mathcal{J}$ and let w be a positive, continuous function on L. Then there is a pair $(f,g) \in \mathcal{P}(L)$ such that for each $x \in L$

$$(2) \qquad \max\left(\left|\int_a^x (f-1)\right|, \left|\int_a^x (g-1)\right| \right) \leq \int_a^x w,$$

$$\max\left(\left|\int_x^b (f-1)\right|, \left|\int_x^b (g-1)\right| \right) \leq \int_x^b w.$$

PROOF. Set $\psi = w/2$. We may suppose that $\psi < 1$ on L. There are numbers $x_n \in L$ $(n=0,\pm 1,\pm 2,\ldots)$ fulfilling (1) such that

$$x_n - x_{n-1} < \min\left(\int_a^{x_{n-1}} \psi, \int_{x_n}^b \psi \right) \text{ for each } n.$$

Let $(f^*,g^*) \in \mathcal{P}(L)$. For each n set $J_n = [x_{n-1},x_n]$. By 3.4 there are pairs $(f_n,g_n) \in \mathcal{P}(J_n)$ such that $f_n = f^*$, $g_n = g^*$ on $\{x_{n-1},x_n\}$ and that

$$\int_{J_n} (1-\psi) < \int_{J_n} f_n \leq |J_n| , \qquad \int_{J_n} (1-\psi) < \int_{J_n} g_n \leq |J_n| .$$

Define functions f and g on L setting $f = f_n$, $g = g_n$ on J_n. Now let $x \in J_n$. Then $\int_a^x f \leq \int_a^{x_n} f \leq x_n - a < \int_a^{x_{n-1}} (1+\psi) < \int_a^x (1+w)$, $\int_a^x f \geq \int_a^{x_{n-1}} f > \int_a^{x_{n-1}} (1-\psi) > \int_a^{x_n} (1-\psi) - (x_n - x_{n-1}) > \int_a^{x_n} (1-w) \geq \int_a^x (1-w)$. Similarly for $\int_x^b f$ and for g. This proves (2). It is obvious that $(f,g) \in \mathcal{P}(L)$.

3.6. THEOREM. Let S be a D-closed ambiguous set. Then $R \in \mathcal{J}$.

PROOF. Let U be the set of all points x such that $x \in I$ for some open interval $I \in \mathcal{J}$. Let $A = R \setminus U$. Then A is closed. Suppose that $A \neq \emptyset$. Let w be a continuous function on R such that $w = 0$ on A and $w > 0$ on U. Since χ_S is a Baire one function, there is a bounded open interval I such that $A \cap I \neq \emptyset$ and that χ_S is constant on $A \cap I$. For each component $L = (a,b)$ of $I \cap U$ we have, by 3.3, $L \in \mathcal{J}$ so that by 3.5 there is a pair $(f,g) \in \mathcal{P}(L)$ fulfilling (2). In this way we construct functions f,g on $I \cap U$. Now we distinguish two cases.

Suppose first that $A \cap I \subset S$. Define $f^* = f$, $g^* = g$ on $I \cap U$, $f^* = g^* = 1$ on $I \cap A$. If $x_1,x_2 \in I$, $x_1 < x_2$ and if $x_1 \in A$ or $x_2 \in A$, then it follows easily from (2) that $|\int_{x_1}^{x_2} (f^*-1)| \leq \int_{x_1}^{x_2} w$. Since w is continuous and $w = 0$ on A, $f^* \in \mathcal{D}(I)$; similarly $g^* \in \mathcal{D}(I)$. It is obvious that $(f^*,g^*) \in \mathcal{P}(I)$ so that $I \in \mathcal{J}$, $I \subset U$ – a contradiction.

Now suppose that $A \cap I \cap S = \emptyset$. Let J be a closed interval with interior V such that $A \cap V \neq \emptyset$ and $J \subset I$. Then $A \cap J$ and S are disjoint, D-closed, G_δ sets. By 2.5 there is an $\alpha \in C_{ap}$ such that $\alpha = 0$ on $A \cap J$, $\alpha = 1$ on S and $0 \leq \alpha \leq 1$ on R. Define $f^* = g^* = 0$ on $V \cap A$, $f^* = \alpha f$, $g^* = \alpha g$ on $V \cap U$. By 2.1 we have f^*, $g^* \in \mathcal{D}(V)$. It is obvious that $(f^*,g^*) \in \mathcal{P}(V)$ so that $V \in \mathcal{J}$, $V \subset U$ – a contradiction.

It follows that $A = \emptyset$. By 3.3 we have $R = U \in \mathcal{J}$.

3.7. COROLLARY. Let $S \subset R$. Then the following four conditions are equivalent:

1) There is a natural number m and functions $f_1,\ldots,f_m \in \mathcal{D}^+$ such that $f_1 \cdots f_m = \chi_S$.

2) S is ambiguous and D-closed.

3) There are $f,g \in \mathcal{D}^+$ such that $f=g=1$ on S, $fg=0$ on $R\setminus S$ and $f<2$, $g<2$ on R.

4) There are $f,g \in \mathcal{T}$ such that $fg=\chi_S$.

PROOF. If 1) holds, then S is ambiguous and it follows easily from 3.1 that S is D-closed. If 2) holds, then, by 3.6, 3) holds as well. The implications 3) => 4) and 4) => 1) are obvious.

4. BOUNDED, BAIRE ONE, NULL FUNCTIONS. The goal of this section is to establish that a bounded, [nonnegative] Baire one function that is zero a.e. can be expressed as the product of two bounded [nonnegative] derivatives. This fact follows easily from Theorem 4.2 whose proof relies mainly on Theorem 3.6. We begin with a proposition based on 2.5 which is used in the proof of 4.2.

4.1. PROPOSITION. Let $A_1,A_2,\ldots$ be pairwise disjoint elements of $\mathfrak{A}$ of measure zero. Then there are $\alpha_1,\alpha_2,\ldots \in C_{ap}$ such that for each j $\alpha_j=1$ on A_j, $0\le\alpha_j\le 1$ on R and $\alpha_i\alpha_j=0$ on R, if $i\neq j$.

PROOF. For each j let $S_j=\bigcup_{i\neq j}A_i$. Then $|S_j|=0$. Let T_j be a G_δ set such that $S_j\subset T_j$ and $|T_j|=0$. Further let $B_j=T_j\setminus A_j$. Then $S_j\subset B_j$; B_j and A_j are disjoint, D-closed G_δ sets. So by 2.5 there are $\alpha_j^*,\beta_j\in C_{ap}$ such that for each j $\alpha_j^*=1$ on A_j, $\beta_j=1$ on B_j, and $0\le\alpha_j^*\le 1$, $0\le\beta_j\le 1$, $\alpha_j^*\beta_j=0$ on R. Let $\alpha_j=\beta_1\cdots\beta_{j-1}\alpha_j^*$. If $i<j$, then $\alpha_i\alpha_j$ is a multiple of $\alpha_i^*\beta_i$ so that $\alpha_i\alpha_j=0$. The other requirements are easily verified.

4.2. THEOREM. Let $B_1,B_2,\ldots$ be pairwise disjoint elements of $\mathfrak{A}$ of measure zero and let $Q\in(2,\infty)$. Let $\varphi_1,\varphi_2,\ldots$ be Baire one functions such that $|\varphi_n|\le 1$ for each n and that $\varphi_n\to 0$ uniformly on R. Then there are $f,g\in\mathcal{D}$ such that $|f|<Q$, $|g|<Q$ and $fg=\Sigma_{n=1}^{\infty}\varphi_n\chi_{B_n}$ on R. If, moreover, $\varphi_n\ge 0$ for each n, we may choose $f\ge 0$ and $g\ge 0$.

PROOF. Let $\epsilon_1,\epsilon_2,\ldots$ be positive numbers such that $\epsilon_1=1$ and that $\Sigma_{j=1}^{\infty}2\epsilon_j<Q$. There are integers r_j such that $0=r_0<r_1<\ldots$ and that $|\varphi_n|\le\epsilon_j^2$ for each $n>r_{j-1}$ $(j=1,2,\ldots)$. Set $S_j=\{r_{j-1}+1,\ldots,r_j\}$, $\psi_j=\Sigma_{n\in S_j}\varphi_n\chi_{B_n}$, and $A_j=\bigcup_{n\in S_j}B_n$. Then $A_1,A_2,\ldots$ are pairwise disjoint elements of $\mathfrak{A}$ of measure zero. Let α_j be as in 4.1. According to 2.3 there are $\gamma_1,\gamma_2,\ldots\in C_{ap}$ such that $\gamma_j=\psi_j$ on A_j. Since $|\psi_j|\le\epsilon_j^2$, we may assume $|\gamma_j| \le \epsilon_j^2$. According to 3.6 there are $f_j, g_j \in \mathcal{T}$ such that $f_j g_j = \chi_{A_j}$. Obviously $|\alpha_j \gamma_j f_j/\epsilon_j| \le 2\epsilon_j$ and $|\alpha_j g_j \epsilon_j| \le 2\epsilon_j$. Define $f = \Sigma_{j=1}^{\infty} \alpha_j \gamma_j f_j/\epsilon_j$ and $g = \Sigma_{j=1}^{\infty} \alpha_j g_j \epsilon_j$. Then $|f| \le \Sigma_{j=1}^{\infty} 2\epsilon_j < Q$ and also $|g| < Q$. The series that define f and g converge uniformly and by 2.2 each term is in $\mathcal{D}$. Thus $f, g \in \mathcal{D}$. Since $A_1, A_2, \ldots$ are pairwise disjoint and $\gamma_j \chi_{A_j} = \psi_j$, we have $fg = \Sigma_{j=1}^{\infty} \alpha_j^2 \gamma_j f_j g_j = \Sigma_{j=1}^{\infty} \alpha_j^2 \psi_j = \Sigma_{k=1}^{\infty} \psi_j = \Sigma_{n=1}^{\infty} \varphi_n \chi_{B_n}$. If $\varphi_n \ge 0$ for each n, then we may choose $\gamma_j \ge 0$ and we obtain $f \ge 0$ and $g \ge 0$ on R.

4.3. COROLLARY. Let φ be a Baire one function, $|\varphi| \le 1$ on R, $\varphi = 0$ a.e. and let $Q \in (2, \infty)$. Then there are $f, g \in \mathcal{D}$ such that $|f| < Q$, $|g| < Q$ and $fg = \varphi$ on R. Moreover, if $\varphi \ge 0$, we may select $f \ge 0$ and $g \ge 0$.

PROOF. Let a_n be numbers such that $a_0 > a_1 > \ldots$, $a_1 = 1$ and $a_n \to 0$. For $n = 1, 2, \ldots$ let $V_n = \{x : a_{n+1} \le |\varphi(x)| \le a_n\}$ and $W_n = \{x : 0 < |\varphi(x)| < a_{n-1}\}$. Then $V_n \subset W_n$, V_n is a G_δ set and W_n an F_σ set. So by 2.4 there are $M_n \in \mathfrak{A}$ such that $V_n \subset M_n \subset W_n$. Let $B_n = M_n \setminus \bigcup_{j=1}^{n-1} M_j$ for $n = 1, 2, \ldots$. We see that $B_1, B_2, \ldots$ are pairwise disjoint elements of $\mathfrak{A}$ of measure zero, $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} M_n = \{x : \varphi(x) \ne 0\}$, and $\varphi = \Sigma_{n=1}^{\infty} \varphi \chi_{B_n}$. Since $B_n \subset W_n$, we have $\varphi \chi_{B_n} \to 0$ uniformly on R. Now we apply 4.2.

The next result shows that we would get a wrong assertion, if we admitted $Q < 2$ in 4.3. If, however, the function φ in 4.3 satisfies the relation $\varphi(R) = \{0,1\}$, then, by 3.6, it can be expressed as the product of two nonnegative derivatives each of which is bounded by 2. We do not know whether Q can be replaced by 2 in 4.3.

4.4. THEOREM. Let $Q \in (0, \infty)$. Let $f, g \in \mathcal{D}$, $|f| \le Q$, $|g| \le Q$ and $fg = 0$ a.e. on R. Then $|fg| \le Q^2/4$ on R.

PROOF. Let $x, y \in R$, $y \ne x$. Let $a = (y-x)^{-1} \int_x^y f$, $b = (y-x)^{-1} \int_x^y g$. Since $|f| + |g| \le Q$ a.e., we have $|a| + |b| \le Q$ so that $4|ab| \le Q^2$. Thus $|f(x)g(x)| \le Q^2/4$.

5. ARBITRARY BAIRE ONE, NULL FUNCTIONS.

Throughout the rest of the paper p will denote a number in $[1, \infty)$. We set $\gamma = 1 - p^{-1}$. If f is a function, $S \subset R$ and if the integral $M = \int_S |f|^p$ is finite, we write $\|f\|_S = M^{1/p}$. If, moreover, $|S| < \infty$, then, by Hölder's inequality,

(3) $$\int_S |f| \leq \|f\|_S \cdot |S|^{\gamma} .$$

If the meaning of S is obvious from the context (if, e.g., S is the domain of definition of f), we write $\|f\|_S = \|f\|$. The class of all functions φ for which to each $\varepsilon \in (0,\infty)$ there correspond $f,g \in \mathcal{D}$ ($\mathcal{D}^+$ resp.) such that $\|f\|_R + \|g\|_R < \varepsilon$ and $fg = \varphi$ on S will be denoted by $\mathcal{R}(S)$ ($\mathcal{R}^+(S)$ resp.). Moreover, $\ell oc\,\mathcal{R}(S)$ ($\ell oc\,\mathcal{R}^+(S)$ resp.) is the family of all functions φ such that for each $x \in S$ there is an open interval I with $x \in I$ and $\varphi \in \mathcal{R}(S \cap I)$ ($\mathcal{R}^+(S \cap I)$ resp.). In keeping with our previous conventions $\mathcal{R}(R)$ and $\mathcal{R}^+(R)$ will be denoted simply by $\mathcal{R}$ and $\mathcal{R}^+$ respectively.

In terms of this notation our objective is to show that if φ is a Baire one function which is zero a.e. on R, then $\varphi \in \mathcal{R}$. If in addition $\varphi \geq 0$, then $\varphi \in \mathcal{R}^+$. We begin with a useful lemma and the treat the case where φ is bounded.

5.1. LEMMA. Let A be a measurable set and let $\varepsilon \in (0,\infty)$. Then there is a closed set $C \subset A$ and a $\lambda \in C_{ap}$ such that $|A \setminus C| < \varepsilon$, $0 \leq \lambda \leq 1$ on R, $\lambda = 1$ on C and $\lambda = 0$ on $R \setminus A$.

PROOF. Denote by S the set of all points in A that are points of density of A. There is an F_σ set $T \subset S$ such that $|S \setminus T| = 0$. There is a closed set $C \subset T$ such that $|T \setminus C| < \varepsilon$. Clearly $|A \setminus C| = |T \setminus C| < \varepsilon$. Since $R \setminus T$ and C are disjoint, D-closed, G_δ sets, by 2.5 there is a $\lambda \in C_{ap}$ such that $\lambda = 1$ on C, $\lambda = 0$ on $R \setminus T$, and $0 \leq \lambda \leq 1$ on R.

5.2. THEOREM. Let φ be a bounded Baire one function such that $\varphi = 0$ a.e. on R. Then $\varphi \in \mathcal{R}$. If in addition $\varphi \geq 0$, then $\varphi \in \mathcal{R}^+$.

PROOF. Assume as we may that $|\varphi| \leq 1$. Let $\varepsilon \in (0,\infty)$. By 4.3 there are $f_1, g_1 \in \mathcal{D}$ such that $|f_1| < 3$, $|g_1| < 3$, and $f_1 g_1 = \varphi$ on R. Let $B = \{x : \varphi(x) \neq 0\}$. It follows easily from 5.1 that there is an open set $U \supset B$ and a $\lambda \in C_{ap}$ such that $|U| = |U \setminus B| < (\varepsilon/6)^p$, $0 \leq \lambda \leq 1$ on R, $\lambda = 1$ on B and $\lambda = 0$ on $R \setminus U$. Let $f = \lambda f_1$ and $g = \lambda g_1$. By 2.2 $f,g \in \mathcal{D}$. Since $\lambda = 1$ on B, $fg = f_1 g_1 = \varphi$. Finally $\|f\| + \|g\| \leq 2 \cdot 3 \cdot |U|^{1/p} < \varepsilon$. If $\varphi \geq 0$, then we may select f_1 and g_1 from $\mathcal{D}^+$ and we have $f,g \in \mathcal{D}^+$.

We now take up the process of showing that the assumption of boundedness can be deleted from 5.2. We start with three assertions the first of which will be used again later.

5.3. LEMMA. Let $J = [a,b]$ and let $\varepsilon \in (0,\infty)$. Let $f_1, g_2, g_1, g_2 \in \mathcal{D}(J)$ ($\mathcal{D}^+(J)$ resp.). Suppose that $f_1 g_1 = f_2 g_2 = \varphi$ on J and that $\varphi = 0$ on a dense subset of J. Then there are a

$c \in (a,b)$ and $f,g \in \mathcal{B}(J)$ ($\mathcal{B}^+(J)$ resp.) such that $fg = \varphi$ on J, $f(c) = g(c) = 0$, $f(a) = f_1(a)$, $g(a) = g_1(a)$, $f(b) = f_2(b)$, $g(b) = g_2(b)$, $|f| \leq |f_1| + \varepsilon$, $|g| \leq |g_1| + \varepsilon$ on $[a,c]$ and $|f| \leq |f_2| + \varepsilon$, $|g| \leq |g_2| + \varepsilon$ on $[c,b]$.

PROOF. There is a $c \in (a,b)$ such that f_1, f_2, g_1, and g_2 are all continuous at c. Since $f_1 g_1 = 0$ on a dense subset of J, $f_1(c) g_1(c) = 0$. Since the roles of f_1 and g_1 are interchangeable, we may assume that $f_1(c) = 0$. There is a $c_1 \in (a,c)$ and a function h continuous on $[a,c]$ such that $h = 1$ on $[a,c_1]$, $h(c) = 0$, $|f_1|/\varepsilon < h \leq 1$ on $[c_1,c)$ and g_1 is bounded on $[c_1,c]$. Set $f = f_1/\sqrt{h}$, $g = g_1\sqrt{h}$ on $[a,c)$ and $f(c) = g(c) = 0$. We define f and g on $[c,b]$ in a similar fashion. On $[c_1,c)$ we have $|f| < \varepsilon\sqrt{h} \leq \varepsilon$; in particular, f is continuous at c. Now it is easy to see that $f \in \mathcal{B}(J)$. The rest of the proof is left to the reader.

5.4. PROPOSITION. Let $-\infty < a_1 < a_2 < b_1 < b_2 < \infty$ and let $\varphi \in \mathcal{R}([a_1,b_1]) \cap \mathcal{R}([a_2,b_2])$ ($\mathcal{R}^+([a_1,b_1]) \cap \mathcal{R}^+([a_2,b_2])$ resp.). Then $\varphi \in \mathcal{R}([a_1,b_2])$ ($\mathcal{R}^+([a_1,b_2])$ resp.).

PROOF. Let $\varepsilon \in (0,\infty)$. There are $f_1, g_1, f_2, g_2 \in \mathcal{B}$ ($\mathcal{B}^+$ resp.) such that $f_1 g_1 = \varphi$ on $[a_1,b_1]$, $f_2 g_2 = \varphi$ on $[a_2,b_2]$ and $\|f_1\|^p + \|g_1\|^p + \|f_2\|^p + \|g_2\|^p < (\varepsilon/2)^p$. Let $J = [a_2,b_1]$. It follows easily from 5.3 that there are $f,g \in \mathcal{B}(J)$ ($\mathcal{B}^+(J)$ resp.) such that $fg = \varphi$ on J, $f(a_2) = f_1(a_2)$, $g(a_2) = g_1(a_2)$, $f(b_1) = f_2(b_1)$, $g(b_1) = g_2(b_1)$ and $\|f\|_J^p + \|g\|_J^p < (\varepsilon/2)^p$. Let $f = f_1$, $g = g_1$ on $(-\infty,a_2)$ and $f = f_2$, $g = g_2$ on (b_1,∞). Then $f,g \in \mathcal{B}$ ($\mathcal{B}^+$ resp.), $fg = \varphi$ on $[a_1,b_2]$ and $\|f\|^p + \|g\|^p < 2(\varepsilon/2)^p$. So $(\|f\| + \|g\|)^p \leq 2^{p-1}(\|f\|^p + \|g\|^p) < \varepsilon^p$.

The next statement follows easily from the above by a routine compactness argument.

5.5. COROLLARY. Let J be a closed, bounded interval and let $\varphi \in \ell oc\,\mathcal{R}(J)$ ($\ell oc\,\mathcal{R}^+(J)$ resp.). Then $\varphi \in \mathcal{R}(J)$ ($\mathcal{R}^+(J)$ resp.).

The preceding result and the next lemma are used in the proofs of propositions 5.7 and 5.7.1.

5.6. LEMMA. Let $J = [a,b]$. Let $M,N \in R$, $f_1, g_1 \in \mathcal{B}(J)$, $f_1 g_1 = 0$ a.e. on J and $\|f_1\| + \|g_1\| < \infty$. Then there are $f,g \in \mathcal{B}(J)$ such that $f = f_1$, $g = g_1$ on $\{a,b\}$, $fg = f_1 g_1$ on J, $\int_J f = M$, $\int_J g = N$, $\|f\| \leq 5\|f_1\| + |M|(4/|J|)^\gamma$, $\|g\| \leq \|g_1\| + |N|(4/|J|)^\gamma$. If in addition $f_1, g_1 \in \mathcal{B}^+(J)$, $M \geq \int_J f_1$ and $N \geq \int_J g_1$, then f and g can be chosen from $\mathcal{B}^+(J)$.

PROOF. Let $B = \{x : f_1(x)g_1(x) \neq 0\} \cup \{x : f_1$ or g_1 is not approximately continuous at $x\} \cup \{a,b\}$. Then $|B| = 0$. There is a $K \in (0,\infty)$ such that $|S| > |J|/2$ where $S = \{x \in J\backslash B : |f_1(x)| + |g_1(x)| \leq K\}$. There is a closed set $A \subset S$ with $|A| > |J|/2$. According to 5.1 there is a closed set $C \subset A$ and a $\lambda \in C_{ap}$ such that $|C| > |J|/2$, $0 \leq \lambda \leq 1$ on R, $\lambda = 1$ on C and $\lambda = 0$ on $R\backslash A$. Note that λf_1 is approximately continuous on A (since $A \subset J\backslash B$) and $\lambda f_1 = 0$ on $J\backslash A$ which is open in J. It follows that λf_1 is approximately continuous on J. Since $|\lambda f_1| \leq K$ on J, $\lambda f_1 \in \mathcal{D}(J)$. Likewise $\lambda g_1 \in \mathcal{D}(J)$. Let $f_2 = f_1 - \lambda f_1$ and $g_2 = g_1 - \lambda g_1$. Then $f_2, g_2 \in \mathcal{D}(J)$ and, as is easily verified from the properties of λ, $f_2 g_2 = f_1 g_1$, $|f_2| \leq |f_1|$ and $|g_2| \leq |g_1|$. Since $\|f_1\| < \infty$, we have $\int_J |f_2| \leq \int_J |f_1| < \infty$; similarly $\int_J |g_2| < \infty$. It follows from 5.1 that there are $\alpha, \beta \in C_{ap}$ and disjoint, closed subsets C_α and C_β of C such that $|C_\alpha| > |J|/4$, $|C_\beta| > |J|/4$, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $\alpha\beta = 0$ on R, $\alpha = \beta = 0$ on $R\backslash C$, $\alpha = 1$ on C_α and $\beta = 1$ on C_β. Let $s = (M - \int_J f_2)/\int_J \alpha$ and $t = (N - \int_J g_2)/\int_J \beta$. Let $f = f_2 + s\alpha$ and $g = g_2 + t\beta$. Obviously $\alpha^p \leq \alpha$ and $\int_J \alpha \geq |J|/4$; by (3), $\int_J |f_2| \leq \|f_1\| \cdot |J|^\gamma$. Thus $(\int_J \alpha^p)^{1/p} / \int_J \alpha \leq (\int_J \alpha)^{p^{-1}-1} \leq (4/|J|)^\gamma$, $\|f\| \leq \|f_2\| + \|\alpha\| (|M| + \int_J |f_2|)/\int_J \alpha \leq \|f_1\| + (4/|J|)^\gamma (|M| + \|f_1\| \cdot |J|^\gamma) \leq 5\|f_1\| + (4/|J|)^\gamma |M|$. A similar estimate is valid for g.

If the additional assumption is fulfilled, then $f_2 \geq 0$, $g_2 \geq 0$, $s \geq 0$, $t \geq 0$ so that $f \geq 0$ and $g \geq 0$.

5.7. PROPOSITION. Let I be an open interval, ω a positive, continuous function on I, $F_0, G_0 \in \Delta(I)$, $\|F_0'\| + \|G_0'\| < \infty$, $\varphi \in \ell oc\ (I)$ and let $\epsilon \in (0,\infty)$. Then there are $F, G \in \Delta(I)$ such that

$$F'G' = \varphi, \quad |F - F_0| + |G - G_0| < \omega \text{ on } I \text{ and} \tag{4}$$

$$\|F'\|^p + \|G'\|^p < \epsilon + 8^{p-1}(\|F_0'\|^p + \|G_0'\|^p). \tag{5}$$

PROOF. There are numbers $y_n \in I$ $(n = 0, \pm 1, \pm 2, \ldots)$ such that $y_n < y_{n+1} < y_n + \frac{1}{2}$, $\inf_n y_n = \inf I$, $\sup_n y_n = \sup I$ and that

$$\int_{y_{n-1}}^{y_{n-1}} (|F_0'| + |G_0'|) | < \mu_n = \min\{\omega(x) : x \in [y_{n-1}, y_{n+1}]\}/7 .$$

Choose $\epsilon_n \in (0, \mu_n^p)$ such that $\Sigma_{n=-\infty}^{\infty} \epsilon_n < \epsilon/2^{p-1}$. Since $\varphi \in \ell oc\,\mathcal{R}(I)$, 5.5 implies that $\varphi \in \mathcal{R}([y_{n-1}, y_{n+1}])$. There are $f_{n,1}, g_{n,1} \in \mathcal{D}$ such that $f_{n,1} g_{n,1} = \varphi$ on $[y_{n-1}, y_{n+1}]$ and $\|f_{n,1}\|^p + \|g_{n,1}\|^p < \epsilon_n/5^p$. It follows from 5.3 that there are $x_n \in (y_{n-1}, y_n)$ and $f_{n,2}, g_{n,2} \in \mathcal{D}([x_n, x_{n+1}])$ such that $f_{n,2} g_{n,2} = \varphi$ on $[x_n, x_{n+1}]$, $f_{n,2} = g_{n,2} = 0$ on $\{x_n, x_{n+1}\}$, and $\|f_{n,2}\|^p + \|g_{n,2}\|^p < \epsilon_n/5^p$. Denote

$[x_n, x_{n+1}]$ by J_n. According to 5.6 there are $f_n, g_n \in \mathcal{D}(J_n)$ such that $f_n = g_n = 0$ on $\{x_n, x_{n+1}\}$, $f_n g_n = \varphi$ on J_n,

(6) $$\int_{J_n} f_n = \int_{J_n} F_0', \quad \int_{J_n} g_n = \int_{J_n} G_0',$$

(7) $$\|f_n\| \le 5\|f_{n,2}\| + |\int_{J_n} F_0'|(4/|J_n|)^\gamma,$$

(8) $$\|g_n\| \le 5\|g_{n,2}\| + |\int_{J_n} G_0'|(4/|J_n|)^\gamma.$$

Using (3), (7), and the relation $|J_n| < 1$ we get $\int_{J_n}|f_n| \le 5|J_n|^\gamma\|f_{n,2}\| + 4^\gamma\int_{J_n}|F_0'| \le \epsilon_n^{1/p} + 4\int_{J_n}|F_0'|$, $\|f_n\|^p \le 2^{p-1}(5^p\|f_{n,2}\|^p + 4^{p\gamma}\|F_0'\|_{J_n}^p)$. Similarly, $\int_{J_n}|g_n| \le \epsilon_n^{1/p} + 4\int_{J_n}|G_0'|$, $\|g_n\|^p \le 2^{p-1}(5^p\|g_{n,2}\|^p + 4^{p\gamma}\|G_0'\|_{J_n}^p)$. There are $F, G \in \mathcal{D}(I)$ such that $F' = f_n$, $G' = g_n$ on J_n, $F(x_n) = F_0(x_n)$, and $G(x_n) = G_0(x_n)$ for each n. If $x \in J_n$, then (since $\epsilon_n < \mu_n^p$ and $J_n \subset [y_{n-1}, y_{n+1}]$) $|F(x) - F_0(x)| + |G(x) - G_0(x)| \le \int_{J_n}(|f_n| + |g_n| + |F_0'| + |G_0'|) < 2\epsilon_n^{1/p} + 4\mu_n + \mu_n \le \omega(x)$. Finally (note that $p\gamma = p-1$) $\|F'\|^p + \|G'\|^p = \Sigma_{n=-\infty}^{\infty}(\|f_n\|^p + \|g_n\|^p) \le 2^{p-1}\Sigma_{n=-\infty}^{\infty}\epsilon_n + 8^{p-1}\Sigma_{n=-\infty}^{\infty}(\|F_0'\|_{J_n}^p + \|G_0'\|_{J_n}^p) < \epsilon + 8^{p-1}(\|F_0'\|_I^p + \|G_0'\|_I^p)$.

The version of the preceding theorem involving nonnegative functions is somewhat different. Consequently we state it separately.

5.7.1. PROPOSITION. Let I be an open interval, ω a positive, continuous function on I, $F_0, G_0 \in \Delta^+(I)$, $F_0'G_0' > 0$ on I, $\|F_0'\| + \|G_0'\| < \infty$, $\varphi \in \mathit{loc}\,\mathcal{R}^+(I)$ and let $\epsilon \in (0,\infty)$. Then there are $F, G \in \Delta^+(I)$ fulfilling (4) and (5).

PROOF. There are $x_n, y_n \in I$ such that $x_n < y_n < x_{n+1} < x_n + \frac{1}{2}$ $(n = 0, \pm 1, \pm 2, \ldots)$, $\inf_n x_n = \inf I$, $\sup_n x_n = \sup I$ and $F_0(x_{n+1}) - F_0(y_{n-1}) + G_0(x_{n+1}) - G_0(y_{n-1}) < \min\{\omega(x) : x \in [y_{n-1}, x_{n+1}]\}$ for each n. Choose $\epsilon_n \in (0,\infty)$ such that $\Sigma_{n=-\infty}^{\infty}\epsilon_n < \epsilon/2^{p-1}$. Since $\varphi \in \mathit{loc}\,\mathcal{R}^+(I)$, 5.5 implies that $\varphi \in \mathcal{R}^+([y_{n-1}, x_{n+1}])$. Consequently there are $f_{n,1}, g_{n,1} \in \mathcal{D}^+$ such that $f_{n,1}g_{n,1} = \varphi$ on $[y_{n-1}, x_{n+1}]$ and $\|f_{n,1}\|^p + \|g_{n,1}\|^p < \zeta_n = \min\{\epsilon_n/5^p, (F_0(y_n) - F_0(x_n))^p, (G_0(y_n) - G_0(x_n))^p\}$. It follows from 5.3 that there are $z_n \in (y_n, x_{n+1})$ and functions $f_{n,2}, g_{n,2} \in \mathcal{D}^+([z_{n-1}, z_n])$ such that $f_{n,2}g_{n,2} = \varphi$ on $[z_{n-1}, z_n]$, $f_{n,2} = g_{n,2} = 0$ on $\{z_{n-1}, z_n\}$, and $\|f_{n,2}\|^p + \|g_{n,2}\|^p < \zeta_n$. Denote $[z_{n-1}, z_n]$ by J_n. Since $|J_n| < 1$, we have (see (3)) $\int_{J_n} f_{n,2} \le$

$\|f_{n,2}\| < F_0(y_n) - F_0(x_n) < F_0(z_n) - F_0(z_{n-1})$; similarly $\int_{J_n} g_{n,2} < G_0(z_n) - G_0(z_{n-1})$. According to 5.6 there are $f_n, g_n \in \mathcal{D}^+(J_n)$ fulfilling (6)-(8) such that $f_n = g_n = 0$ on $\{z_{n-1}, z_n\}$ and $f_n g_n = \varphi$ on J_n. There are $F, G \in \mathcal{D}^+(I)$ such that $F' = f_n$, $G' = g_n$ on J_n, $F(z_n) = F_0(z_n)$ and $G(z_n) = G_0(z_n)$ for each n. The estimate (5) follows just as it did in the proof of 5.7. It is clear that $F, G \in \Delta^+(I)$ and that $F'G' = \varphi$ on I. If $x \in J_n$, then $F_0(z_{n-1}) \le F(x) \le F_0(z_n)$ so that $|F(x) - F_0(x)| \le F_0(z_n) - F_0(z_{n-1}) < F_0(x_{n+1}) - F_0(y_{n-1})$. Likdwise $|G(x) - G_0(x)| < G_0(x_{n+1}) - G_0(y_{n-1})$. This completes the proof of (4).

5.8. PROPOSITION. Let C be a closed set, $U = R \setminus C$, $f_0, g_0 \in \mathcal{D}$ ($\mathcal{D}^+$ resp.), $\|f_0\| + \|g_0\| < \infty$, $\varphi \in \mathit{loc}\,\mathcal{R}(U)$ ($\mathit{loc}\,\mathcal{R}^+(U)$ resp.) and let $\eta \in (0, \infty)$. Then there are $f, g \in \mathcal{D}$ ($\mathcal{D}^+$ resp.) with $f = f_0$, $g = g_0$ on C, $fg = \varphi$ on U and

(9) $$\|f\|^p + \|g\|^p < \eta + 8^{p-1}(\|f_0\|^p + \|g_0\|^p) .$$

PROOF. Let $B = 8^{p-1}$. There is a $\delta \in (0, \infty)$ with $B((\|f_0\|_U + \delta)^p + (\|g_0\|_U + \delta)^p) < \frac{\eta}{2} + B(\|f_0\|_U^p + \|g_0\|_U^p)$. Choose an $\omega \in \Delta$ with $\omega > 0$ on U, $\omega = \omega' = 0$ on C and $\|\omega\| < \delta$. Let $F_0' = f_0 + \omega$, $G_0' = g_0 + \omega$. If $f_0, g_0 \in \mathcal{D}^+$, then $F_0'G_0' > 0$ on U. Applying 5.7 (5.7.1 resp.) to each component of U we get $F, G \in \Delta(U)$ ($\Delta^+(U)$ resp.) such that $F'G' = \varphi$, $|F - F_0| + |G - G_0| < \omega$ on U and $\|F'\|_U^p + \|G'\|_U^p < \frac{\eta}{2} + B(\|F_0'\|_U^p + \|G_0'\|_U^p)$. Since $\|F_0'\|_U < \|f_0\|_U + \delta$ and $\|G_0'\|_U < \|g_0\|_U + \delta$, we have

(10) $$\|F'\|_U^p + \|G'\|_U^p < \eta + B(\|f_0\|_U^p + \|g_0\|_U^p) .$$

Define $F = F_0$ and $G = G_0$ on C. Since $\omega = \omega' = 0$ on C, we have $F, G \in \Delta$ and $F' = F_0'$, $G' = G_0'$ on C. Let $f = F'$ and $g = G'$. Then $F' = f_0$, $G' = g_0$ on C and (9) follows at once from (10). If $f_0, g_0 \in \mathcal{D}^+$, then $f, g \in \mathcal{D}^+$ as well.

5.9. THEOREM. Let φ be a function on R. Suppose that for each nonempty closed set, C, there is an open interval I such that $C \cap I \ne \emptyset$ and $\varphi \in \mathcal{R}(C \cap I)$ ($\mathcal{R}^+(C \cap I)$ resp.). Then $\varphi \in \mathcal{R}$ ($\mathcal{R}^+$ resp.).

PROOF. Let $U = \{x :$ there is an open interval, I, with $x \in I$ and $\varphi \in \mathcal{R}(I)\}$, and let $C = R \setminus U$. Our objective is to show that $C = \emptyset$, for if so, then by 5.8 $\varphi \in \mathcal{R}$. So suppose $C \ne \emptyset$. Then there is an open interval, I, with $C \cap I \ne \emptyset$ and $\varphi \in \mathcal{R}(C \cap I)$. Let $\varepsilon \in (0, \infty)$. Then there are $f_0, g_0 \in \mathcal{D}$ such that $8^{p-1}(\|f_0\|^p + \|g_0\|^p) < \varepsilon$ and $f_0 g_0 = \varphi$ on $C \cap I$. By 5.8 there are $f, g \in \mathcal{D}$ such that $f = f_0$,

$g = g_0$ on C, $fg = \varphi$ on U, and $\|f\|^p + \|g\|^p < \varepsilon$. Since $fg = \varphi$ on $(C \cap I) \cup U$, we have $\varphi \in \mathcal{R}(I)$ and hence $I \subset U$ contrary to $C \cap I \neq \emptyset$. Consequently $C = \emptyset$. The alternate assertion can be proved in the same way.

We finally come to the objective of this section which, with the help of 5.2, is an easy consequence of 5.9.

5.10. COROLLARY. Let φ be a Baire one function with $\varphi = 0$ a.e. on R. Then $\varphi \in \mathcal{R}$. If in addition $\varphi \geq 0$, then $\varphi \in \mathcal{R}^+$.

PROOF. Let C be a nonempty, closed set. There is an open interval, I, such that $C \cap I \neq \emptyset$ and φ is bounded on $S = C \cap I$. By 5.2 $\varphi\chi_S \in \mathcal{R}$. So by definition $\varphi \in \mathcal{R}(C \cap I)$. Thus 5.9 implies that $\varphi \in \mathcal{R}$. The proof of the additional assertion is similar.

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